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Nagoya Math. J.
Vol. 70 (1978), 81-104

INNOVATION PROCESSES ASSOCIATED WITH STATIONARY GAUSSIAN PROCESSES WITH APPLICATION TO THE PROBLEM OF PREDICTION

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§ 1. Introduction

As a continuation of the previous paper [7], we shall consider in this paper the problem of prediction given bounded intervals and obtain integral representations of predictors and prediction errors. For that purpose we shall introduce innovation processes well matched with bounded intervals. We follow the notation and terminology in [6].

Let $X = (X(t); t \in \boldsymbol{R})$ be a real separable and measurable stationary Gaussian process on a probability space $(\Omega, \boldsymbol{F}, P)$ with expectation zero which is continuous in the mean and purely nondeterministic. Furthermore we suppose that X has the N-ple Markovian property in the broad sense ([7]). We then know that the spectral measure of X has a Hardy density Δ whose outer part h is expressed in the form

$$(1.1)\quad \begin{cases} h(\lambda) = \dfrac{Q(-\lambda)}{P(-\lambda)} \qquad (\lambda \in \boldsymbol{R}) , \\ P(\lambda) = \sum_{n=0}^{N} c_n(-i\lambda)^n , \quad Q(\lambda) = \sum_{n=0}^{N-1} b_n(-i\lambda)^n , \quad c_n, b_n \in \boldsymbol{R} , \quad c_N \neq 0 \\ \text{and} \\ V_P \subset \boldsymbol{C}^+ , \quad V_Q \subset \boldsymbol{C}^+ \cup \boldsymbol{R} , \quad V_p \cap V_Q = \phi , \end{cases}$$

where V_S denotes the set of zero points of a polynomial S.

In [7], we have constructed an N-dimensional stationary Gaussian process $\mathscr{X} = (\mathscr{X}(t); t \in \boldsymbol{R})$ satisfying

$$(1.2)\qquad \boldsymbol{F}_X^{+/-}(t) = \boldsymbol{F}_{\mathscr{X}}^{+/-}(t) = \sigma(\mathscr{X}(t)) \qquad (t \in \boldsymbol{R}) .$$

Similarly we can obtain an N-dimensional stationary Gaussian process $\mathscr{Y} = (\mathscr{Y}(t); t \in \boldsymbol{R})$ satisfying

Received July 8, 1976.

(1.3) $$F_X^{-/+}(t) = F_{\mathscr{Y}}^{-/+}(t) = \sigma(\mathscr{Y}(t)) \qquad (t \in \boldsymbol{R}) .$$

Using these processes $\mathscr{X}$ and $\mathscr{Y}$, we shall define in §2 for any $a \in \boldsymbol{R}$ innovation processes $\boldsymbol{\nu}_a^{\pm} = (\nu_a^{\pm}(t)\,;\, t \geq 0)$ which are standard $(\boldsymbol{F}_a^{\pm}(t)\,;\, t \geq 0)$-Brownian motions, where $\boldsymbol{F}_a^{+}(t) = \partial \boldsymbol{F}_X(a)$ $(t = 0)$, $\boldsymbol{F}_X((a, a + t))$ $(t > 0)$ and $\boldsymbol{F}_a^{-}(t) = \partial \boldsymbol{F}_X(a)$ $(t = 0)$, $\boldsymbol{F}_X((a - t, a))$ $(t > 0)$.

In §3 we shall obtain integral representations of the predictors $E(\mathscr{X}(a + T) | \boldsymbol{F}_X((a, a + T)))$ and $E(X(a + T + t) | \boldsymbol{F}_X((a, a + T)))$ (resp. $E(\mathscr{Y}(a - T) | \boldsymbol{F}_X((a - T, a)))$ and $E(X(a - T - t) | \boldsymbol{F}_X((a - T, a))))$ in terms of innovation processes $\boldsymbol{\nu}_a^{+}$ (resp. $\boldsymbol{\nu}_a^{-}$) $(a \in \boldsymbol{R},\ t > 0,\ T > 0)$. As an application of these results, we shall find that Gaussian processes $\boldsymbol{Y}_{\pm} = (Y_{\pm}(t)\,;\, t \geq 0) = (X(\pm t) - E(X(\pm t) | \partial \boldsymbol{F}_X(0))\,;\, t \geq 0)$ have canonical representations ([3]).

We shall obtain in §4 integral representations of the predictors $E(\mathscr{X}(a - t) | \boldsymbol{F}_X((a, a + T)))$ and $E(X(a - t) | \boldsymbol{F}_X((a, a + T)))$ (resp. $E(\mathscr{Y}(a + t) | \boldsymbol{F}_X((a - T, a)))$ and $E(X(a + t) | \boldsymbol{F}_X((a - T, a))))$ in terms of innovation processes $\boldsymbol{\nu}_a^{+}$ (resp. $\boldsymbol{\nu}_a^{-}$) $(a \in \boldsymbol{R},\ t > 0,\ T > 0)$. Representation kernels in representation theorems in §3 and §4 can be written using the solution of matrix Riccati equation.

In §5 we shall prove orthogonal decomposition theorems of integral representations of the predictors $E(\mathscr{X}(-a - t) | \boldsymbol{F}_X((-a, a)))$ and $E(X(-a - t) | \boldsymbol{F}_X((-a, a)))$ (resp. $E(\mathscr{Y}(a + t) | \boldsymbol{F}_X((-a, a)))$ and $E(X(a + t) | \boldsymbol{F}_X((-a, a))))$ in terms of innovation processes $\boldsymbol{\nu}_0^{-}$ and $\boldsymbol{\nu}_{-a}^{+}$ (resp. $\boldsymbol{\nu}_0^{+}$ and $\boldsymbol{\nu}_a^{-}$) $(a > 0,\ t > 0)$.

In §6 we shall give concrete computations in the space $\boldsymbol{Z}_{\Delta}$ of representation kernels in representation theorems in §3 and §4 and then obtain explicit representations of prediction errors of $E(X(a + t) | \boldsymbol{F}_X((a - T, a)))$, $E(X(a - t) | \boldsymbol{F}_X((a, a + T)))$ $(a \in \boldsymbol{R},\ t > 0,\ T > 0)$ and $E(X(\pm(a + t)) | \boldsymbol{F}_X((-a, a)))$ $(a > 0,\ t > 0)$.

Using the results of the previous section, we shall obtain in §7 integral representations of the predictors $E(\mathscr{X}(a + t) | \boldsymbol{F}_X((a - T, a)))$ (resp. $E(\mathscr{Y}(a - t) | \boldsymbol{F}_X((a, a + T))))$ in terms of innovation processes $\boldsymbol{\nu}_a^{-}$ (resp. $\boldsymbol{\nu}_a^{+}$) $(a \in \boldsymbol{R},\ t > 0,\ T > 0)$ and then the predictors $E(\mathscr{X}(a + t) | \boldsymbol{F}_X((-a, a)))$ (resp. $E(\mathscr{Y}(-a - t) | \boldsymbol{F}_X((-a, a))))$ in terms of innovation processes $\boldsymbol{\nu}_0^{+}$ and $\boldsymbol{\nu}_a^{-}$ (resp. $\boldsymbol{\nu}_0^{-}$ and $\boldsymbol{\nu}_{-a}^{+}$) $(a > 0,\ t > 0)$.

§2. Innovation processes $\boldsymbol{\nu}_a^{+}$ and $\boldsymbol{\nu}_a^{-}$ $(a \in \boldsymbol{R})$

We denote by E the Fourier transform of h in (1.1). Then we have the following canonical representation:

https://doi.org/10.1017/S0027763000021796 Published online by Cambridge University Press

(2.1) $$X(t) = \sqrt{2\pi}^{-1} \int_{-\infty}^{t} E(t-s)dB(s) ,$$

where $(B(t); t \in \boldsymbol{R})$ is a standard Brownian motion satisfying

(2.2) $$\boldsymbol{F}_X^-(t) = \sigma(B(s_1) - B(s_2); s_1, s_2 < t) \qquad (t \in \boldsymbol{R}) .$$

We define an $N \times N$-matrix A and an $N \times 1$-vector $\boldsymbol{b}$ by

(2.3) $$A = \begin{pmatrix} 0 & & & & a_0 \\ -1 & 0 & & 0 & a_1 \\ & -1 & 0 & & a_2 \\ & & \ddots & \ddots & \vdots \\ 0 & & \ddots & 0 & a_{N-2} \\ & & & -1 & a_{N-1} \end{pmatrix} \quad \text{and} \quad \boldsymbol{b} = \begin{pmatrix} b_0 \\ b_1 \\ \vdots \\ b_{N-1} \end{pmatrix} ,$$

where $a_n = c_n c_N^{-1}$ $(0 \leq n \leq N-1)$. Since the characteristic equation of A is equal to $(-1)^N c_N^{-1} P(i^{-1}\lambda)$ and so all eigenvalues of A have negative real parts by (1.1), we can define N real L^2-functions E_n $(0 \leq n \leq N-1)$ and a real L^2-function F by

(2.4) $$E_n(t) = \sqrt{2\pi}^{-1} \chi_{(0,\infty)}(t)(e^{tA} \cdot \boldsymbol{b})_n \qquad (t \in \boldsymbol{R})$$

and

(2.5) $$F(t) = -\sqrt{2\pi} c_N^{-1} \chi_{(0,\infty)}(t) \left(e^{tA} \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \right)_{N-1} \qquad (t \in \boldsymbol{R}) .$$

Using these L^2-functions E_n, we define an N-dimensional stationary Gaussian process $\mathscr{X} = (\mathscr{X}(t); t \in \boldsymbol{R}) = ((X_0(t), \cdots, X_{N-1}(t))^*; t \in \boldsymbol{R})$ by

(2.6) $$X_n(t) = \sqrt{2\pi}^{-1} \int_{-\infty}^{t} E_n(t-s)dB(s) \qquad (0 \leq n \leq N-1,\ t \in \boldsymbol{R}) .$$

Then we see from the results in [7] that

THEOREM 2.1 (i) $X_{N-1}(t) = (-2\pi)^{-1} c_N X(t)$ $(t \in \boldsymbol{R})$.
(ii) $\{X_n(t); 0 \leq n \leq N-1\}$ *is linearly independent in* $\boldsymbol{M}_X$ *for any* $t \in \boldsymbol{R}$.
(iii) $\boldsymbol{M}_X^-(t) = \boldsymbol{M}_{\mathscr{X}}^-(t)$ *and* $\boldsymbol{F}_X^-(t) = \boldsymbol{F}_{\mathscr{X}}^-(t)$ $(t \in \boldsymbol{R})$.
(iv) $\boldsymbol{M}_X^{+/-}(t)$ *is equal to the linear hull of* $\{X_n(t); 0 \leq n \leq N-1\}$ $(t \in \boldsymbol{R})$.
(v) $\boldsymbol{F}_X^{+/-}(t) = \boldsymbol{F}_{\mathscr{X}}^{+/-}(t) = \sigma(\mathscr{X}(t))$ $(t \in \boldsymbol{R})$.

(vi) $\mathscr{X}(t) - \mathscr{X}(s) = \sqrt{2\pi}^{-1}(B(t) - B(s))\boldsymbol{b} + \int_s^t A\mathscr{X}(u)du \ (s < t)$.

(vii) $\mathscr{X}(t) = e^{(t-s)A}\mathscr{X}(s) + \sqrt{2\pi}^{-1}\int_s^t e^{(t-u)A}\cdot \boldsymbol{b}dB(u) \ (s < t)$.

(viii) $E(X(t)\,|\,\boldsymbol{F}_X^-(s)) = \sum_{n=0}^{N-1}(-1)^n F^{(n)}(t-s)X_n(s) \ (s < t)$.

(ix) $E(\mathscr{X}(t)\,|\,\boldsymbol{F}_{\mathscr{X}}^-(s)) = e^{(t-s)A}\mathscr{X}(s) \ (s < t)$.

[2.1] Now we fix any $a \in \boldsymbol{R}$ and define the σ-fields $\boldsymbol{F}_a^+(t)$ $(t \geq 0)$ by

$$\boldsymbol{F}_a^+(t) = \begin{cases} \partial \boldsymbol{F}_X(a) & (t = 0)\,, \\ \boldsymbol{F}_X((a, a+t)) & (t > 0)\,. \end{cases} \tag{2.7}$$

Then we shall show

THEOREM 2.2. *There exists a standard Brownian motion* $\boldsymbol{\nu}_a^+ = (\nu_a^+(t)\,;\, t \geq 0)$ *such that*

(i) $\nu_a^+(0) = 0$,

(ii) $\boldsymbol{F}_a^+(t) = \partial\boldsymbol{F}_X(a) \vee \sigma(\nu_a^+(s)\,;\, 0 \leq s \leq t) \ (t \geq 0)$,

(iii) $\boldsymbol{\nu}_a^+$ *is independent of the* σ*-field* $\partial\boldsymbol{F}_X(a)$,

(iv) $X_{n_0}(t+a) - X_{n_0}(a) = \sqrt{2\pi}^{-1}b_{n_0}\nu_a^+(t) + b_{n_0}\boldsymbol{e}\cdot\int_a^{a+t} E(\mathscr{X}(s)\,|\,\boldsymbol{F}_X((a,s)))ds$

$(t \geq 0)$,

where $n_0 = \max\{n \in \{0, 1, \cdots, N-1\}\,;\, b_n \neq 0\}$ *and* $b_{n_0}\boldsymbol{e} =$ *the* $n_0 + 1$*-th row of the matrix* A.

Proof. By (iv) we define a stochastic process $\boldsymbol{\nu}_a^+ = (\nu_a^+(t)\,;\, t \geq 0)$ with continuous paths. It then follows from (2.2), Theorem 2.1 (iii) and (iv) that $\boldsymbol{\nu}_a^+$ is a square integrable $(\boldsymbol{F}_a^+(t)\,;\, t \geq 0)$-martingale with expectation zero. We put $M = N - 1 - n_0$. It is easy to see from Lemma 2.4, Theorem 2.1 (i) and Lemma 4.1 (i) in [7] that $E^{(n)}(0+) = 0$ $(0 \leq n \leq M-1)$. This implies by (2.1) that $X(t)$ is M-times differentiable in the mean and a stationary Gaussian process $\boldsymbol{X}^{(M)} = (X^{(M)}(t)\,;\, t \in \boldsymbol{R})$ has the same property as $\boldsymbol{X}$. Applying Theorem 2.1 in this paper to the process $\boldsymbol{X}^{(M)}$, we have an N-dimensional stationary Gaussian process $\mathscr{X}_M = (\mathscr{X}_M(t)\,;\, t \in \boldsymbol{R})$ such that $\mathscr{X}_{M,N-1}(t) = (-2\pi)^{-1}c_N X^{(M)}(t)$ and

$$\begin{aligned} &\mathscr{X}_{M,N-1}(t+a) - \mathscr{X}_{M,N-1}(a) \\ &\quad = (-1)^M b_{n_0}\sqrt{2\pi}^{-1}(B(t+a) - B(a)) + \int_a^{a+t}(A\mathscr{X}_M(u))_{N-1}du \qquad (t \geq 0)\,. \end{aligned}$$

Using this process $\mathscr{X}_M$, we define an $\boldsymbol{R}^N$-valued stochastic process

https://doi.org/10.1017/S0027763000021796 Published online by Cambridge University Press

$(\mathscr{X}(t); t \geq 0)$ and a real valued stochastic process $(Y(t); t \geq 0)$ by

$$\begin{cases} \mathscr{X}(t) = \mathscr{X}_M(t + a) - E(\mathscr{X}_M(t + a) \,|\, \partial \boldsymbol{F}_X(a)) \ , \\ Y(t) = (-1)^M \sqrt{2\pi} \boldsymbol{e} \cdot \int_0^t \mathscr{X}(s) ds + B(t + a) - B(a) \ , \end{cases}$$

and then a real valued stochastic process $(\eta_a^+(t); t \geq 0)$ by

$$\eta_a^+(t) = Y(t) - (-1)^M \sqrt{2\pi} \boldsymbol{e} \int_0^t E(\mathscr{X}(s) \,|\, \boldsymbol{F}_X((a, a + s))) ds \ .$$

Then it follows from the results of [4] and [5] that $(\eta_a^+(t); t \geq 0)$ is a standard $(\boldsymbol{F}_a^+(t); t \geq 0)$-Brownian motion for which $\sigma(\eta_a^+(s); 0 \leq s \leq t)$ is equal to $\sigma(Y(s); 0 \leq s \leq t)$ and $\sigma(\eta_a^+(u) - \eta_a^+(v); u, v > t)$ is independent of $\boldsymbol{F}_a^+(t)$. By the definition of the process $(Y(t); t \geq 0)$, $Y(t) = (-1)^M b_{n_0}^{-1}(\mathscr{X}_{M,N-1}(t + a) - E(\mathscr{X}_{M,N-1}(t + a) \,|\, \partial \boldsymbol{F}_X(a)))$. Therefore, noting that $\partial \boldsymbol{F}_X(a) = \sigma(X^{(n)}(a); 0 \leq n \leq M)$ and $\mathscr{X}_{M,N-1}(t) = (-2\pi)^{-1} c_N X^{(M)}(t)$, we find that $\sigma(Y(s); 0 \leq s \leq t) \vee \partial \boldsymbol{F}_X(a) = \boldsymbol{F}_X((a, a + t))$ and so $\boldsymbol{F}_a^+(t) = \partial \boldsymbol{F}_X(a) \vee \sigma(\eta_a^+(s); 0 \leq s \leq t)$. On the other hand, by the definition of the processes $(\nu_a^+(t); t \geq 0)$ and $(\eta_a^+(t); t \geq 0)$, we see that $(\nu_a^+(t) - \eta_a^+(t); t \geq 0)$ is a bounded variation process. Since $(\nu_a^+(t); t \geq 0)$ and $(\eta_a^+(t); t \geq 0)$ are continuous $(\boldsymbol{F}_a^+(t); t \geq 0)$-martingales, we find that $\nu_a^+(t) = \eta_a^+(t)$ for any $t \in [0, \infty)$ and so this completes the proof of Theorem 2.2. (Q.E.D.)

[2.2] Next we define the σ-fields $\boldsymbol{F}_a^-(t)$ $(t \geq 0)$ by

$$\boldsymbol{F}_a^-(t) = \begin{cases} \partial \boldsymbol{F}_X(a) & (t = 0) \ , \\ \boldsymbol{F}_X((a - t, a)) & (t > 0) \ . \end{cases} \tag{2.8}$$

Noting that $\widehat{\check{h}} = \check{E}$, we see that there exists a standard Brownian motion $(B_-(t); t \in \boldsymbol{R})$ for which the followings hold:

$$X(t) = \sqrt{2\pi}^{-1} \int_t^\infty E(s - t) dB_-(s) \ , \tag{2.9}$$

$$\boldsymbol{F}_X^+(t) = \sigma(B_-(s_1) - B_-(s_2); s_1, s_2 > t) \qquad (t \in \boldsymbol{R}) \ . \tag{2.10}$$

Using this Brownian motion $(B_-(t); t \in \boldsymbol{R})$ and N real L^2-functions E_n $(0 \leq n \leq N - 1)$ in (2.4), we define an N-dimensional stationary Gaussian process $\mathscr{Y} = (\mathscr{Y}(t); t \in \boldsymbol{R}) = ((Y_0(t), \cdots, Y_{N-1}(t))^*; t \in \boldsymbol{R})$ by

$$Y_n(t) = \sqrt{2\pi}^{-1} \int_t^\infty E_n(s - t) B_-(s) \qquad (0 \leq n \leq N - 1, \ t \in \boldsymbol{R}) \ . \tag{2.11}$$

Similarly as in Theorem 2.1, we have

THEOREM 2.3. (i) $Y_{N-1}(t) = X_{N-1}(t) = (-2\pi)^{-1}c_N X(t)$ $(t \in \boldsymbol{R})$.

(ii) $\{Y_n(t); 0 \leq n \leq N-1\}$ *is linearly independent in* $\boldsymbol{M}_X$ *for any* $t \in \boldsymbol{R}$.

(iii) $\boldsymbol{M}_X^+(t) = \boldsymbol{M}_{\mathscr{Y}}^+(t)$ *and* $\boldsymbol{F}_X^+(t) = \boldsymbol{F}_{\mathscr{Y}}^+(t)$ $(t \in \boldsymbol{R})$.

(iv) $\boldsymbol{M}_X^{-/+}(t)$ *is equal to the linear hull of* $\{Y_n(t); 0 \leq n \leq N-1\}$ $(t \in \boldsymbol{R})$.

(v) $\boldsymbol{F}_X^{-/+}(t) = \boldsymbol{F}_{\mathscr{Y}}^{-/+}(t) = \sigma(\mathscr{Y}(t))$ $(t \in \boldsymbol{R})$.

(vi) $\mathscr{Y}(s) - \mathscr{Y}(t) = \sqrt{2\pi}^{-1}(B_-(t) - B_-(s))\boldsymbol{b} + \int_s^t A\mathscr{Y}(u)du$ $(s < t)$.

(vii) $\mathscr{Y}(s) = e^{(t-s)A}\mathscr{Y}(t) + \sqrt{2\pi}^{-1}\int_s^t e^{(u-s)A}\cdot \boldsymbol{b}dB_-(u)$ $(s < t)$.

(viii) $E(X(s)\,|\,\boldsymbol{F}_X^+(t)) = \sum_{n=0}^{N-1}(-1)^n F^{(n)}(t-s)Y_n(t)$ $(s < t)$.

(ix) $E(\mathscr{Y}(s)\,|\,\boldsymbol{F}_{\mathscr{Y}}^+(t)) = e^{(t-s)A}\mathscr{Y}(t)$ $(s < t)$.

By virtue of Theorem 2.3, in the same way as Theorem 2.2, we obtain

THEOREM 2.4. *There exists a standard Brownian motion* $\boldsymbol{\nu}_a^- = (\nu_a^-(t); t \geq 0)$ *such that*

(i) $\nu_a^-(0) = 0$,

(ii) $\boldsymbol{F}_a^-(t) = \partial\boldsymbol{F}_X(a) \vee \sigma(\nu_a^-(s); 0 \leq s \leq t)$ $(t \geq 0)$,

(iii) $\boldsymbol{\nu}_a^-$ *is independent of the* σ*-field* $\partial\boldsymbol{F}_X(a)$,

(iv) $Y_{n_0}(a-t) - Y_{n_0}(a) = \sqrt{2\pi}^{-1}b_{n_0}\nu_a^-(t) + b_{n_0}\boldsymbol{e}\cdot\int_{a-t}^a E(\mathscr{Y}(s)\,|\,\boldsymbol{F}_X((s,a)))ds$ $(t \geq 0)$.

DEFINITION 2.1. We call the standard Brownian motions $\boldsymbol{\nu}_a^+$ (resp. $\boldsymbol{\nu}_a^-$) $(\boldsymbol{F}_a^+(t); t \geq 0)$- (resp. $(\boldsymbol{F}_a^-(t); t \geq 0)$-) innovation processes associated with the stationary Gaussian process $\boldsymbol{X}$.

[2.3] Finally in this subsection we shall give a relation between the family of innovation processes $\boldsymbol{\nu}_a^{\pm}$ $(a \in \boldsymbol{R})$. We have the unitary transformation group $(U(t); t \in \boldsymbol{R})$ acting on the space $\boldsymbol{M}_X$ defined by

$$U(t)X(s) = X(t+s) \qquad (t, s \in \boldsymbol{R})\,. \tag{2.12}$$

Then we shall show

THEOREM 2.5. $U(t)\boldsymbol{\nu}_a^{\pm} = \boldsymbol{\nu}_{a+t}^{\pm}$ *for any* $a \in \boldsymbol{R}$ *and* $t \in \boldsymbol{R}$.

Proof. It is easy to see that $U(t)\boldsymbol{M}_X((a,b)) = \boldsymbol{M}_X((a+t, b+t))$, $U(t)\partial\boldsymbol{M}_X(a) = \partial\boldsymbol{M}_X(a+t)$ and $U(t)\boldsymbol{M}_X^{+/-}(a) = \boldsymbol{M}_X^{+/-}(a+t)$. We define an N-dimensional stationary Gaussian process $\mathscr{X}(t) = (U(t)X_0(0), \cdots, U(t)X_{N-1}(0))^*$

$(t \in \boldsymbol{R})$. Then it follows from Theorem 2.1 (ii) and (iv) that $\mathscr{Z}(t)$ is continuous in the mean, each component of $\mathscr{Z}(t)$ belongs to the space $\boldsymbol{M}_X^{+/-}(t)$ and $\{U(t)X_n(0); 0 \leq n \leq N-1\}$ is linearly independent in $\boldsymbol{M}_X$ for any $t \in \boldsymbol{R}$. Therefore we see from Theorem 5.1 in [7] that there uniquely exists a constant $N \times N$-matrix $\tilde{T}$ for which $\mathscr{Z}(t) = \tilde{T}\mathscr{X}(t)$ $(t \in \boldsymbol{R})$. Since $\mathscr{Z}(0) = \mathscr{X}(0)$, we find that $\tilde{T}$ is the unit matrix and so $\mathscr{Z}(t) = \mathscr{X}(t)$. By Theorem 2.2 (iv) this implies that $U(t)\nu_a^+(s) = \nu_{a+t}^+(s)$. Similarly, we have $U(t)\nu_a^-(s) = \nu_{a+t}^-(s)$. (Q.E.D.)

§ 3. Integral representations of the predictors (I)

[3.1] In this subsection we shall obtain integral representations of the predictors $E(\mathscr{X}(a+T)\,|\,\boldsymbol{F}_a^+(T))$ $(a \in \boldsymbol{R},\ T \geq 0)$. For any $a \in \boldsymbol{R}$ we define $N \times N$-matrices $P_a(t)$ $(t \geq 0)$ by

$$(3.1)\qquad \begin{aligned} P_a(t) = E\{(&\mathscr{X}(a+t) - E(\mathscr{X}(a+t)\,|\,\boldsymbol{F}_a^+(t))\cdot(\mathscr{X}(a+t) \\ &- E(\mathscr{X}(a+t)\,|\,\boldsymbol{F}_a^+(t)))^*\} \end{aligned}$$

and then $N \times 1$-vectors $\boldsymbol{f}_a(t,s)$ $(0 \leq s \leq t < \infty)$ by

$$(3.2)\qquad \boldsymbol{f}_a(t,s) = e^{(t-s)A}\cdot(P_a(s)\boldsymbol{e}^* + \sqrt{2\pi}^{-1}\boldsymbol{b}) .$$

At first we shall show

LEMMA 3.1. $\boldsymbol{f}_a(t,s) = (\partial/\partial s)E(\nu_a^+(s)\cdot\mathscr{X}(a+t))$ $(0 \leq s \leq t < \infty)$.

Proof. We put $\tilde{\mathscr{X}}(s) = \mathscr{X}(a+s) - E(\mathscr{X}(a+s)\,|\,\boldsymbol{F}_a^+(s))$. It then follows from Theorems 2.1 (vi) and 2.2 (iv) that

$$(3.3)\qquad \nu_a^+(t) = \boldsymbol{e}\cdot\int_0^t \tilde{\mathscr{X}}(s)ds + B(a+t) - B(a) \qquad (t \geq 0) .$$

Therefore we see from (2.2) and Theorem 2.1 (vii) that

$$\begin{aligned} E(\nu_a^+(s)\cdot\mathscr{X}(a+t)) &= \int_0^s E(\mathscr{X}(a+t)\cdot\tilde{\mathscr{X}}(\iota)^*)\boldsymbol{e}^*\cdot d\iota \\ &\quad + E((B(a+s) - B(a))\cdot\mathscr{X}(a+t)) \\ &= \int_0^s e^{(t-\iota)A}E(\mathscr{X}(a+\tau)\cdot\tilde{\mathscr{X}}(\iota)^*)\boldsymbol{e}^* d\iota \\ &\quad + \sqrt{2\pi}^{-1}\int_a^{a+s} e^{(a+t-\iota)A}\boldsymbol{b}d\iota \\ &= \int_0^s e^{(t-\iota)A}(P_a(\iota)\boldsymbol{e}^* + \sqrt{2\pi}^{-1}\boldsymbol{b})d\iota . \end{aligned}$$

On the other hand, by Corollaries 2.1 and 2.2 in [6] and the results in [7], we find that $P_a(\iota)$ is continuous in ι and so this implies Lemma 3.1. (Q.E.D.)

LEMMA 3.2. *For any $a \in \boldsymbol{R}$ and any $T \in (0, \infty)$,*

$$E(\mathscr{X}(a+T) \mid \boldsymbol{F_X}((a, a+T))) = E(\mathscr{X}(a+T) \mid \partial \boldsymbol{F_X}(a)) + \int_0^T f_a(T, s) d\nu_a^+(s) .$$

Proof. We put $Y = \mathscr{X}(a+T) - E(\mathscr{X}(a+T) \mid \partial \boldsymbol{F_X}(a)) - \int_0^T f_a(T, s) d\nu_a^+(s)$. It then follows from Theorem 2.2 (i), (iii) and Lemma 3.1 that $(d/ds)E(\boldsymbol{Y} \cdot \nu_a^+(s)) = 0$ and so $E(\boldsymbol{Y} \cdot \nu_a^+(s)) = E(\boldsymbol{Y} \cdot \nu_a^+(0)) = 0$ for any $s \in [0, T]$. Since $\boldsymbol{Y}$ is orthogonal to the space $\partial \boldsymbol{M_X}(a)$, we see that $\boldsymbol{Y}$ is orthogonal to the closed linear hull of $\{\nu_a^+(s) ; 0 \leq s \leq T\} \cup \partial \boldsymbol{M_X}(a)$ and so $\boldsymbol{Y}$ is independent of the σ-field generated by $\{\nu_a^+(s) ; 0 \leq s \leq T\} \cup \partial \boldsymbol{M_X}(a)$. Therefore, by virtue of Theorem 2.2 (ii), we find that $E(\boldsymbol{Y} \mid \boldsymbol{F_X}((a, a+T))) = 0$ and this implies Lemma 3.2. (Q.E.D.)

Next we shall derive a differential equation which $P_a(t)$ satisfies.

LEMMA 3.3. *For any $a \in \boldsymbol{R}$, $P_a(t)$ is the unique solution of the following matrix Riccati equation:*

$$\begin{cases} \dfrac{dP_a(t)}{dt} = (A - \sqrt{2\pi}^{-1}\boldsymbol{b} \cdot \boldsymbol{e}) \cdot P_a(t) + P_a(t)(A - \sqrt{2\pi}^{-1}\boldsymbol{b} \cdot \boldsymbol{e})^* - P_a(t)\boldsymbol{e}^* \cdot \boldsymbol{e} P_a(t) \\ \hfill (t > 0) , \\ P_a(0) = K_{\mathfrak{x}}(0) - \Sigma_a(0) , \end{cases}$$

where $\Sigma_a(0) = E\{E(\mathscr{X}(a) \mid \partial \boldsymbol{F_X}(a)) \cdot E(\mathscr{X}(a) \mid \partial \boldsymbol{F_X}(a))^*\}$.

Proof. We put $\Sigma_a(t) = E\{E(\mathscr{X}(a+t) \mid \boldsymbol{F}_a^+(t)) \cdot E(\mathscr{X}(a+t) \mid \boldsymbol{F}_a^+(t))^*\}$. Then it follows from Theorems 2.1 (xi), 2.2 (iii) and Lemma 3.2 that

$$\Sigma_a(t) = e^{tA} \Sigma_a(0) e^{tA^*} + \int_0^t f_a(t, s) f_a^*(t, s) ds .$$

Noting that $P_a(t) = K_{\mathfrak{x}}(0) - \Sigma_a(t)$, we see from Lemma 5.2 in [7] that

$$\begin{aligned} \frac{dP_a(t)}{dt} &= -A \cdot \Sigma_a(t) - \Sigma_a(t) A^* - P_a(t) \boldsymbol{e}^* \cdot \boldsymbol{e} \cdot P_a(t) \\ &\quad - \sqrt{2\pi}^{-1} P_a(t) \boldsymbol{e}^* \cdot \boldsymbol{b}^* - \sqrt{2\pi}^{-1} \boldsymbol{b} \cdot \boldsymbol{e} \cdot P_a(t) - (2\pi)^{-1} \boldsymbol{b} \cdot \boldsymbol{b}^* \\ &= A \cdot (K_{\mathfrak{x}}(0) - \Sigma_a(t)) + (K_{\mathfrak{x}}(0) - \Sigma_a(t)) A^* \\ &\quad - P_a(t) \cdot \boldsymbol{e}^* \cdot \boldsymbol{e} \cdot P_a(t) - \sqrt{2\pi}^{-1} P_a(t) \boldsymbol{e}^* \cdot \boldsymbol{b}^* - \sqrt{2\pi}^{-1} \boldsymbol{b} \cdot \boldsymbol{e} \cdot P_a(t) \end{aligned}$$

$$= A \cdot P_a(t) + P_a(t)A^* - P_a(t)\boldsymbol{e}^* \cdot \boldsymbol{e} \cdot P_a(t)$$
$$- \sqrt{2\pi}^{-1} P_a(t)\boldsymbol{e}^* \cdot \boldsymbol{b}^* - \sqrt{2\pi}^{-1}\boldsymbol{b} \cdot \boldsymbol{e} \cdot P_a(t)$$
$$= (A - \sqrt{2\pi}^{-1}\boldsymbol{b} \cdot \boldsymbol{e})P_a(t) + P_a(t)(A - \sqrt{2\pi}^{-1}\boldsymbol{b} \cdot \boldsymbol{e})^* - P_a(t)\boldsymbol{e}^* \cdot \boldsymbol{e} \cdot P_a(t) .$$

(Q.E.D.)

We define an $N \times (N - n_0)$-matrix J by

$$J = (K_{\mathscr{X}}(0)_{mn})_{\substack{0 \le m \le N-1 \\ n_0 \le n \le N-1}} \cdot (K_{\mathscr{X}}(0)_{mn})^{-1}_{n_0 \le m,\, n \le N-1} . \tag{3.4}$$

Then we shall show

LEMMA 3.4. $\Sigma_a(0) = J \cdot (K_{\mathscr{X}}(0)_{mn})_{\substack{n_0 \le m \le N-1 \\ 0 \le n \le N-1}} = J \cdot (K_{\mathscr{X}}(0)_{mn})_{n_0 \le m,\, n \le N-1} \cdot J^*$.
In particular, $\Sigma_a(0)$ is independent of a.

Proof. Since the dimension of the space $\partial M_X(a)$ is $N - n_0$ ([1]), it follows from Theorem 2.1 (i) (ii) (vi) that $\partial M_X(a)$ equals the linear hull of $\{X_n(a);\, n_0 \le n \le N-1\}$. Therefore there uniquely exists an $N \times (N - n_0)$-matrix $J(a)$ such that $E(\mathscr{X}(a) \mid \partial \boldsymbol{F}_X(a)) = J(a)(X_{n_0}(a) \cdots X_{N-1}(a))^*$. This implies that

$$J(a)$$
$$= E\{\mathscr{X}(a) \cdot (X_{n_0}(a) \cdots X_{N-1}(a))\} \cdot E\{(X_{n_0}(a) \cdots X_{N-1}(a))^* \cdot (X_{n_0}(a) \cdots X_{N-1}(a))\}^{-1}$$
$$= E\{\mathscr{X}(0) \cdot (X_{n_0}(0) \cdots X_{N-1}(0))\} \cdot E\{(X_{n_0}(0) \cdots X_{N-1}(0))^* \cdot (X_{n_0}(0) \cdots X_{N-1}(0))\}^{-1}$$

and so we have Lemma 3.4. (Q.E.D.)

By the uniqueness of local solutions, we see from Lemmas 3.3 and 3.4 that

LEMMA 3.5. *For any $a, b \in \boldsymbol{R}$ $P_a(t) = P_b(t)$ $(t \ge 0)$.*

Consequently, combining above lemmas, we obtain

THEOREM 3.1. *For any $a \in \boldsymbol{R}$ and any $T \in (0, \infty)$,*

$$E(\mathscr{X}(a + T) \mid \boldsymbol{F}_X((a, a + T)))$$
$$= E(\mathscr{X}(a + T) \mid \partial \boldsymbol{F}_X(a)) + \int_0^T e^{(T-s)A}(P(s)\boldsymbol{e}^* + \sqrt{2\pi}^{-1}\boldsymbol{b})d\nu_a^+(s)$$
$$= e^{TA}J \cdot (X_{n_0}(a) \cdots X_{N-1}(a))^* + \int_0^T e^{(T-s)A}(P(s)\boldsymbol{e}^* + \sqrt{2\pi}^{-1}\boldsymbol{b})d\nu_a^+(s) ,$$

where J is the $N \times (N - n_0)$-matrix given by (3.4) *and $P(t)$ is the unique solution of the following matrix Riccati equation:*

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$$(3.5)\quad \begin{cases} \dfrac{dP(t)}{dt} = (A - \sqrt{2\pi}^{-1}\boldsymbol{b}\cdot\boldsymbol{e})P(t) + P(t)(A - \sqrt{2\pi}^{-1}\boldsymbol{b}\cdot\boldsymbol{e})^* \\ \qquad - P(t)\boldsymbol{e}^*\cdot\boldsymbol{e}\cdot P(t) \qquad (t > 0)\,, \\ P(0) = K_{\mathscr{X}}(0) - J\cdot(K_{\mathscr{X}}(0)_{mn})_{n_0\le m,\ n\le N-1}J^*\,. \end{cases}$$

[3.2] In this subsection we shall obtain integral representations of the predictors $E(\mathscr{Y}(a-T)\,|\,\boldsymbol{F}_a^-(T))$ $(a\in\boldsymbol{R},\ T\ge 0)$. Similarly as in Lemmas 3.1 and 3.2, we see from Theorems 2.3 and 2.4 that

LEMMA 3.6. *For any $a\in\boldsymbol{R}$ and any $T\in(0,\infty)$,*

$$\begin{aligned} &E(\mathscr{Y}(a-T)\,|\,\boldsymbol{F}_X((a-T,a))) \\ &\quad = E(\mathscr{Y}(a-T)\,|\,\partial\boldsymbol{F}_X(a)) + \int_0^T e^{(T-s)A}(Q_a(s)\boldsymbol{e}^* + \sqrt{2\pi}^{-1}\boldsymbol{b})d\nu_a^-(s)\,, \end{aligned}$$

where $Q_a(t) = E\{(\mathscr{Y}(a-t) - E(\mathscr{Y}(a-t)\,|\,\boldsymbol{F}_a^-(t)))\cdot(\mathscr{Y}(a-t) - E(\mathscr{Y}(a-t)\,|\,\boldsymbol{F}_a^-(t)))^*\}$.

In the same way as Lemma 3.3, by Theorems 2.3, 2.4 and Lemma 3.6, we have

LEMMA 3.7. *For any $a\in\boldsymbol{R}$, $Q_a(t)$ satisfies the following matrix Riccati equation*

$$\begin{cases} \dfrac{dQ_a(t)}{dt} = (A - \sqrt{2\pi}^{-1}\boldsymbol{b}\cdot\boldsymbol{e})Q_a(t) + Q_a(t)(A - \sqrt{2\pi}^{-1}\boldsymbol{b}\cdot\boldsymbol{e})^* - Q_a(t)\cdot\boldsymbol{e}^*\cdot\boldsymbol{e}\cdot Q_a(t), \\ Q_a(0) = K_{\mathscr{Y}}(0) - \Pi_a(0)\,, \end{cases}$$

where $\Pi_a(0) = E\{E(\mathscr{Y}(a)\,|\,\partial\boldsymbol{F}_X(a))\cdot E(\mathscr{Y}(a)\,|\,\partial\boldsymbol{F}_X(a))^*\}$.

Now we shall show

LEMMA 3.8. *For any $a\in\boldsymbol{R}$ $Q_a(0) = P(0)$.*

Proof. Similarly as in Lemma 3.4, we see that

$$\Pi_a(0) = \tilde{J}\cdot(K_{\mathscr{Y}}(0)_{mn})_{n_0\le m,\ n\le N-1}\cdot\tilde{J}^*\,,$$

where $\tilde{J} = (K_{\mathscr{Y}}(0)_{mn})_{\substack{0\le m\le N-1\\ n_0\le n\le N-1}}\cdot(K_{\mathscr{Y}}(0)_{mn})^{-1}_{n_0\le m,\ n\le N-1}$. On the other hand, it follows from (2.6) and (2.11) that $K_{\mathscr{Y}}(0) = K_{\mathscr{X}}(0)$. This implies that $Q_a(0) = P(0)$. (Q.E.D.)

Therefore, we find from Lemmas 3.6, 3.7 and 3.8 that

THEOREM 3.2. *For any $a\in\boldsymbol{R}$ and any $T\in(0,\infty)$,*

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$$E(\mathscr{Y}(a-T)\,|\,\boldsymbol{F}_X((a-T,a)))$$
$$=E(\mathscr{Y}(a-T)\,|\,\partial\boldsymbol{F}_X(a))+\int_0^T e^{(T-s)A}(P(s)\boldsymbol{e}^*+\sqrt{2\pi}^{-1}\boldsymbol{b})d\nu_a^-(s)$$
$$=e^{TA}J\cdot(Y_{n_0}(a)\cdots Y_{N-1}(a))^*+\int_0^T e^{(T-s)A}(P(s)\boldsymbol{e}^*+\sqrt{2\pi}^{-1}\boldsymbol{b})d\nu_a^-(s)\;,$$

where J is the $N\times(N-n_0)$-matrix given by (3.4) *and $P(t)$ is the unique solution of the matrix Riccati equation* (3.5).

[3.3] As an application of Theorems 3.1 and 3.2, we shall show

THEOREM 3.3. *For any $a\in\boldsymbol{R}$, $t\in(0,\infty)$ and $T\in(0,\infty)$,*

(i) $E(X(a+T+t)\,|\,\boldsymbol{F}_X((a,a+T)))$
$$=E(X(a+T+t)\,|\,\partial\boldsymbol{F}_X(a))$$
$$+(0\cdots0(-c_N)^{-1}2\pi)\cdot\int_0^T \boldsymbol{f}(T+t,s)d\nu_a^+(s)$$

(ii) $E(X(a-T-t)\,|\,\boldsymbol{F}_X((a-T,a)))$
$$=E(X(a-T-t)\,|\,\partial\boldsymbol{F}_X(a))$$
$$+(0\cdots0(-c_N)^{-1}2\pi)\cdot\int_0^T \boldsymbol{f}(T+t,s)d\nu_a^-(s)\;,$$

where $\boldsymbol{f}(t,s)$ is the $N\times 1$-vector function for $a=0$ in (3.2).

Proof. By Theorems 2.1 (xi) and 3.1, we have

$$E(\mathscr{X}(a+T+t)\,|\,\boldsymbol{F}_X((a,a+T)))$$
$$=E(e^{tA}\mathscr{X}(a+T)\,|\,\boldsymbol{F}_X((a,a+T)))$$
$$=e^{tA}(E(\mathscr{X}(a+T)\,|\,\partial\boldsymbol{F}_X(a))+\int_0^T \boldsymbol{f}(T,s)d\nu_a^+(s))$$
$$=E(\mathscr{X}(a+T+t)\,|\,\partial\boldsymbol{F}_X(a))+\int_0^T \boldsymbol{f}(T+t,s)d\nu_a^+(s)\;.$$

Therefore we obtain (i) noting Theorem 2.1 (i). By Theorem 2.3 (i) (xi) and Theorem 3.2, (ii) is similarly proved. (Q.E.D.)

Immediately from Theorem 3.3, we have

THEOREM 3.4. *For any $a\in\boldsymbol{R}$ and $t\in(0,\infty)$,*

(i) $X(a+t)=E(X(a+t)\,|\,\partial\boldsymbol{F}_X(a))$
$$+(0\cdots0(-c_N)^{-1}2\pi)\cdot\int_0^t \boldsymbol{f}(t,s)d\nu_a^+(s)\;,$$

(ii) $X(a-t)=E(X(a-t)\,|\,\partial\boldsymbol{F}_X(a))$
$$+(0\cdots0(-c_N)^{-1}2\pi)\cdot\int_0^t \boldsymbol{f}(t,s)d\nu_a^-(s)\;.$$

We define two Gaussian processes $Y_{\pm} = (Y_{\pm}(t)\,;\, t \geq 0)$ by

(3.6) $$Y_{\pm}(t) = X(\pm t) - E(X(\pm t)\,|\,\partial F_X(0)) .$$

From Theorem 3.4, we have the following representations

(3.7) $$Y_{\pm}(t) = (0\cdots 0(-c_N)^{-1}2\pi)\cdot\int_0^t f(t, s)d\nu_0^{\pm}(s) .$$

It is easy to see from Theorems 2.2 (ii) and 2.4 (ii) that $\sigma(Y_{\pm}(s)\,;\, 0 \leq s \leq t) \vee \partial F_X(0) = \sigma(\nu_0^{\pm}(s)\,;\, 0 \leq s \leq t) \vee \partial F_X(0)$. Moreover $Y_{\pm}$ and $\nu_0^{\pm}$ are independent of $\partial F_X(0)$. Therefore we obtain

(3.8) $$\sigma(Y_{\pm}(s)\,;\, 0 \leq s \leq t) = \sigma(\nu_0^{\pm}(s)\,;\, 0 \leq s \leq t) \qquad (t \geq 0) .$$

This implies that representations (3.7) are canonical ([3]).

§ 4. Integral representations of the predictors (II)

In the previous section we have obtained integral representations of $E(X(b + t)\,|\,F_X((a, b))) - E(X(b + t)\,|\,\partial F_X(a))$ $(a < b,\ t > 0)$. The aim of this section is to obtain integral representations of $E(X(b + t)\,|\,F_X((a, b))) - E(X(b + t)\,|\,\partial F_X(b))$ $(a < b,\ t > 0)$.

For any $a \in \boldsymbol{R}$ we define $N \times 1$-vectors $\boldsymbol{g}_a(t, s)$ $(0 \leq s,\ t < \infty)$ by

(4.1) $$\boldsymbol{g}_a(t, s) = E\{\mathscr{Y}(a + t)\cdot(\mathscr{Y}(a - s) - E(\mathscr{Y}(a - s)\,|\,F_a^-(s)))^*\}\cdot \boldsymbol{e}^* .$$

Then we shall prove

LEMMA 4.1. $\boldsymbol{g}_a(t, s) = (\partial/\partial s)E(\nu_a^-(s)\mathscr{Y}(a + t))$.

Proof. We put $\tilde{\mathscr{Y}}(s) = \mathscr{Y}(a - s) - E(\mathscr{Y}(a - s)\,|\,F_a^-(s))$. It then follows from Theorems 2.3 (vi) and 2.4 (iv) that

(4.2) $$\nu_a^-(t) = \boldsymbol{e}\cdot\int_0^t \tilde{\mathscr{Y}}(s)ds + B_-(a) - B_-(a - t) \qquad (t \geq 0) .$$

Therefore, by (2.10) and Theorem 2.3 (v) (vii), we have

$$\begin{aligned} &E(\nu_a^-(s)\mathscr{Y}(a + t)) \\ &\quad = e^{-tA}E\Big\{\nu_a^-(s)(\mathscr{Y}(a) - \sqrt{2\pi}^{-1}\int_a^{a+t} e^{(u-a)A}\boldsymbol{b}dB_-(u))\Big\} \\ &\quad = e^{-tA}E\Big\{\boldsymbol{e}\cdot\int_0^s \tilde{\mathscr{Y}}(\iota)d\iota\cdot(\mathscr{Y}(a) - \sqrt{2\pi}^{-1}\cdot\int_a^{a+t}(e^{(u-a)A}\boldsymbol{b}\cdot dB_-(u))\Big\} \\ &\quad = E\Big(\boldsymbol{e}\cdot\int_0^s \tilde{\mathscr{Y}}(\iota)d\iota\cdot\mathscr{Y}(a + t)\Big) . \end{aligned}$$

This implies Lemma 4.1. (Q.E.D.)

Similarly as in Lemma 3.2, we can see from Theorem 2.4 and Lemma 4.1 that

LEMMA 4.2. *For any* $a \in \boldsymbol{R}$, $t \in (0, \infty)$ *and* $T \in (0, \infty)$,

$$E(\mathscr{Y}(a+t) \,|\, \boldsymbol{F}_X((a-T, a))) = E(\mathscr{Y}(a+t) \,|\, \partial \boldsymbol{F}_X(a)) + \int_0^T \boldsymbol{g}_a(t, s) d\nu_a^-(s) .$$

Next we shall obtain an explicit form of $\boldsymbol{g}_a(t, s)$. We define for any $t \in [0, \infty)$ an $N \times N$-matrix $R(t)$ by

$$R(t) = E\{\mathscr{Y}(t) \cdot (\mathscr{Y}(0) - E(\mathscr{Y}(0) \,|\, \partial \boldsymbol{F}_X(0)))^*\} . \tag{4.3}$$

LEMMA 4.3. *For any* $a \in \boldsymbol{R}$

$$\boldsymbol{g}_a(t, s) = R(t) \Phi^*(s) \boldsymbol{e}^* \qquad (0 \leq s,\ t < \infty) ,$$

where $\Phi(s)$ *is the unique solution of the following linear differential equation*

$$\begin{cases} \dfrac{d\Phi(s)}{ds} = (A - P(s)\boldsymbol{e}^*\boldsymbol{e} - \sqrt{2\pi}^{-1}\boldsymbol{b} \cdot \boldsymbol{e})\Phi(s) & (s > 0) , \\ \Phi(0) = I_N . \end{cases} \tag{4.4}$$

In particular, $\boldsymbol{g}_a(t, s)$ *are independent of* a.

Proof. We put $R_a(t, s) = E\{\mathscr{Y}(a+t) \cdot (\mathscr{Y}(a-s) - E(\mathscr{Y}(a-s) \,|\, \boldsymbol{F}_a^-(s)))^*\}$. By Theorem 3.2,

$$\begin{aligned} R_a(t, s) &= E\{\mathscr{Y}(a+t)(\mathscr{Y}(a-s) - E(\mathscr{Y}(a-s) \,|\, \partial \boldsymbol{F}_X(a)))^*\} \\ &\quad - E(\mathscr{Y}(a+t) \int_0^s \boldsymbol{f}^*(s, \iota) d\nu_a^-(\iota)) \\ &= I - II . \end{aligned}$$

It is easy to see from Theorem 2.4 and Lemma 4.2 that $II = \int_0^s \boldsymbol{g}_a(t, \iota) \boldsymbol{f}^*(s, \iota) d\iota$. On the other hand, by Theorem 2.3 (iii) (ix), $I = E\{\mathscr{Y}(a+t) \cdot (\mathscr{Y}(a) - E(\mathscr{Y}(a) \,|\, \partial \boldsymbol{F}_X(a))^*\} \cdot e^{sA^*}$. Since $K_{\mathscr{Y}}(0) = K_{\mathscr{X}}(0)$ and $\partial \boldsymbol{F}_X(a) = \sigma(Y_n(a);\, n_0 \leq n \leq N-1)$, it can be seen that $E(\mathscr{Y}(a) \,|\, \partial \boldsymbol{F}_X(a)) = J \cdot (Y_{n_0}(a) \cdots Y_{N-1}(a))^*$, where J is the $N \times (N - n_0)$-matrix in (3.4). Therefore we find that $I = E\{\mathscr{Y}(t)(\mathscr{Y}(0) - E(\mathscr{Y}(0) \,|\, \partial \boldsymbol{F}_X(0))^*\} e^{sA^*} = R(t) e^{sA^*}$. Consequently, we have

$$R_a(t, s) = R(t) e^{sA^*} - \int_0^s R_a(t, \iota) \boldsymbol{e}^* \cdot \boldsymbol{f}^*(s, \iota) d\iota .$$

Since $(\partial/\partial s)\boldsymbol{f}^*(s, \iota) = \boldsymbol{f}^*(s, \iota) A^*$ by (3.2), we obtain the following linear

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differential equation

$$\begin{cases} \dfrac{\partial}{\partial s} R_a(t, s) = R_a(t, s)(A - P(s)e^*e - \sqrt{2\pi}^{-1}b \cdot e)^* & (s > 0) , \\ R_a(t, 0) = R(t) . \end{cases}$$

Thus, using the unique solution $\Phi(t)$ of equation (4.4), we find that $R_a(t, s) = R(t)\Phi^*(s)$ and this completes the proof of Lemma 4.3. (Q.E.D.)

In the proof of Lemma 4.3, we have shown

$$(4.5) \qquad R(t) = K_{\mathscr{Y}}(0)e^{tA^*} - E(\mathscr{Y}(t) \cdot (Y_{n_0}(0) \cdots Y_{N-1}(0))) \cdot J^* ,$$

where J is the $N \times (N - n_0)$-matrix in (3.4). Furthermore we define for any $t \in \boldsymbol{R}$ an $N \times (N - n_0)$-matrix $J(t)$ by

$$(4.6) \qquad J(t) = (K_{\mathscr{Y}}(t)_{mn})_{\substack{0 \le m \le N-1 \\ n_0 \le n \le N-1}} \cdot (K_{\mathscr{Y}}(0)_{mn})^{-1}_{n_0 \le m,\, n \le N-1} .$$

Immediately from Lemmas 4.2 and 4.3, we have

THEOREM 4.1. *For any* $a \in \boldsymbol{R}$, $t \in (0, \infty)$ *and* $T \in (0, \infty)$,

$$E(\mathscr{Y}(a + t) \mid \boldsymbol{F}_X((a - T, a))) = J(t) \cdot (Y_{n_0}(a) \cdots Y_{N-1}(a))^* + \int_0^T R(t)\Phi^*(s)e^* d\nu_a^-(s) .$$

Since $K_{\mathscr{X}}(-t) = K_{\mathscr{Y}}(t)$ $(t \ge 0)$, it is easy to see from (4.3) and (4.5) that $E\{\mathscr{X}(-t)(\mathscr{X}(0) - E(\mathscr{X}(0) \mid \partial \boldsymbol{F}_{\mathscr{X}}(0)))^*\} = R(t)$ $(t \ge 0)$. Therefore, similarly as in Theorem 4.1, we can show from Theorems 2.1, 2.2, 3.1 and (3.3) that

THEOREM 4.2. *For any* $a \in \boldsymbol{R}$, $t \in (0, \infty)$ *and* $T \in (0, \infty)$,

$$E(\mathscr{X}(a - t) \mid \boldsymbol{F}_X((a, a + T))) = J(t)(X_{n_0}(a) \cdots X_{N-1}(a))^* + \int_0^T R(t)\Phi^*(s)e^* d\nu_a^+(s) .$$

For any $t \in [0, \infty)$ we define a $1 \times N$-vector $\boldsymbol{r}(t)$ by

$$(4.7) \qquad \boldsymbol{r}(t) = E\{X(t) \cdot (\mathscr{Y}(0) - E(\mathscr{Y}(0) \mid \partial \boldsymbol{F}_X(0)))^*\} .$$

Since $X(t) = (-2\pi)c_N^{-1}X_{N-1}(t) = (-2\pi)c_N^{-1}Y_{N-1}(t)$, it can be seen from Theorems 4.1 and 4.2 that

THEOREM 4.3. *For any* $a \in \boldsymbol{R}$, $t \in (0, \infty)$ *and* $T \in (0, \infty)$,

(i) $E(X(a + t) \mid \boldsymbol{F}_X((a - T, a))$

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$$= E(X(a+t)\,|\,\partial \boldsymbol{F}_X(a)) + \int_0^T \boldsymbol{r}(t)\Phi^*(s)\boldsymbol{e}^* d\nu_a^-(s)\ ,$$

(ii) $E(X(a-t)\,|\,\boldsymbol{F}_X((a, a+T)))$

$$= E(X(a-t)\,|\,\partial \boldsymbol{F}_X(a)) + \int_0^T \boldsymbol{r}(t)\Phi^*(s)\boldsymbol{e}^* d\nu_a^+(s)\ .$$

More generally, we define for any $Y \in \boldsymbol{M}_X$ two $1 \times N$-vectors $\boldsymbol{r}_\pm(Y)$ by

$$\boldsymbol{r}_+(Y) = E\{Y\cdot(\mathscr{X}(0) - E(\mathscr{X}(0)\,|\,\partial \boldsymbol{F}_X(0)))^*\} \tag{4.8}$$

and

$$\boldsymbol{r}_-(Y) = E\{Y\cdot(\mathscr{Y}(0) - E(\mathscr{Y}(0)\,|\,\partial \boldsymbol{F}_X(0)))^*\}\ . \tag{4.9}$$

Note that $\boldsymbol{r}(t) = \boldsymbol{r}_+(X(-t)) = \boldsymbol{r}_-(X(t))$ $(t \in [0, \infty))$. Using the unitary operator $U(-a)$ in (2.12), we can prove, by Theorem 4.3,

THEOREM 4.4. *Let any $a \in \boldsymbol{R}$ and $T \in (0, \infty)$ be fixed.*

(i) *For any $Y \in \boldsymbol{M}_X^+(a)$*

$$E(Y\,|\,\boldsymbol{F}_X((a-T, a))) = E(Y\,|\,\partial \boldsymbol{F}_X(a)) + \int_0^T \boldsymbol{r}_-(U(-a)Y)\Phi^*(s)\boldsymbol{e}^* d\nu_a^-(s)\ .$$

(ii) *For any $Y \in \boldsymbol{M}_X^-(a)$*

$$E(Y\,|\,\boldsymbol{F}_X((a, a+T))) = E(Y\,|\,\partial \boldsymbol{F}_X(a)) + \int_0^T \boldsymbol{r}_+(U(-a)Y)\Phi^*(s)\boldsymbol{e}^* d\nu_a^+(s)\ .$$

§ 5. Integral representations of the predictors (III)

In this section we shall obtain integral representations of $E(X(a+t)\,|\,\boldsymbol{F}_X((-a, a))) - E(X(a+t)\,|\,\partial \boldsymbol{F}_X(0))$ $(a > 0,\ t > 0)$. We define for any $t \in [0, \infty)$ two $N \times N$-matrices $S_\pm(t)$ by

$$S_+(t) = E(\mathscr{Y}(t)\cdot\mathscr{X}^*(0))\cdot K_{\mathscr{X}}(0)^{-1} \tag{5.1}$$

and

$$S_-(t) = E(\mathscr{X}(-t)\cdot\mathscr{Y}^*(0))\cdot K_{\mathscr{X}}(0)^{-1}\ . \tag{5.2}$$

As an application of Theorem 4.1, we shall prove

THEOREM 5.1. *For any $a \in (0, \infty)$ and $t \in (0, \infty)$,*

$E(\mathscr{Y}(a+t)\,|\,\boldsymbol{F}_X((-a, a)))$

$$= E(\mathscr{Y}(a+t)\,|\,\partial \boldsymbol{F}_X(0)) + \int_0^a S_+(t)e^{(a-s)A}(P(s)\boldsymbol{e}^* + \sqrt{2\pi}^{-1}\boldsymbol{b})d\nu_0^+(s)$$

$$+ \int_a^{2a} R(t)\Phi^*(s)\boldsymbol{e}^* d\nu_a^-(s) .$$

Proof. It is easy to see from Theorem 4.1 that

$$E(\mathscr{Y}(a+t)\,|\,\boldsymbol{F}_X((-a,a))) = E(\mathscr{Y}(a+t)\,|\,\boldsymbol{F}_X((0,a))) + \int_a^{2a} R(t)\Phi^*(s)\boldsymbol{e}^* d\nu_a^-(s) .$$

By Theorems 2.1 (iii) (iv) and 2.3 (iii) (iv), we can show that

$$E(\mathscr{Y}(a+t)\,|\,\boldsymbol{F}_X^-(a)) = S_+(t)\mathscr{X}(a) .$$

Therefore, it follows from Theorem 3.1 that

$$\begin{aligned} &E(\mathscr{Y}(a+t)\,|\,\boldsymbol{F}_X((0,a))) \\ &\quad = S_+(t)\cdot\Big(E(\mathscr{X}(a)\,|\,\partial\boldsymbol{F}_X(0)) + \int_0^a \boldsymbol{f}(a,s)d\nu_0^+(s)\Big) \\ &\quad = E(\mathscr{Y}(a+t)\,|\,\partial\boldsymbol{F}_X(0)) + \int_0^a S_+(t)\boldsymbol{f}(a,s)d\nu_0^+(s) , \end{aligned}$$

where $\boldsymbol{f}(a,s)$ are $N \times 1$-vectors in (3.2). Thus we have proved Theorem 5.1. (Q.E.D.)

Similarly, we find from Theorems 3.2 and 4.2 that

Theorem 5.2. *For any $a \in (0,\infty)$ and $t \in (0,\infty)$,*

$$\begin{aligned} &E(\mathscr{X}(-a-t)\,|\,\boldsymbol{F}_X((-a,a))) \\ &\quad = E(\mathscr{X}(-a-t)\,|\,\partial\boldsymbol{F}_X(0)) + \int_0^a S_-(t)e^{(a-s)A}(P(s)\boldsymbol{e}^* + \sqrt{2\pi}^{-1}\boldsymbol{b})d\nu_0^-(s) \\ &\qquad + \int_a^{2a} R(t)\Phi^*(s)\boldsymbol{e}^* d\nu_{-a}^+(s) . \end{aligned}$$

Remark 5.1. By Theorems 2.2 (ii) (iii) and 2.4 (ii) (iii) we note that the decompositions in Theorems 5.1 and 5.2 are orthogonal.

For any $t \in [0,\infty)$ we define two $1 \times N$-vectors $\boldsymbol{S}_\pm(t)$ by

$$\boldsymbol{S}_+(t) = E(X(t)\cdot\mathscr{X}^*(0))\cdot K_{\mathscr{X}}(0)^{-1} \tag{5.3}$$

and

$$\boldsymbol{S}_-(t) = E(X(-t)\cdot\mathscr{Y}^*(0))\cdot K_{\mathscr{X}}(0)^{-1} . \tag{5.4}$$

Since $X(t) = (-2\pi)c_N^{-1}X_{N-1}(t) = (-2\pi)c_N^{-1}Y_{N-1}(t)$, we can show from Theorems 5.1 and 5.2 that

Theorem 5.3. *For any $a \in (0,\infty)$ and $t \in (0,\infty)$,*

(i) $E(X(a+t)\,|\,\boldsymbol{F}_X((-a,a)))$

$$= E(X(a+t)\,|\,\partial\boldsymbol{F}_X(0)) + \int_0^a \boldsymbol{S}_+(t)e^{(a-s)A}(P(s)\boldsymbol{e}^* + \sqrt{2\pi}^{-1}\boldsymbol{b})d\nu_0^+(s)$$
$$+ \int_a^{2a} \boldsymbol{r}(t)\Phi^*(s)\boldsymbol{e}^* d\nu_a^-(s)\ ,$$

(ii) $E(X(-a-t)\,|\,\boldsymbol{F}_X((-a,a)))$

$$= E(X(-a-t)\,|\,\partial\boldsymbol{F}_X(0))$$
$$+ \int_0^a \boldsymbol{S}_-(t)e^{(a-s)A}(P(s)\boldsymbol{e}^* + \sqrt{2\pi}^{-1}\boldsymbol{b})d\nu_0^-(s)$$
$$+ \int_a^{2a} \boldsymbol{r}(t)\Phi^*(s)\boldsymbol{e}^* d\nu_{-a}^+(s)\ .$$

More generally, we define for any $Y \in \boldsymbol{M}_X$ two $1 \times N$-vectors $\boldsymbol{S}_\pm(Y)$ by

$$\boldsymbol{S}_+(Y) = E(Y\cdot\mathscr{X}^*(0))\cdot K_{\mathscr{X}}(0)^{-1} \tag{5.5}$$

and

$$\boldsymbol{S}_-(Y) = E(Y\cdot\mathscr{Y}^*(0))\cdot K_{\mathscr{X}}(0)^{-1}\ . \tag{5.6}$$

Using the unitary operators $U(\pm a)$ in (2.12), we can generalize Theorem 5.3 as follows.

THEOREM 5.4. *Let any $a \in (0,\infty)$ be fixed.*

(i) *For any $Y \in \boldsymbol{M}_X^+(a)$*

$E(Y\,|\,\boldsymbol{F}_X((-a,a)))$

$$= E(Y\,|\,\partial\boldsymbol{F}_X(0)) + \int_0^a \boldsymbol{S}_+(U(-a)Y)e^{(a-s)A}(P(s)\boldsymbol{e}^* + \sqrt{2\pi}^{-1}\boldsymbol{b})d\nu_0^+(s)$$
$$+ \int_a^{2a} \boldsymbol{r}_-(U(-a)Y)\Phi^*(s)\boldsymbol{e}^* d\nu_a^-(s)\ .$$

(ii) *For any $Y \in \boldsymbol{M}_X^-(-a)$*

$E(Y\,|\,\boldsymbol{F}_X((-a,a)))$

$$= E(Y\,|\,\partial\boldsymbol{F}_X(0)) + \int_0^a \boldsymbol{S}_-(U(a)Y)\cdot e^{(a-s)A}(P(s)\boldsymbol{e}^* + \sqrt{2\pi}^{-1}\boldsymbol{b})d\nu_0^-(s)$$
$$+ \int_a^{2a} \boldsymbol{r}_+(U(a)Y)\cdot\Phi^*(s)\boldsymbol{e}^* d\nu_{-a}^+(s)\ .$$

Remark 5.2. We note that the decompositions in Theorems 5.3 and 5.4 are orthogonal.

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§ 6. Prediction errors

In [6] we have obtained the following commutative diagram

(6.1)
$$\begin{array}{ccc} M_X & \xrightarrow{U} & Z_\Delta \\ {\scriptstyle U_1}\uparrow & & \uparrow{\scriptstyle V} \\ K & \xrightarrow{K} & L^2(\boldsymbol{R}) \end{array}$$

$$U_1(k(\cdot - t) = X(t)\,, \quad K(k(\cdot - t)) = \sqrt{2\pi}^{-1}E(t - \cdot) \quad \text{and} \quad U(X(t)) = e^{it\cdot}\,.$$

Similarly we have the following commutative diagram

(6.2)
$$\begin{array}{ccc} M_X & \xrightarrow{U} & Z_\Delta \\ {\scriptstyle U_1}\uparrow & & \uparrow{\scriptstyle \check{V}} \\ K & \xrightarrow{\check{K}} & L^2(\boldsymbol{R})\,, \end{array} \qquad \check{K}(k(\cdot - t)) = \sqrt{2\pi}^{-1}E(\cdot - t)\,.$$

We note that

(6.3)
$$f = \sqrt{2\pi}^{-1}(Vf\cdot\check{h})^\wedge = \sqrt{2\pi}^{-1}(\check{V}f\cdot h)^\wedge \qquad (f \in L^2(\boldsymbol{R}))\,.$$

Using L^2-functions E_n in (2.4) we define N functions φ_n in Z_Δ $(0 \le n \le N-1)$ by

(6.4)
$$\varphi_n = \sqrt{2\pi}^{-1}V(\check{E}_n)\,.$$

It is easy to see from (6.3) that

(6.5)
$$V(\sqrt{2\pi}^{-1}E_n(t - \cdot)) = e^{it\cdot}\cdot\varphi_n \quad \text{and} \quad \check{V}(\sqrt{2\pi}^{-1}E_n(\cdot - t)) = e^{it\cdot}\cdot\check{\varphi}_n\,.$$

Therefore it can be shown from (2.1), (2.6), (2.9) and (2.11) that

(6.6)
$$U(X_n(t)) = e^{it\cdot}\cdot\varphi_n \quad \text{and} \quad U(Y_n(t)) = e^{it\cdot}\cdot\check{\varphi}_n\,.$$

Furthermore we are able to prove by Theorems 2.2 (iv) and 2.4 (iv) that

(6.7)
$$U(\nu_a^+(t))(\lambda) = \sqrt{2\pi}\int_a^{a+t}\{b_{n_0}^{-1}\varphi_{n_0}(\lambda)i\lambda e^{is\lambda} - P_{Z_\Delta((a,s))}(\boldsymbol{e}\cdot U\mathscr{X}(s))(\lambda)\}ds$$

and

(6.8)
$$U(\nu_a^-(t))(\lambda) = \sqrt{2\pi}\int_{a-t}^{a}\{b_{n_0}^{-1}\varphi_{n_0}(-\lambda)(-i\lambda)e^{is\lambda} - P_{Z_\Delta((s,a))}(\boldsymbol{e}\cdot U\mathscr{Y}(s))(\lambda)\}ds\,,$$

where $U\mathscr{X}(s) = (UX_0(s), \cdots, UX_{N-1}(s))^*$ and $U\mathscr{Y}(s) = (UY_0(s), \cdots, UY_{N-1}(s))^*$. We define functions $D_a^\pm(t, \lambda)$ by

$$(6.9)\quad \begin{cases} D_a^+(t,\lambda) = \sqrt{2\pi}\{b_{n_0}^{-1}\varphi_{n_0}(\lambda)\cdot i\lambda e^{i(t+a)\lambda} - P_{Z_\Delta((a,a+t))}(\boldsymbol{e}\cdot U\mathscr{X}(t+a))(\lambda)\}\,, \\ D_a^-(t,\lambda) = \sqrt{2\pi}\{b_{n_0}^{-1}\varphi_{n_0}(-\lambda)(-i\lambda)\cdot e^{i(a-t)\lambda} - P_{Z_\Delta((a-t,a))}(\boldsymbol{e}\cdot U\mathscr{Y}(a-t))(\lambda)\}\,. \end{cases}$$

By (6.7), (6.8) and (6.9), it follows from Theorems 3.1 and 3.2 that

$$(6.10)\quad \begin{cases} D_a^+(t,\lambda) + \sqrt{2\pi}\int_0^t \boldsymbol{e}\cdot \boldsymbol{f}(t,s)D_a^+(s,\lambda)ds \\ \qquad = \sqrt{2\pi}\{b_{n_0}^{-1}\varphi_{n_0}(\lambda)i\lambda e^{i(t+a)\lambda} - P_{\partial Z_\Delta(a)}(\boldsymbol{e}\cdot U\mathscr{X}(t+a))(\lambda)\}\,, \\ D_a^-(t,\lambda) + \sqrt{2\pi}\int_0^t \boldsymbol{e}\cdot \boldsymbol{f}(t,s)D_a^-(s,\lambda)ds \\ \qquad = \sqrt{2\pi}\{b_{n_0}^{-1}\varphi_{n_0}(-\lambda)(-i\lambda)e^{i(a-t)\lambda} - P_{\partial Z_\Delta(a)}(\boldsymbol{e}\cdot U\mathscr{Y}(a-t))(\lambda)\}\,. \end{cases}$$

In [6] we have introduced the function $P(\lambda,\varphi)(\lambda\in \boldsymbol{C},\ \varphi\in\mathscr{O}(\{0\}))$ defined by

$$(6.11)\qquad P(\lambda,\varphi) = (2\pi)^{-1}\sum_{n=0}^{N-1}\left(\sum_{k=0}^{N-n-1} c_{n+k+1}\varphi^{(k)}(0)\right)(-i\lambda)^n\,.$$

We then define N functions P_n $(0\le n\le N-1)$ by

$$(6.12)\qquad P_n(\lambda) = (n\,!)^{-1}P(\lambda, x^n)\,.$$

Firstly we shall prove

LEMMA 6.1. *$\varphi_n = P_n$ for any $n\in\{n_0, n_0+1,\cdots,N-1\}$.*

Proof. We define N real L^2-functions F_n $(0\le n\le N-1)$ by

$$(6.13)\qquad F_n(t) = \sqrt{2\pi}^{-1}\chi_{(0,\infty)}(t)\left(e^{tA}\begin{pmatrix}1\\0\\\vdots\\0\end{pmatrix}\right)_n.$$

By Lemma 8.2 in [6] and (2.3), (6.8) in [7], we have

$$(6.14)\qquad F_n = (n\,!)^{-1}\{P(-\cdot, x^n)\cdot P(-\cdot)^{-1}\}^\wedge = (\check{P}_n\check{P}^{-1})^\wedge\,.$$

Since P_n $(n_0\le n\le N-1)$, P and Q are polynomials of at most order $N-n_0-1$, N and n_0, respectively, we see from (2.7) in [7] and Lemma 4.1 in [7] that

$$(6.15)\qquad E_n = (\check{P}_n\cdot\check{Q}\cdot\check{P}^{-1})^\wedge = (\check{P}_n\cdot h)^\wedge \qquad (n_0\le n\le N-1)\,.$$

This implies Lemma 6.1 by (6.3) and (6.5). (Q.E.D.)

Noting that $P_{N-1} = -(2\pi)^{-1}c_N$, we note by (6.11), (6.12) and Lemma 6.1 that

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$$(6.16)\qquad b_{n_0}^{-1}\varphi_{n_0}(\lambda)i\lambda = \begin{cases} \boldsymbol{e}\cdot(P_0(\lambda)\cdots P_{N-1}(\lambda))^* & \text{if } n_0 \geq 1\,, \\ (2\pi b_0)^{-1}\sum_{k=1}^{N} c_k(-i\lambda)^k & \text{if } n_0 = 0\,. \end{cases}$$

Moreover, by Theorem 3.1 and (6.6),

$$(6.17)\qquad P_{\partial Z_\Delta(a)}(\boldsymbol{e}\cdot U\mathscr{X}(t+a))(\lambda) = e^{ia\lambda}\boldsymbol{e}\cdot e^{tA}J\cdot(\varphi_{n_0}(\lambda)\cdots\varphi_{N-1}(\lambda))^*\,.$$

Therefore, defining functions $\psi(t,\lambda)$ by

$$(6.18)\qquad \psi(t,\lambda) = \sqrt{2\pi}\{b_{n_0}^{-1}e^{it\lambda}\varphi_{n_0}(\lambda)i\lambda - \boldsymbol{e}\cdot e^{tA}J\cdot(\varphi_{n_0}(\lambda)\cdots\varphi_{N-1}(\lambda))^*\}$$

we find from (6.10) that

$$(6.19)\qquad D_a^+(t,\lambda) + \sqrt{2\pi}\int_0^t \boldsymbol{e}\cdot\boldsymbol{f}(t,s)D_a^+(s,\lambda)ds = e^{ia\lambda}\psi(t,\lambda)\,.$$

By the uniqueness of solution of Volterra equation (6.19), we have

$$(6.20)\qquad D_a^+(t,\lambda) = e^{ia\lambda}D_0^+(t,\lambda)\,.$$

Similarly, by Theorem 3.2, (6.6) and (6.10),

$$(6.21)\qquad D_a^-(t,\lambda) + \sqrt{2\pi}\int_0^t \boldsymbol{e}\cdot\boldsymbol{f}(t,s)D_a^-(s,\lambda)ds = e^{ia\lambda}\psi(t,-\lambda)$$

and

$$(6.22)\qquad D_a^-(t,\lambda) = e^{ia\lambda}D_0^-(t,\lambda)\,.$$

In particular, we see from (6.19) and (6.21) that

$$(6.23)\qquad D_0^+(t,\lambda) = D_0^-(t,-\lambda)\,.$$

Next we shall obtain explicit representations of functions $\boldsymbol{f}(t,s)$ and $\boldsymbol{g}(t,s)$ in (3.2) and (4.1), respectively.

LEMMA 6.2. (i) $\boldsymbol{f}(t,s) = (D_0^+(s,\cdot), e^{it\cdot}\cdot\varphi)_\Delta$,
(ii) $\boldsymbol{g}(t,s) = R(t)\Phi^*(s)\boldsymbol{e}^* = (D_0^-(s,\cdot), e^{it\cdot}\cdot\check{\varphi})_\Delta$,
where $\varphi = (\varphi_0\cdots\varphi_{N-1})^*$.

Proof. By (6.7), (6.8) and (6.9),

$$(6.24)\qquad U(\nu_a^\pm(s))(\lambda) = \pm\int_a^{a\pm s} D_a^\pm(\pm(\iota-a),\lambda)d\iota\,.$$

Therefore it follows from Lemma 3.1 and (6.6) that

$$\boldsymbol{f}(t,s) = \frac{\partial}{\partial s}E(\nu_0^+(s)\cdot\mathscr{X}(t)) = (D_0^+(s,\cdot), e^{it\cdot}\cdot\varphi)_\Delta\,.$$

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Similarly we find from Lemmas 4.1, 4.3 and (6.6) that

$$g(t, s) = R(t)\Phi^*(s)e^* = \frac{\partial}{\partial s}E(\nu_0^-(s)\mathscr{Y}(t)) = (D_0^-(s, \cdot), e^{it\cdot}\cdot\check{\varphi})_\Delta .$$

(Q.E.D.)

LEMMA 6.3. $S_\pm(t)f(a, s) = (D_0^+(s, \cdot), e^{i(a+t)\cdot})_\Delta$.

Proof. By (5.3), (5.4), Theorems 2.1 (ix) and 2.3 (ix),

$$S_\pm(t) = (0\cdots 0\ (-c_N)^{-1}2\pi)e^{tA} . \tag{6.25}$$

Therefore it follows from (3.2) and Lemma 6.2 that

$$\begin{aligned} S_\pm(t)f(a, s) &= (0\cdots 0\,(-c_N)^{-1}2\pi)f(a + t, s) \\ &= (0\cdots 0\,(-c_N)^{-1}2\pi)(D_0^+(s, \cdot), e^{i(a+t)\cdot}\cdot\varphi)_\Delta . \end{aligned}$$

Noting that $\varphi_{N-1} = P_{N-1} = -(2\pi)^{-1}c_N$, we have Lemma 6.3. (Q.E.D.)

LEMMA 6.4. $r(t)\Phi^*(s)e^* = (D_0^-(s, \cdot), e^{it\cdot})_\Delta$.

Proof. By (4.3), (4.7) and Lemma 6.2,

$$\begin{aligned} r(t)\Phi^*(s)e^* &= (0\cdots 0\,(-c_N)^{-1}2\pi)R(t)\Phi^*(s)e^* \\ &= (0\cdots 0\,(-c_N)^{-1}2\pi)(D_0^-(s, \cdot), e^{it\cdot}\cdot\check{\varphi})_\Delta \\ &= (D_0^-(s, \cdot), e^{it\cdot})_\Delta . \end{aligned}$$

(Q.E.D.)

Now we shall obtain explicit integral representations of prediction errors.

THEOREM 6.1. *For any* $a \in \mathbf{R}$, $t \in (0, \infty)$ *and* $T \in (0, \infty)$,

$$\begin{aligned} &\|X(a + t) - E(X(a + t)\,|\,\mathbf{F}_X((a - T, a)))\|^2 \\ &\quad = \|X(a - t) - E(X(a - t)\,|\,\mathbf{F}_X((a, a + T)))\|^2 \\ &\quad = \int_0^t (D_0^+(s, \cdot), e^{it\cdot})_\Delta^2 ds - \int_0^T (D_0^-(s, \cdot), e^{it\cdot})_\Delta^2 ds . \end{aligned}$$

Proof. Using Lemmas 6.4 and 6.2, we see from Theorems 4.3 and 3.4 that

$$\begin{aligned} &\|X(a + t) - E(X(a + t)\,|\,\mathbf{F}_X((a - T, a)))\|^2 \\ &\quad = \|X(a + t)\|^2 - \|E(X(a + t)\,|\,\mathbf{F}_X((a - T, a)))\|^2 \\ &\quad = \|X(a + t)\|^2 - \|E(X(a + t)\,|\,\partial\mathbf{F}_X(a))\|^2 - \int_0^T (D_0^-(s, \cdot), e^{it\cdot})_\Delta^2 ds \\ &\quad = \|X(a + t) - E(X(a + t)\,|\,\partial\mathbf{F}_X(a))\|^2 - \int_0^T (D_0^-(s, \cdot), e^{it\cdot})_\Delta^2 ds \end{aligned}$$

$$= \int_0^t (D_0^+(s, \cdot), e^{it\cdot})_\Delta^2 ds - \int_0^T (D_0^-(s, \cdot), e^{it\cdot})_\Delta^2 ds .$$

The rest is similarly proved. (Q.E.D.)

THEOREM 6.2. *For any* $a \in (0, \infty)$ *and* $t \in (0, \infty)$,

$$\|X(\pm(a+t)) - E(X(\pm(a+t) | F_X((-a, a))))\|^2$$
$$= \int_a^{a+t} (D_0^+(s, \cdot), e^{i(a+t)\cdot})_\Delta^2 ds - \int_a^{2a} (D_0^-(s, \cdot), e^{it\cdot})_\Delta^2 ds .$$

Proof. Using Lemmas 6.3, 6.4 and 6.2, we find from Theorems 5.3 and 3.4 that

$$\begin{aligned}
&\|X(\pm(a+t)) - E(X(\pm(a+t)) | F_X((-a, a)))\|^2 \\
&= \|X(\pm(a+t))\|^2 - \|E(X(\pm(a+t)) | F_X((-a, a)))\|^2 \\
&= \|X(\pm(a+t))\|^2 - \|E(X(\pm(a+t)) | \partial F_X(0))\|^2 \\
&\quad - \int_0^a (D_0^+(s, \cdot), e^{i(a+t)\cdot})_\Delta^2 ds - \int_a^{2a} (D_0^-(s, \cdot), e^{it\cdot})_\Delta^2 ds \\
&= \|X(\pm(a+t)) - E(X(\pm(a+t)) | \partial F_X(0))\|^2 \\
&\quad - \int_0^a (D_0^+(s, \cdot), e^{i(a+t)\cdot})_\Delta^2 ds - \int_a^{2a} (D_0^-(s, \cdot), e^{it\cdot})_\Delta^2 ds \\
&= \int_a^{a+t} (D_0^+(s, \cdot), e^{i(a+t)\cdot})_\Delta^2 ds - \int_a^{2a} (D_0^-(s, \cdot), e^{it\cdot})_\Delta^2 ds .
\end{aligned}$$
(Q.E.D.)

§ 7. Integral representations of the predictors (IV)

In this section we shall give explicit integral representations of prediction formulas in § 3, § 4 and § 5 using the section of the previous section. By Theorem 3.3 and Lemma 6.2,

THEOREM 7.1. *For any* $a \in \boldsymbol{R}$, $t \in (0, \infty)$ *and* $T \in (0, \infty)$,

(i) $E(X(a+T+t) | F_X((a, a+T)))$

$$= E(X(a+T+t) | \partial F_X(a)) + \int_0^T (D_0^+(s, \cdot), e^{i(T+t)\cdot})_\Delta d\nu_a^+(s)$$

(ii) $E(X(a-t-T) | F_X((a-T, a)))$

$$= E(X(a-T-t) | \partial F_X(a)) + \int_0^T (D_0^+(s, \cdot), e^{i(T+t)\cdot})_\Delta d\nu_a^-(s).$$

Similarly, we see from Theorem 3.4 and Lemma 6.2 that

THEOREM 7.2. *For any* $a \in \boldsymbol{R}$ *and* $t \in (0, \infty)$,

$$X(a \pm t) = E(X(a \pm t) | \partial F_X(a)) + \int_0^t (D_0^+(s, \cdot), e^{it\cdot})_\Delta d\nu_a^\pm(s) .$$

By Theorem 4.3 and Lemma 6.4,

THEOREM 7.3. *For any* $a \in \boldsymbol{R}$, $t \in (0, \infty)$ *and* $T \in (0, \infty)$,

(i) $E(X(a+t) \mid \boldsymbol{F}_X((a-T, a)))$

$$= E(X(a+t) \mid \partial \boldsymbol{F}_X(a)) + \int_0^T (D_0^-(s, \cdot), e^{it\cdot})_\Delta d\nu_a^-(s)$$

(ii) $E(X(a-t) \mid \boldsymbol{F}_X((a, a+T)))$

$$= E(X(a-t) \mid \partial \boldsymbol{F}_X(a)) + \int_0^T (D_0^-(s, \cdot), e^{it\cdot})_\Delta d\nu_a^+(s).$$

From Theorem 5.3, Lemmas 6.3 and 6.4, it follows that

THEOREM 7.4. *For any* $a \in (0, \infty)$ *and* $t \in (0, \infty)$,

$$E(X(\pm(a+t)) \mid \boldsymbol{F}_X((-a, a)))$$

$$= E(X(\pm(a+t)) \mid \partial \boldsymbol{F}_X(0)) + \int_0^a (D_0^+(s, \cdot), e^{i(a+t)\cdot})_\Delta d\nu_0^\pm(s)$$

$$+ \int_a^{2a} (D_0^-(s, \cdot), e^{it\cdot})_\Delta d\nu_{\pm a}^\mp(s)\ .$$

By (6.6), (6.20), (6.22) and (6.24), we have

$$\frac{\partial}{\partial s} E(\mathscr{X}(a+t)\nu_a^-(s)) = \frac{\partial}{\partial s} E(\mathscr{Y}(a-t)\nu_a^+(s)) = (D_0^-(s, \cdot), e^{it\cdot}\varphi)_\Delta\ . \tag{7.1}$$

Therefore, similarly as in Lemma 3.2, we obtain the following Theorem 7.5 as a supplement of Theorems 4.1 and 4.2.

THEOREM 7.5. *For any* $a \in \boldsymbol{R}$, $t \in (0, \infty)$ *and* $T \in (0, \infty)$,

(i) $E(\mathscr{X}(a+t) \mid \boldsymbol{F}_X((a-T, a))$

$$= E(\mathscr{X}(a+t) \mid \partial \boldsymbol{F}_X(a)) + \int_0^T (D_0^-(s, \cdot), e^{it\cdot}\varphi)_\Delta d\nu_a^-(s),$$

(ii) $E(\mathscr{Y}(a-t) \mid \boldsymbol{F}_X((a, a+T)))$

$$= E(\mathscr{Y}(a-t) \mid \partial \boldsymbol{F}_X(a)) + \int_0^T (D_0^-(s, \cdot), e^{it\cdot}\varphi)_\Delta d\nu_a^+(s)\ .$$

Furthermore, as a supplement of Theorems 5.1 and 5.2, we shall prove

THEOREM 7.6. *For any* $a \in (0, \infty)$ *and* $t \in (0, \infty)$,

(i) $E(\mathscr{X}(a+t) \mid \boldsymbol{F}_X((-a, a)))$

$$= E(\mathscr{X}(a+t) \mid \partial \boldsymbol{F}_X(0)) + \int_0^a (D_0^+(s, \cdot), e^{i(a+t)\cdot}\varphi)_\Delta d\nu_0^+(s)$$

$$+ \int_a^{2a} (D_0^-(s, \cdot), e^{it\cdot}\varphi)_\Delta d\nu_a^-(s),$$

(ii) $E(\mathscr{Y}(-a-t) \mid \boldsymbol{F}_X((-a, a)))$

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$$= E(\mathscr{Y}(-a-t)\,|\,\partial\boldsymbol{F}_X(0)) + \int_0^a (D_0^+(s,\cdot),\, e^{i(a+t)\cdot}\varphi)_\Delta d\nu_0^-(s)$$
$$+ \int_a^{2a} (D_0^-(s,\cdot),\, e^{it\cdot}\varphi)_\Delta d\nu_{-a}^\pm(s)\ .$$

Proof. By Theorems 7.5 (i), 2.1 (ix), 3.1 and Lemma 6.2,

$$E(\mathscr{X}(a+t)\,|\,\boldsymbol{F}_X((-a,a)))$$
$$= E(\mathscr{X}(a+t)\,|\,\partial\boldsymbol{F}_X(a)) + \int_0^{2a} (D_0^-(s,\cdot),\, e^{it\cdot}\varphi)_\Delta d\nu_a^-(s)$$
$$= E(\mathscr{X}(a+t)\,|\,\boldsymbol{F}_X((0,a)) + \int_a^{2a} (D_0^-(s,\cdot),\, e^{it\cdot}\varphi)_\Delta d\nu_a^-(s)$$
$$= E(\mathscr{X}(a+t)\,|\,\partial\boldsymbol{F}_X(0)) + \int_0^a (D_0^-(s,\cdot),\, e^{i(a+t)\cdot}\varphi)_\Delta d\nu_0^+(s)\ .$$

Similarly, we have (ii) from Theorems 7.5 (ii), 2.3 (ix), 3.2 and Lemma 6.2. (Q.E.D.)

Remark 7.1. Theorem 7.3 (resp. Theorem 7.4) follows immediately from Theorem 7.5 (resp. Theorem 7.6).

Remark 7.2. The decompositions in Theorems 7.4 and 7.6 are orthogonal.

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https://doi.org/10.1017/S0027763000021796 Published online by Cambridge University Press