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A DIOPHANTINE INEQUALITY WITH PRIME VARIABLES

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Let λ_1, λ_2, λ_3 be non-zero reals, not all of the same sign and such that at least one ratio λ_i/λ_j is irrational. Then it is proved that for any given integer $k \geqslant 1$ and real η, the inequaltiy

$$\left|\lambda_1 p_1 + \lambda_2 p_2 + \lambda_3 p_3^k + \eta\right| < \varepsilon$$

is solvable for every $\varepsilon > 0$. More general and sharper results are also proved.

INTRODUCTION

Here we are concerned with the solvability of the diophantine inequality

$$(1) \qquad \left|\eta + \sum_{j=1}^{s} \lambda_j x_j^k\right| < \varepsilon, \qquad (\eta \text{ an arbitrary, but fixed, real number})$$

for every $\varepsilon > 0$ in *primes* x_j, where $k \geqslant 2$ is any given integer, under the assumption that $s \geqslant s(k)$ is suitably large and $\lambda_1, \ldots, \lambda_s$ are *any* non-zero reals, not all of the same sign and with λ_1/λ_2 irrational. For details about earlier work in this topic we refer to Vaughan ([4], [5]), from where we get $s(k) \leqq 2^k + 1 (k = 1, 2, 3)$, and smaller values for $s(k)$ for $k \geqslant 4$ (in fact $s(k) \leqslant ck \log k$ with a certain constant c); also, we can impose the condition that ε is a negative power of $\max x_j$.

For $k = 2$, Bambah [**1**] has shown, combining some ideas of Watson with the method of Daveport-Heilbronn (when x_i's are natural numbers), that in (1) one can replace $\lambda_5 x_5^2$ by $\lambda_5 x_5^K$, where K is any given natural number. Here we prove that one can, analogously, replace any kth power in (1), x_i^k say, by x_i^K for any given natural number K, and also can replace ε by a negative power (depending on k, K) of $\max x_j$ while taking $s = s(k)$, the value given by the results mentioned above. We obtain this by adding a simple idea to the method of Davenport-Heilbronn as extended by Vaughan and so avoid the use of Watson's work. We prove the

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THEOREM. *Let k and K be any two given natural numbers and let $s = s(k)$ be as above. Let $\lambda_1, \dots, \lambda_s$ be any set of non-zero reals, not all of the same sign and with λ_1/λ_2 irrational. Fix an i, $1 \leqslant i \leqslant s$. Let η be any given real number. Then, for a suitable $\delta > 0$ (depending on η, k, K and the λ's), the inequaltiy*

$$\left|\sum_{j\neq i} \lambda_j p_j^k + \lambda_i p_i^K + \eta\right| < (\max p_j)^{-\delta}. \tag{2}$$

has infinitely many solutions in primes $p_1, \dots, p_s$.

In particular, since $s(1) \leqq 3$, we have the following extension of a result of Danicic [**2**]:

THEOREM'. *Let k be any given natural number. Let λ, μ be non-zero reals, not both negative and at least one of them irrational. Then both the sets of (real) numbers*

$$[\lambda p_1 + \mu p_2^k], [\lambda p_1 + \mu p_2]^{1/k} \qquad (p_1, p_2 \text{ primes})$$

contain infinitely many primes, where, as usual, $[x]$ denotes the largest integer not exceeding x.

This result gives immediately the following well-known assertion:
Let α be any (positive) irrational. Then, for every integer $k \geqslant 1$, the sequence $[n\alpha]$, $n = 1, 2, 3, \dots$, contains infinitely many kth powers of primes.

2. Notation

Symbols with or without suffices have the same connotation. The letter p denotes prime numbers. The letters K, b, j, k, m, n, q, r and s denote positive integers. μ, η, x and the λ's are reals. ε, δ denote sufficiently small positive numbers. Like the implied constants in the 'order notation' the positive numbers a, c, A, B and C depend at most on the λ's, δ's, k's. As usual, $e(x) = \exp(2\pi i x)$ and $[x]$ denotes the integral part of x, $L = \log X$. Set

$$S_k(x) = \sum_{\delta X^{1/k} < p \leqslant X^{1/k}} e(xp^k), \; I_k(x) = \int_{\delta X^{1/k}}^{X^{1/k}} (\log u)^{-1} e(xu^k)\, du,$$

and $K_\varepsilon(x) = \pi^{-2} x^{-2} \sin^2(\varepsilon \pi x)$ for $x \neq 0$; $K_\varepsilon(x) = \varepsilon^2$ for $x = 0$. For any function $\Phi(x)$ of a real variable we write $\Phi_{(j)}(x)$ to mean $\Phi(\lambda_j x)$.

3. Some Lemmas

We shall prove completely the case $k \leqslant 3$ of the Theorem and conclude its proof by indicating how to adapt the argument in the remaining case. However, we have freely referred to results from [**4**] and [**5**]. We begin by noting two lemmas.

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LEMMA 1. *Let λ be any non-zero real number and let $0 < \varepsilon < |\lambda|$. Then, for every $m \geqslant 1$, we have*

$$\int_{-\infty}^{\infty} |S_k(\lambda x)|^{2m} K_\varepsilon(x)dx = \varepsilon \int_0^1 |S_k(x)|^{2m}\, dx. \tag{3}$$

PROOF: By Lemma 1 of [4], the lefthand-side of (3) is equal to

$$\sum_{\delta X^{1/k} < p, p' \leqslant X^{1/k}} \max\left(0, \varepsilon - \left|\lambda\left(\sum_{j=1}^{m}\left(p_j^k - p'^k_j\right)\right)\right|\right).$$

Since $|\lambda| > \varepsilon > 0$ the non-zero terms here correspond to the solutions of $p_1^k + \cdots + p_m^k = p'^k_1 + \cdots + p'^k_m$ and then each such term $= \varepsilon$. Thus this quantity is precisely the expression on the right in (3). ∎

LEMMA 2. *For every integer $k \geqslant 1$ we have*

$$\int_0^1 \left|\sum_{n \leqslant X_1} e(xn^k)\right|^{2^k} dx = 0\left(X_1^{2^k - k}(\log X_1)^B\right) \tag{4}$$

for some B, depending only on k.

PROOF: This is a special case of Theorem 4 of Hua [3]. ∎

REMARK: For our purposes the easier estimate $0_\delta\left(X_1^{2^k - k + \delta}\right)$, $\delta > 0$, suffices, but we use (4) instead to avoid some minor complications in details.

4. Proof of the Theorem

We divide this section into four parts.

4.1. The neighbourhood of the origin.

The results of this sub-section are derived, analogously to those in Section 5 of [4], by the method of Vaughan (particularly Lemma 3 below). The proofs are included only for completeness.

LEMMA 3. *Let $n \geqslant 3$ and let k, K be any two natural numbers. Fix an i, $1 \leqslant i \leqslant n$. Let $\lambda_1, \ldots, \lambda_n$ be a set of non-zero reals (not necessarily distinct). Then, there exists a $\delta_0 = \delta_0(k, K) > 0$ such that for all sufficiently large X*

$$\left\{\begin{aligned} &\int_{|x| \leqslant X^{-1+\delta_0}} \left| S_{(i),K}(x) \prod_{j \neq i} S_{(j),k}(x) - I_{(i),K}(x) \prod_{j \neq i} I_{(j),k}(x)\right| dx \\ &\qquad << X^{(n-1)/k + K^{-1} - 1} L^{-n-1}. \end{aligned}\right. \tag{5}$$

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(Here, and in the sequel, n will be bounded in terms of k, K and the λ's).

PROOF: Introducing the functions J, B of (5.11), (5.12) of [5] and making their dependence on the degree (k) explicit we see that the integrand in (5) is

$$= \left| \sum_{j=1}^{n} \left(\prod_{h<j} S_{(h),k(h)}(x) \right) \left(B_{(j),k(j)}(x) - J_{(j),k(j)}(x) \right) \left(\prod_{j<h} I_{(h),k(h)}(x) \right) \right|,$$

where $k(b) = K$ or k according as $b = i$ or not. Obviously,

$$\left| S_{(h),k(h)}(x) \right| \leqslant X^{1/k(h)}, \left| I_{(h),k(h)}(x) \right| \leqslant X^{1/k(h)}; \quad 1 \leqslant h \leqslant n.$$

Using this estimate to replace all but one of S or I corresponding to a $h \neq i$ (possible since $n \geqslant 3$) we get the last sum in absolute value

$$<< \sum_{j=1}^{n} \sum_{h \neq i} X^{\sigma(j)} \left| B_{(j),k(j)}(x) - J_{(j),k(j)}(x) \right| \left(\left| S_{(h),k(h)}(x) \right| + \left| I_{(h),k(h)}(x) \right| \right),$$

where $\sigma(j) = (n-2)k^{-1} + K^{-1} - k(j)^{-1}$. Now we note that if δ_0 is small enough, depending on k and K, then the bounds (5.15)–(5.18) of [5] with τ replaced by $X^{-1+\delta_0}$ are available to us, for the given values of k, K. Hence integrating the double sum above over $|x| \leqslant X^{-1+\delta_0}$, applying Schwarz's inequality and using the above bounds we see that the integral in (5) is

$$\begin{aligned} &<< \sum_{j=1}^{n} X^{\sigma(j) - \frac{1}{2} + \frac{1}{k(j)}} \exp\left(-(\log X)^{\frac{1}{10}} \right) X^{-\frac{1}{2} + \frac{1}{k}} \\ &<< X^{(n-1)/k + K^{-1} - 1} L^{-n-1}. \end{aligned}$$

This proves the lemma. ∎

LEMMA 4. *Under the conditions of Lemma 3, for any* $\delta_0 > 0$

$$(6) \qquad \left\{ \begin{aligned} &\int_{|x| \geqslant X^{-1+\delta_0}} \left| I_{(i),K}(x) \prod_{j \neq i} I_{(j),k}(x) \right| K_\varepsilon(x) dx \\ &\qquad << \varepsilon^2 X^{(n-1)k^{-1} + K^{-1} - 1} L^{-n-1}. \end{aligned} \right.$$

Further supposing that λ*'s are not all of the same sign we have, for any given real* η,

$$(7) \qquad \left\{ \begin{aligned} &\int_{-\infty}^{\infty} I_{(i),K}(x) \prod_{j \neq i} I_{(j),k}(x) K_\varepsilon(x) e(x\eta) dx \\ &\qquad >> \varepsilon^2 X^{(n-1)k^{-1} + K^{-1} - 1} L^{-n}. \end{aligned} \right.$$

PROOF: We have $K_\varepsilon(x) < \varepsilon^2$ for all x and, by partial intergration, also

$$I_k(x) << X^{1/k} \min\left(1, (X|x|)^{-1}\right).$$

These give (6). By Lemma 1 of [4], the integral in (7) can be written as

$$\int_{\delta^{k(1)}X}^{X} \cdots \int_{\delta^{k(n)}X}^{X} \frac{u_1^{-1+k(1)^{-1}} \ldots u_n^{-1+k(n)^{-1}}}{\log u_1 \ldots \log u_n} (*) du_1 \ldots du_n$$

with $(*) \equiv \max\left(0, \varepsilon - \left|\eta + \sum_{j=1}^{n} u_j \lambda_j\right|\right)$, where $k(h) = K$ or k according as $h = i$ or not. Since λ's are not all of the same sign $\lambda_h > 0 > \lambda_j$ for some h, j. Now for $(u_1, \ldots, u_n)$ with $\delta X \leqslant u_b \leqslant 2\delta X (1 \leqslant b \leqslant n,\ b \neq h,\ b \neq j)$, and for a suitably chosen A

$$nA\delta X\, |\lambda_h/\lambda_j| \leqslant u_j \leqslant 2nA\delta X\, |\lambda_h/\lambda_j|$$

we see that, when δ is sufficiently small,

$$\delta X + \frac{1}{2}\varepsilon\lambda_h^{-1} \leqslant -\left(\eta + \sum_{b \neq h} \lambda_b u_b\right)\lambda_h^{-1} \leqslant X - \frac{1}{2}\varepsilon\lambda_h^{-1}.$$

This shows that the box $\delta X \leqslant u_j \leqslant X \quad (1 \leqslant j \leqslant n)$ contains a region with volume $>> \varepsilon X^{n-1}$ such that for each $(u_1, \ldots, u_n)$ in it

$$\left|\eta + \sum_{j=1}^{n} \lambda_j u_j\right| < \varepsilon/2.$$

So the multiple integral above is

$$>> \varepsilon^2 X^{(n-1)+(n-1)\left(-1+k^{-1}\right)+K^{-1}-1} L^{-n},$$

because $\min k(j) \geqslant 1$. This proves (7). ∎

The next lemma follows immediately from Lemmas 3 and 4.

LEMMA 5. *Under the hypotheses of Lemmas 3 and 4 we have for any* δ_0, $0 < \delta_0 \leqslant \delta_0(k, K)$,

$$(8) \qquad \left\{ \begin{aligned} &\left| \int_{|x| \leqslant X^{-1+\delta_0}} S_{(i),K}(x) \prod_{j \neq i} S_{(j),k}(x) e(x\eta) K_\varepsilon(x)\, dx \right| \\ &\qquad\qquad >> \varepsilon^2 X^{(n-1)k^{-1}+K^{-1}-1} L^{-n}. \end{aligned} \right.$$

We also require

LEMMA 6. *Let $n \geqslant 2^k+1$ and let $\delta_1 > 0$. Then, under the hypotheses of Lemma 3, we have*

(9) $$\begin{cases} \int_{|x|\geqslant X^{\delta_1}} |S_{(i),K}(x)\prod_{j\neq i} S_{(j),k}(x)|K_\varepsilon(x)dx \\ \qquad << X^{(n-1)k^{-1}+K^{-1}-1-\delta_1}L^B \end{cases}$$

with $B = B(k)$, provided $4X^{-\delta_1} < \varepsilon < \min|\lambda_j|$, for all sufficiently large X.

PROOF: Obviously $\left|S_{(j),k}(x)\right| \leqslant X^{1/k}$ for all x, j, k. So it suffices to prove (9) with $n = 2^k+1$ and further assuming (permuting λ's if necessary) $i = 2^k+1$. Thus we need to show that

$$\int_{|x|\geqslant X^{\delta_1}} \prod_{j=1}^{2^k} \left|S_{(j),k}(x)\right| K_\varepsilon(x)dx << X^{2^k k^{-1}-1-\delta_1}L^B.$$

By Hölder's inequality (with respect to many factors), Lemmas 1, 2, and Lemma 13 of [5], we get the integral here as

$$\begin{aligned} &\leqslant \prod_{j=1}^{2^k}\left(\left(\int_{|x|\geqslant X^{\delta_1}} \left|S_{(j),k}(x)\right|^{2^k} K_\varepsilon(x)dx\right)^{2^{-k}}\right) \\ &<< \prod_{j=1}^{2^k}\left(\left(\varepsilon^{-1}X^{-\delta_1}\int_{-\infty}^{\infty} \left|S_{(j),k}(x)\right|^{2^k} K_\varepsilon(x)dx\right)^{2^{-k}}\right) \\ &<< X^{-\delta_1}\int_0^1 |S_k(x)|^{2^k}dx << X^{(2^k-k)k^{-1}-\delta_1}L^B. \end{aligned}$$

This proves the assertion made above, and hence also (9). ∎

4.2 The Intermediate Region.

LEMMA 7. *Let λ, μ be two non-zero reals with λ/η irrational. Let $C > 1$ be any fixed number. Let positive δ_0, δ_1 be such that $\delta_0+\delta_1 < 1$. Set $\delta_2 = (1-\delta_0-\delta_1)/6$ and for sufficiently large Y define $X = Y^{1/(3\delta_2+\delta_1)}$. Suppose that h/q is a convergent to the continued fraction of λ/μ satisfiying $(h,q) = 1$ and $Y \leqslant q \leqslant CY$. Then for every x in the intervals $X^{-1+\delta_0} \leqslant |x| \leqslant X^{\delta_1}$ one has the approximations*

(10) $$\left|\lambda x - \frac{h_1}{q_1}\right| \leqslant q_1^{-1}X^{-1+\frac{1}{2}\delta_0}, \left|\mu x - \frac{h_2}{q_2}\right| \leqslant q_2^{-1}X^{-1+\frac{1}{2}\delta_0}$$

with $(h_j, q_j) = 1 (j = 1, 2)$ and

(11) $$X^{\delta_2} \leqslant \max(q_1, q_2) \leqslant X^{1-\frac{1}{2}\delta_0}.$$

PROOF: By Dirichlet's approximation theorem, we have integers $h_j, q_j (j = 1,2)$, for any given x, such that (10) holds with $(h_j, q_j) = 1$ and $1 \leqslant q_j \leqslant X^{1-\frac{1}{2}\delta_0} (j = 1,2)$. For $|x| \geqslant X^{-1+\delta_0}$ we see easily that $h_1 h_2 \neq 0$; otherwise, $|x| \geqslant X^{-1+\delta_0} > \max\left(|\lambda|^{-1}, |\mu|^{-1}\right) X^{-1+\frac{1}{2}\delta_0}$ leads to a contradiction. Now it suffices to show that $\max(q_1, q_2) < X^{\delta_2}$ gives a contradiction. Under this assumption we will have (using $h_1 h_2 \neq 0$)

$$(12) \qquad \begin{cases} \left|q_1 h_2 \lambda \mu^{-1} - q_2 h_1\right| = \left|h_2 (q_2 \mu x)^{-1} q_1 q_2 \left(\lambda x - h_1 q^{-1}\right)\right. \\ \qquad\qquad \left. + h_1 (q_1 \mu x)^{-1} q_1 q_2 \left(h_2 q_2^{-1} - \mu x\right)\right| \leqslant 4X^{\delta_2 - 1 + 3\delta_0/4}, \end{cases}$$

X being sufficiently large. Further $Y^{-1} = X^{-3\delta_2 - \delta_1} > 12 C X^{\delta_2 - 1 + 3\delta_0/4}$, since $4\delta_2 + \delta_1 + 3\delta_0/4 < 1$. So (12) implies

$$\left|q_1 h_2 \lambda \mu^{-1} - q_2 h_1\right| < (2CY)^{-1} \leqslant (2q)^{-1}.$$

This implies, by Legendre's law of best approximation, that (since $h_1 h_2 \neq 0$)$q < q_1 |h_2|$. But, on the other hand, using $|x| \leqslant X^{\delta_1}$ one has

$$q_1 |h_2| \leqslant 10 |\mu| X^{\delta_1} q_1 q_2 \leqslant 10 |\mu| X^{\delta_1 + 2\delta_2} < Y \leqslant q,$$

a contradiction. Hence $\max(q_1, q_2) \geqslant X^{\delta_2}$ and the Lemma is proved. ∎

Let b and m be two given natural numbers, and δ_0 satisfy, in the notation of Lemma 3, $0 < \delta_0 \leqslant \delta_0(b, m)$. Now, with the notation and definitions of Lemma 7, denote by J_1 the part of the interval $X^{-1+\frac{1}{2}\delta_0} \leqslant |x| \leqslant X^{\delta_1}$ corresponding (via Lemma 7) to $q_1 = \max(q_1, q_2)$, and by J_2 the remaining part. Then we prove

LEMMA 8. *We have*

$$(13) \qquad S_b(\lambda x) = 0\left(X^{b^{-1} - \delta_b'}\right), x \in J_1; S_m(\mu x) = 0\left(X^{m^{-1} - \delta_m'}\right), x \in J_2,$$

where $\delta_k' = \left(2^{2k+2}(k+1)\right)^{-1} \min(1/3k, \delta_2, \delta_0/2)$, *for* $k \geqslant 1$.

PROOF: We prove only the first part of (13), the other part being obtained likewise. We have, by Lemma 7, for $x \in J_1$

$$\left|\lambda x - h_1 q_1^{-1}\right| \leqslant q_1^{-2}, X^{\delta_2} \leqslant q_1 \leqslant X^{1-\frac{1}{2}\delta_0}.$$

From this it easily follows that

$$\log\left(\min\left(\left(\delta^b X\right)^{1/3b}, q_1, \delta^b X q_1^{-1}\right)\right) \geqslant \left(2^{6b-2}(2b+1)\right) \log\log X,$$

and hence by Lemma 10 of [5] (twice)

$$S_b(\lambda x) = 0\left(X^{b^{-1} - \delta_b'}\right),$$

with δ_b' as defined in the statement of the lemma. ∎

4.3 Proof of the Theorem $(k \leqslant 3)$.

We have $s = 2^k + 1$. We treat the cases $i \leqslant 2$ and $i > 2$ separately. Let $0 < \varepsilon < \min |\lambda_j|$.

(a) $i \leqslant 2$. Without loss of generality we can assume $i = 2$. Taking, in Section 4.2, $\lambda = \lambda_1$, $\mu = \lambda_2$; $b = k$, $m = K$ and $q = Y$ we get, by Lemma 8, for $X = q^{1/(3\delta_2+\delta_1)}$ (where q is a sufficiently large denominator of a convergent to the continued fraction of λ/μ)

$$(14) \qquad S_{(1),k}(x) = 0\left(X^{k^{-1}-\delta_k'}\right),\ x \in J_1;\ S_{(2),K}(x) = 0\left(X^{K^{-1}-\delta_K'}\right),\ x \in J_2.$$

We use these bounds to estimate

$$(15) \qquad \int_{X^{-1+\delta_0}}^{X^{\delta_1}} \left| S_{(2),K}(x) \prod_{j \neq 2} S_{(j),k}(x) \right| K_\varepsilon(x)dx.$$

By (14), the part of the integral over J_2 is

$$<< X^{K^{-1}-\delta_K'} \int_{-\infty}^{\infty} \left| \prod_{j \neq 2} S_{(j),k}(x) \right| K_\varepsilon(x)dx$$

which, by Hölder's inequality, Lemmas 1 and 2 (as in the proof of Lemma 6) is $<< \varepsilon X^{a_1} L^B$, where $a_1 = (2^k - k)k^{-1} + K^{-1} - \delta_K'$.

Again, by (14), the part of (15) over J_1 is, with $c = 2^k\left(\max\left(2^k, 2^K\right)\right)^{-1} \leqslant 1$,

$$<< \left(X^{k^{-1}-\delta_k'}\right)^c \int_{-\infty}^{\infty} \left| S_{(2),K}(x)\left(S_{(1),k}(x)\right)^{1-c} \prod_{j \geqslant 3} S_{(j),k}(x) \right| K_\varepsilon(x)dx.$$

By Hölder's inequality, Lemmas 1 and 2, this expression is

$$<< X^{(k^{-1}-\delta_k')c}\left(\left(\prod_{j \geqslant 3} \left(\int_{-\infty}^{\infty} \left|S_{(j),k}(x)\right|^{2^k} K_\varepsilon(x)dx \right)^{2^{-k}} \right) \right.$$

$$\left. \left(\int_{-\infty}^{\infty} \left| S_{(2),K}(x) S_{(1),k}^{1-c}(x) \right|^{2^k} K_\varepsilon(x)dx \right)^{2^{-k}} \right)$$

$$<< \left(\varepsilon L^B\right)^{(2^k-1)2^{-k}} X^{\sigma_1} \left(\int_{-\infty}^{\infty} \left| S_{(2),K}(x) S_{(1),k}^{1-c}(x) \right|^{2^k} K_\varepsilon(x)dx \right)^{2^{-k}},$$

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where $\sigma_1 = c(k^{-1} - \delta_k') + (1 - 2^{-k})(2^k - k)k^{-1}$. The integral here is by Hölder's inequaltiy,

$$<< \left(\int_{-\infty}^{\infty} |S_{(2),K}(x)|^{2^k c^{-1}} K_\varepsilon(x)dx\right)^c \left(\int_{-\infty}^{\infty} |S_{(1),k}(x)|^{2^k} K_\varepsilon(x)dx\right)^{1-c}$$

Noting that $2^k c^{-1} \geqslant 2^K$ and using Lemmas 1 and 2, we see that this quantity is $<< \varepsilon L^{B_1} X^{\sigma_2}$, for some $B_1 = B_1(k,K)$, where $\sigma_2 = c(2^k c^{-1} - K)K^{-1} + (1-c)(2^k - k)k^{-1}$. Hence the part of the integral (15) over J_1 is $<< \varepsilon L^{B_2} X^{a_2}$, for some $B_2 = B_2(k,K)$ and $a_2 = \sigma_1 + 2^{-k}\sigma_2 = 2^k k^{-1} + K^{-1} - 1 - c\delta_k'$.

Thus the integral (15) is

$$(14)\ldots \qquad << \varepsilon L^B X^{2^k k^{-1} - 1 + K^{-1} - \delta''},$$

where $B = B(k,K)$ and $\delta'' = \min(\delta_K', c\delta_k')$.

(b) $i > 2$. In this case we take $\lambda = \lambda_1$, $\mu = \lambda_2$, and $b = m = k$ in Section 4.2 and argue as in the case (a) for the part of (15) over J_1 there. This leads again to a similar bound for (15).

To complete the proof of the Theorem in this case $(s = 2^k + 1)$ we need, for given $\delta_1 < 1$, only to show that ε satisfies

$$4X^{-\delta_1} \leqslant \varepsilon < \min|\lambda_j|;\, L^B X^{-\delta_1} \leqslant \varepsilon^2 L^{-s-1};\, L^B X^{-\delta''} \leqslant \varepsilon L^{-s-1}.$$

These conditions are satisfied by $\varepsilon = X^{-\alpha}$, where $\alpha = \min\left(\frac{1}{4}\delta_1, \frac{\delta''}{2}\right)$ (say). Thus, with this choice of ε, we get, using Lemmas 5 and 6, (14) that under the hypotheses of the theorem, for a seuence of $X \to \infty$,

$$\int_{-\infty}^{\infty} S_{(i),K}(x) \prod_{j \neq i} S_{(j),k}(x) e(x\eta) K_\varepsilon(x)dx$$
$$>> \varepsilon^2 X^{(2^k k^{-1} + K^{-1} - 1)} L^{-(2^k+1)}.$$

Since, by Lemma 1, the left-side here is $\leqslant \varepsilon$ times the number of solutions of (with $k(j) = K$ or $= k$, according as $j = i$ or not)

$$\left|\eta + \sum_{j \neq i} \lambda_j p_j^k + \lambda_i p_i^K\right| < X^{-\alpha},\ p_j \leqslant X^{1/k(j)} < \delta^{-1} p_j,$$

$1 \leqslant j \leqslant 2^k + 1$. Hence this inequality has $>> X^a L^{-(2^k+1)}$, $a = 2^k k^{-1} + K^{-1} - 1 - \alpha$, solutions in prime p_j, for a suitable sequence of $X \to \infty$. This completes the proof of the theorem for $k \leqslant 3$.

4.4 Proof of the Theorem $(k > 3)$.

Here we indicate the changes required to deal with this case using the results of [5]. We have $s = 2r + 2m + 1$. Analogously we work with the product

$$I_{(i),K}(x) \prod_{\substack{j=1 \\ j \neq i}}^{2r+1} I_{(j),k}(x) F_1^{(k)}(x) F_2^{(k)}(x)$$

(since one can assume $i \leqslant 2r+1$), where $F_t^{(k)}(x)$ are exponential sums $F_t(x)(t = 1, 2)$ of [5]. It is apparent from earlier considerations, in view of Theorem 1 of [5] and its analogue in Section 6 of [5], that the problem is to estimate on the intermediate range only; that is we are to estimate the integrals, for $t = 1, 2$,

$$\int_{X^{-1+\delta_0} \leqslant |x| \leqslant X^{\delta_1}} \left| S_{(i),K}(x) \prod_{j \neq i} S_{(j),k}(x) \right| \left| F_t^2(x) \right| K_\epsilon(x) dx.$$

This can be done as in Section 4.3 above with $c = \frac{2r}{\max(2r, 2^K)}$, using Theorem 1 of [5] for $k > 4$ and its analogue in Section 6 for $k = 4$.

Thus the Theorem is completely proved. Theorem' is an immediate consequence from it.

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