J. Austral. Math. Soc. (*Series A*) **48** (1990), 214–222

LINEAR OPERATORS IN l_2 WITH A UNIVERSALITY PROPERTY

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(Received 7 March 1988; revised 19 September 1988)

Communicated by W. Moran

Abstract

A collection $\mathscr{P}$ of bounded linear operators in l_2 is constructed in such a manner that given any separable metric space X, and any countable collection $\mathscr{F}$ of continuous self-maps of X, there is a homeomorphism h of X onto a subset of l_2 such that for each $f \in \mathscr{F}$ there is $P \in \mathscr{P}$ with $hf = Ph$.

While similar results were obtained by Baayen and De Groot, our construction makes it possible to impose additional conditions on h (depending on $\mathscr{F}$). For example, if all the members of $\mathscr{F}$ are uniformly continuous then h too can be made uniformly continuous.

1980 *Mathematics subject classification* (*Amer. Math. Soc.*) (1985 *Revision*): 46 C 05, 54 C 05.

1. Introduction

By a well-known result of Urysohn [3], every separable metric space X is homeomorphic to a precompact subset of l_2. If f is a self-map of X and $h\colon X \to l_2$ is a homeomorphism onto $h[X]$ then a self-map F of $h[X]$ is induced, where $F = hfh^{-1}$. Of course F is topologically indistinguishable from f. However, in the setting of a linear space the natural preference is to have F related to a linear mapping. Thus, in l_2, we may wish to choose h in such a manner that F become the restriction of a bounded linear operator. Somewhat surprisingly, a much stronger requirement can be satisfied. For, as shown by DeGroot and McDowell [2], one can produce a bounded linear operator in Hilbert space, such that an appropriate h exists, making the induced map a restriction of that operator.

https://doi.org/10.1017/S1446788700035643 Published online by Cambridge University Press

Along similar lines Baayen and DeGroot [1] proved existence theorems for semigroups of mappings on a metric space.

In this paper we construct a concrete collection $\mathscr{P}$ of bounded linear operators on l_2, which generates a free semigroup, and has the following universality property: given a separable metric space X and a countable family of self-maps f of X there is an h such that to any $f \in \mathscr{F}$ there is a $P \in \mathscr{P}$ with $hf = Ph$. Since there is considerable latitude in the manner in which h can be chosen one may impose some additional requirements on it. It is in this direction that our discussion is aimed. Thus, we show that if all members of the family of maps on X are uniformly continuous, or the semigroup generated by it is equicontinuous, then h, as constructed, is uniformly continuous. Similarly, if each map satisfies a Lipschitz condition then h can be made to satisfy a similar condition (with coefficient 1).

2. Construction of the set $\mathscr{P}$

2.1. We begin by defining a sequence $\{P_n : n = 1, 2, \dots\}$ of linear operators on l_2. The action of each P_n on an arbitrary $x = (x_0, x_1, \dots) \in l_2$ could be roughly described as a generalized shift. Hence the need for setting up appropriate correspondences between the indices of the coordinates x_k of x and those of $P_n(x)$. If k, m are nonnegative integers set

$$r(k; m) = \tfrac{1}{2}(k + m)(k + m + 1) + m$$

and write $r(k)$ for $r(k; 0)$.

2.2. **Lemma** 1. *For every nonnegative integer n there is exactly one ordered pair of nonnegative integers k, m such that*

$$n = r(k; m). \tag{1}$$

Proof. For $0 \leq n \leq 1$ the result is trivially true (with $0 = r(0; 0)$ and $1 = r(1; 0)$ uniquely). Proceeding by induction, suppose that for some $n \geq 1$ (1) is satisfied uniquely, and consider the mutually exclusive cases: (i) $k \geq 1$; and (ii) $k = 0$. In case (i) we have, by direct calculation, $n + 1 = r(k; m) + 1 = r(k - 1; m + 1)$. And, if $n + 1 = r(k' - 1; m' + 1)$ for $k', m' \geq 0$ then we must have $n = r(k', m')$ implying $k' = k$ and $m' = m$. (To rule out the other alternative, namely $n + 1 = r(k'; 0)$, we observe that $r(k'; 0) = r(k - 1; m + 1)$ clearly implies $k' \geq k \geq 1$. Hence $r(k'; 0) = r(k' - 1; 1) + 1$ and, therefore, $n = r(k' - 1; 1) = r(k; m)$. Uniqueness for n being assured, it follows that $k' = k + 1$ and $m = 1$. However, then

$$r(k'; 0) = r(k + 1; 0) = \frac{(k + 1)(k + 2)}{2} < \frac{(k + 1)(k + 2)}{2} + 2 = r(k - 1; 2),$$

a contradiction).

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In case (ii) $n+1 = r(0;m)+1 = r(m+1;0) = \frac{1}{2}(m+1)(m+2)$. Suppose $n+1 = r(m';m'')$ with $m'' \geq 1$. Then $n+1 = r(m'+1;m''-1)+1$ and $n = r(m'+1;m''-1)$ which is impossible as $m'+1 \neq 0$. Hence $m'' = 0$ and $m' = m+1$, proving that $n+1 = r(m+1;0)$ uniquely. Thus, in both cases, $n+1$ satisfies the conclusion of the lemma, and every integer $n \geq 0$ is uniquely representable by (1).

2.3. To define $P_n: l_2 \to l_2$ $(n = 1, 2, \ldots)$ set

$$P_n(x) = (x_{r(0;n)}, x_{r(1;n)}, x_{r(2;n)}, \ldots) \qquad (x \in l_2). \tag{2}$$

Because the coordinates of $P_n(x)$ are chosen from those of x without repetition it is clear that each P_n is well defined. Furthermore, P_n is linear, with $\|P_n\| \leq 1$.

2.4. Let $\{\lambda_n: n = 1, 2, \ldots\}$ be a given sequence of real numbers with $\lambda_n \geq 2^n$ $(n = 1, 2, \ldots)$, and set

$$\mathscr{P} = \{\lambda^n P_n: n = 1, 2, \ldots\}. \tag{3}$$

2.5. In the sequel we shall need an extended definition of r. Thus we define $r(k;m_1,m_2) = r(r(k;m_1);m_2)$, and inductively

$$r(k;m_1,\ldots,m_{n-1},m_n) = r(r(k;m_1,\ldots,m_{n-1});m_n). \tag{1'}$$

Here $k, m_1, \ldots, m_n$ are arbitrary nonnegative integers.

LEMMA 2. *Let $k, m_1, \ldots, m_n, m'_1, \ldots, m'_j$ be integers with $1 \leq n \leq j$, $k \geq 0$, $m_p \geq 1$ $(1 \leq p \leq n)$, $m'_q \geq 1$ $(1 \leq q \leq j)$. If $r(k;m_1,\ldots,m_n) = r(k;m'_1,\ldots,m'_j)$ then $j = n$ and $m'_i = m_i$ $(1 \leq i \leq n)$.*

PROOF. We proceed by induction on the positive integer n. For $n = 1$ the result is an easy consequence of Lemma 1. Suppose the assertion is true for $n = l-1$. Then $r(k;m_1,\ldots,m_{l-1},m_l) = r(k;m'_1,\ldots,m'_{j-1},m'_j)$ with k, m_p, m'_q satisfying the hypotheses of the lemma. It follows that

$$r(r(k;m_1,\ldots,m_{l-1});m_l) = r(r(k;m'_1,\ldots,m'_{j-1});m'_j)$$

and, by Lemma 1, $r(k;m_1,\ldots,m_{l-1}) = r(k;m'_1,\ldots,m'_{j-1})$ as well as $m_l = m'_j$. This, together with the inductive hypothesis shows that, as asserted, $j = l$ and $m'_i = m_i$ $(1 \leq i \leq l)$, completing the induction and the proof of the lemma.

Next we define $s(k;m_1,\ldots,m_n)$ by setting $s(k) = r(k)$ and $s(k;m_1,\ldots,m_n) = r(k;0,m_1,\ldots,m_n)$, where $k, m_1, \ldots, m_n$ are integers; the first nonnegative; the rest positive.

LEMMA 3. *For every nonnegative integer p there exists an integer $k \geq 0$ and an ordered set (possibly empty) of positive integer $m_1, \ldots, m_n$ such that*

$$p = s(k; m_1, \ldots, m_n). \tag{1''}$$

Furthermore the representation in (1″) is unique.

PROOF. Suppose, for a contradiction, that there exist nonnegative integers for which no such representation is possible and let p be the smallest among them. By Lemma 1, $p = r(k; m)$ for a unique ordered pair of nonnegative integers k, m; and, clearly, $m \geq 1$. Now since $k < p$ we have, by the defining property of p, $k = r(k'; 0, m_1, \ldots, m_j) = s(k'; m_1, \ldots, m_j)$, with $k' \geq 0$ and $m_1, \ldots, m_j$ positive. It follows that, contrary to the assumption, $p = s(k'; m_1, \ldots, m_j, m)$. Finally, the uniqueness property of the representation (1″) follows directly from Lemma 2.

2.6. We observe that the set $\{P_n\}$ is free of relations and, therefore, the semigroup $\overline{\mathscr{P}}$ generated by it is a free semigroup. Indeed, by definition, $(P_m(x))_k = x_{r(k;m)}$. Hence

$$\begin{aligned}(P_{m_1} P_{m_2} \cdots P_{m_n}(x))_k &= (P_{m_2} \cdots P_{m_n}(x))_{r(k;m_1)} \\ &= (P_{m_3} \cdots P_{m_n}(x))_{r(r(k;m_1);m_2)} \\ &= \cdots = x_{r(k;m_1,m_2,\ldots,m_n)}.\end{aligned}$$

If then $P_{m_1} P_{m_2} \cdots P_{m_n} = P_{i_1} P_{i_2} \cdots P_{i_j}$ with, say, $j \leq n$ then $r(k; m_1, m_2, \ldots, m_n) = r(k; i_1, i_2, \ldots, i_j)$. By Lemma 2 we have $n = j$, $m_1 = i_1, \ldots, m_n = i_n$ $(= i_j)$, and the only relation is the identity.

3. Construction of a homeomorphism

Let X be a subset of l_2 and $f_n \colon X \to X$, $n = 1, 2, \ldots$, continuous mappings. For $x = (x_0, x_1, x_2, \ldots) \in l_2$ set $\phi_{s(k)}(x) = x_k$ and, inductively on n,

$$\phi_{s(k;m_1,m_2,\ldots,m_n)}(x) = \lambda_{m_n}^{-m_n} \phi_{s(k;m_1,m_2,\ldots,m_{n-1})}(f_{m_n}(x)). \tag{4}$$

Having thus defined $\phi_i(x)$ for $i = 0, 1, 2, \ldots$ we set

$$h(x) = (\phi_0(x), \phi_1(x), \phi_2(x), \ldots). \tag{5}$$

PROPOSITION 1. *Let (X, d) be a bounded separable metric space and let $f_n \colon X \to X$, $n = 1, 2, \ldots$, be continuous self-maps of X. Let $\mathscr{B}$ be a countable base for the topology of X consisting of balls $B_n = B_n(x^{(n)}, r_n)$, centered at points $x^{(n)} \in X$ and of radii r_n, with*

$$r_n < \lambda_n^{-1} \leq 2^{-n} \qquad (n = 1, 2, \ldots). \tag{6}$$

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where $\{\lambda_n: n = 1, 2, \dots\}$ *is a given sequence of reals. Let*

$$(7) \qquad \phi_{s(k)}(x) = \begin{cases} 2^{-k-1} \inf\{d(x,y): y \in X \backslash B_k\} & \text{if } B_k \neq X, \\ 0 \quad \text{if } B_k = X, \end{cases}$$

and $\phi_{s(k;m_1,m_2,\dots,m_n)}(x)$, *inductively, as in* (4). *Then* h (*as defined by* (5)) *maps* X *homeomorphically onto a precompact set in* l_2; *and with* P_n, *as given by* (2),

$$(8) \qquad h(f_n(x)) = \lambda_n P_n(h(x)) \qquad (x \in X;\ n = 1, 2, \dots).$$

Proof. From (4) it readily follows that

$$\phi_{s(k;m_1,\dots,m_n)}(x) = \lambda_{m_1}^{-1}\lambda_{m_2}^{-1}\cdots\lambda_{m_n}^{-1}\phi_{s(k)}(f_{m_1}\cdots f_{m_n}(x)).$$

Also, $\phi_{s(k)}(u) \leq 2^{-k-1}$ where $\delta = \sup\{d(v,w): v, w \in X\}$ $(u \in X)$, and $\lambda_n^{-1} \leq 2^{-n}$. Hence

$$\|h(x)\|^2 \leq \sum_{k=0}^{\infty} 2^{-2k-2}\delta^2 \sum_{n=0}^{\infty}\left[\left(\sum_{m_1=1}^{\infty} 2^{-2m_1}\right)\cdots\left(\sum_{m_n=1}^{\infty} 2^{-2m_n}\right)\right] < \infty,$$

showing that h is well defined (and $h[X]$ is bounded). To prove that h is continuous and $h[X]$ is precompact let $h^{(N)}: X \to l_2$ be defined by setting

$$(h^{(N)}(x))_{s(k;m_1,\dots,m_n)} = \phi_{s(k;m_1,\dots,m_n)}(x)$$

if $t = k + \sum_{i=1}^{n} m_i \leq N$, and zero otherwise. It readily follows that $\|h(x) - h^{(N)}(x)\|^2 \leq \delta^2 \sum_{t=N+1}^{\infty} 2^{-t}$. Hence $h^{(N)}(x) \to h(x)$ uniformly over X proving that h is continuous. Further, $h^{(N)}[X]$ is precompact since it is a bounded subset of a finite dimensional subspace of l_2. Given $\varepsilon > 0$ an N exists such that $\|h(x) - h^{(N)}(x)\| < \varepsilon/3$ $(x \in X)$. Let $\{x_1, x_2, \dots, x_n\} \subset X$ have the property that, to any $x \in X$, there is a $j = j(x)$, $1 \leq j \leq n$, such that $\|h^{(N)}(x) - h^{(N)}(x_j)\| < \varepsilon/3$. Then

$$\|h(x) - h(x_j)\| \leq \|h(x) - h^{(N)}(x)\| + \|h^{(N)}(x) - h^{(N)}(x_j)\| \\ + \|h^{(N)}(x_j) - h(x_j)\| \leq \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon,$$

showing that $h[X]$ is precompact. That h is one-to-one is obvious. [If $x, y \in X$, $x \neq y$, then there is an integer j such that $x \in B_j$, $y \in X \backslash B_j$, so that $\phi_{s(k)}(y) = 0$ but $\phi_{s(k)}(x) \neq 0$.]

It is also clear that h^{-1} is continuous. [If not then there must be a sequence $\{h(x^{(n)}): n = 1, 2, \dots\}$ converging to some $h(x)$ in l_2 with $\{x^{(n)}: n = 1, 2, \dots\}$ having no convergent subsequence. But then there is an open ball B_j such that $x \in B_j$ and $\{x^{(n)}: n = 1, 2, \dots\} \subset X \backslash B_j$, so that $\phi_{s(j)}(x^{(n)}) \not\to \phi_{s(j)}(x)$; a contradiction.] Finally,

$$h(f_n(x)) = (\dots, \phi_{s(k;m_1,\dots,m_j)}(f_n(x)), \dots) \\ = (\dots, \lambda_n \phi_{s(k;m_1,\dots,m_j,n)}(x), \dots) = \lambda_n P_n(h(x)) \quad (x \in X;\ n = 1, 2, \dots).$$

3.2. PROPOSITION 2. *Let X be a separable (not necessarily bounded) metric space and $f_n\colon X \to X$, $n = 1, 2, \ldots$, continuous. Let $\mathscr{P}$ and $\{\lambda_n\colon n = 1, 2, \ldots\}$ be as defined before. Then, as in the conclusion of Proposition* 1, *there is a homeomorphism $h\colon X \to h[X] \subset l_2$ such that $h[X]$ is precompact and*

$$h(f_n(x)) = \lambda_n P_n(h(x)) \qquad (x \in X;\ n = 1, 2, \ldots).$$

PROOF. Let Y be a bounded metric space which is homeomorphic to X. Then Y is separable too. Let $h_1\colon X \to Y$ be a homeomorphism and, for $n = 1, 2, \ldots$, set $\tilde{f}_n = h_1 f_n h_1^{-1}$. Then, by Proposition 1, there is a homeomorphism h_2 of Y onto a precompact subset of l_2 such that $h_2 \tilde{f}_n = \lambda_n P_n h_2$ $(n = 1, 2, \ldots)$. Let $h = h_2 h_1$. Then h maps homeomorphically X onto a precompact subset of l_2 and

$$\begin{aligned} h f_n &= h_2 h_1 f_n = h_2 h_1 h_1^{-1} \tilde{f}_n h_1 = h_2 \tilde{f}_n h_1 \\ &= \lambda_n P_n h_2 h_1 = \lambda_n P_n h \qquad (n = 1, 2, \ldots). \end{aligned}$$

4. Uniformly continuous homeomorphisms

4.1. THEOREM 1. *Let $\{f_n\colon n = 1, 2, \ldots\}$ be uniformly continuous self-maps on the separable metric space (X, d). Then the conclusions of Proposition* 2 *hold and the homeomorphism h is uniformly continuous.*

PROOF. Since the identity mapping from (X, d) to (X, δ) with $\delta(x, y) = \min\{1, d(x, y)\}$ is a uniformly continuous homeomorphism, we may assume that (X, d) is bounded with $\operatorname{diam} X = 1$. Let $h\colon X \to l_2$ be defined by (2) with $\lambda_n = 2^n$. In view of the conclusions of Proposition 2 it remains to be shown that h is uniformly continuous. Let $\varepsilon > 0$ be given, and observe that, by (7),

$$|\phi_{s(k)}(u) - \phi_{s(k)}(v)| \le |D(u, X \backslash B_k) - D(v, X \backslash B_k)|$$

where, for $\{z\}$, $S \subset X$, $D(z, S) = \inf\{d(z, w)\colon w \in S\}$. Thus, since D is Lipshitzian with coefficient 1,

$$|\phi_{s(k)}(u) - \phi_{s(k)}(v)| \le d(u, v) \le \operatorname{diam} X = 1.$$

Hence the series expansion for $\|h(x) - h(y)\|^2$ is dominated by a convergent positive series and therefore uniformly convergent. Now let N be such that

$$\|h(x) - h(y)\|^2 \le \sum_{k,n=0}^{N} 2^{-2k-2} \sum_{m_1, \ldots, m_n = 1}^{N} 2^{-2(m_1 + \cdots + m_n)} \times (d(f_{m_1} \cdots f_{m_n}(x), f_{m_1} \cdots f_{m_n}(y)))^2 + \frac{\varepsilon^2}{2}.$$

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Since this sum involves finitely many continuous mappings a $\delta > 0$ exists such that it is less than $\varepsilon^2/2$ whenever $d(x,y) < \delta$. Thus $d(x,y) < \delta$ implies that $\|h(x) - h(y)\| < \varepsilon$ proving the assertion of the theorem.

REMARK. The following example serves to show that the assumption of uniform continuity for each f_n, $n = 1, 2, \ldots$, is essential for the conclusion of the theorem. Let $X = \{1/n: n = 2, 3, \ldots\} \cup \{1, 2, 3, \ldots\}$ with the usual metric and take $f: X \to X$ to be defined by $f(1/n) = n$, $n = 2, 3, \ldots$, $f(n) = 1$, n even, and $f(n) = 2$, n odd. If there exists a bounded linear operator $P: l_2 \to l_2$ and a homeomorphism h of X into l_2 such that $h(f(x)) = P(h(x))$ with h uniformly continuous then the sequence $\{h(1/n): n = 1, 2, \ldots\}$ must converge to some point $u \in l_2$. Now $h(f(n)) = P(h(n)) = P(h(f(1/n))) = P^2(h(1/n)) \to P^2(u)$. But $h(f(n)) = h(1)$ if n is even and $h(2)$ if n is odd implying $h(1) = h(2)$, a contradiction.

4.2. THEOREM 2. *Let $\mathscr{F} = \{f_n: n = 1, 2, \ldots\}$ be a countable and equicontinuous semigroup of self-maps on a separable metric space (X, d). Then there is a uniformly continuous homeomorphism of X into l_2 satisfying the conclusion of the preceding theorem.*

PROOF. As in the proof of Theorem 1 we proceed by assuming, as we may, that (X, d) is a bounded metric space with $\operatorname{diam} X = 1$. Let $h: X \to l_2$ be defined by (4) and (5) with $\lambda_n = 2^n$. By (4),

$$\begin{aligned}
&|\phi_{s(k;m_1,\ldots,m_n)}(x) - \phi_{s(k;m_1,\ldots,m_n)}(y)| \\
&\quad = 2^{-m_n}|\phi_{s(k;m_1,\ldots,m_{n-1})}(f_{m_n}(x)) - \phi_{s(k;m_1,\ldots,m_{n-1})}(f_{m_n}(x))| \\
&\quad \le 2^{-(m_1+\cdots+m_k)} d(f_{m_1} \cdots f_{m_n}(x), f_{m_1} \cdots f_{m_n}(y)).
\end{aligned}$$

By the equicontinuity of $\mathscr{F}$, given $\varepsilon > 0$, there is a $\delta > 0$ such that the above distance is less than ε if $d(x,y) < \delta$. It follows that $\|h(x) - h(y)\| \le M\varepsilon$ for some positive M, independent of x, y, proving the theorem.

5. Nonexpansive homeomorphisms

5.1. THEOREM 3. *Let $\{f_n: n = 1, 2, \ldots\}$ be continuous self-maps of the separable metric space (X, d) satisfying the Lipschitz condition*

$$d(f_n(x), f_n(y)) \le L_n d(x,y) \qquad (x, y \in X,\ n = 1, 2, \ldots). \tag{9}$$

Then, with $\lambda_n \ge \max(2^n, 2^n L_n)$, there is a homeomorphism h of X onto a precompact subset of l_2 such that

$$\|h(x) - h(y)\| \le d(x,y) \qquad (x, y \in X), \tag{10}$$

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and

$$h(f_n(x)) = \lambda_n P_n(h(x)) \qquad (x \in X;\ n = 1, 2, \dots).$$

PROOF. In view of Proposition 2 only (10) has to be verified. From (9) it follows that

$$d(f_{m_1} \cdots f_{m_n}(x), f_{m_1} \cdots f_{m_n}(y)) \le \frac{L_{m_1}}{\lambda_{m_1}} \cdots \frac{L_{m_n}}{\lambda_{m_n}} d(x,y) \le 2^{-(m_1+\cdots+m_n)} d(x,y).$$

Hence

$$\|h(x) - h(y)\|^2 \le \sum_{k=0}^{\infty} 2^{-2k-2} \sum_{n=0}^{\infty} \sum_{m_1,\dots,m_n=1}^{\infty} 2^{-2(m_1+\cdots+m_n)}(d(x,y))^2 \le (d(x,y))^2;$$

and so $\|h(x) - h(y)\| \le d(x,y)$ as claimed.

6. Unboundedness

The universal set $\mathscr{P}$ consists of operators $2^n P_n$ where $\|P\| = 1$; hence $\mathscr{P}$ is not uniformly bounded. As shown below all such universal sets of operators share this property.

6.1. PROPOSITION 3. *Let Q be a countable set of linear operators in a separable Hilbert space H having the following universal property. Given a separable metric space X and a countable collection $\mathscr{F}$ of self-maps of X there is a homeomorphism h of X into H such that for any $f \in \mathscr{F}$ there is a $q \in Q$ with $hf = qh$. Then Q is not uniformly bounded.*

PROOF. Choose $X = l_2$ and $\mathscr{F} = \mathscr{P}$. Then there is a homeomorphism h of l_2 into H such that $h(2^n P_n) = q_n h$, $n = 1, 2, \dots$, for some $\{q_n\} \subset Q$. Suppose Q is uniformly bounded; that is, $\|q\| \le M$ for all $q \in Q$ and some $M \ge 0$. Let y be an arbitrary, but fixed, nonzero point in l_2. For each $n \ge 1$, $P_n^{-1}(y)$ contains exactly one point z_n with $\|z_n\| = \|y\|$. (The coordinates of z_n are equal to the corresponding ones of y in those coordinate positions selected by P_n, and zeros elsewhere.) If $x_n = 2^{-n} z_n$ then $x_n \to 0$ and $2^n P_n(x_n) = y$. Since $\lim_{n\to\infty} h(x_n) = h(0)$, $\|h(x_n) - h(0)\| < \varepsilon$ for given $\varepsilon > 0$ and all $n \ge N$ for sufficiently large N. Hence

$$\varepsilon M > \|q_n(h(x_n)) - q_n(h(0))\| = \|h(2^n P_n(x_n) - h(2^n P_n(0))\| = \|h(y) - h(0)\|.$$

Thus $h(y) = h(0)$, implying $y = 0$, a contradiction.

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