2
Operators in Hilbert spaces

In this chapter we recall basic properties of operators on topological vector spaces. We concentrate on Hilbert spaces, which play the central role in quantum physics.

2.1 Convergence and completeness

We start with a discussion of various topics related to convergence and completeness.

2.1.1 Nets

Nets are generalizations of sequences. In this subsection we briefly recall this useful concept.

Definition 2.1 *A* directed set *is a set I equipped with a partial order relation $\leq$ such that for any $i, j \in I$ there exists $k \in I$ such that $i \leq k$, $j \leq k$.*

We will often use the following directed set:

Definition 2.2 *Let I be a set. We denote by $2^I_{\rm fin}$ the family of finite subsets of I. It becomes a directed set when we equip it with the inclusion.*

Definition 2.3 *Let $\mathcal{S}$ be a set. A* net in $\mathcal{S}$ *is a mapping from a directed set I to $\mathcal{S}$, denoted by $\{x_i\}_{i\in I}$.*

Definition 2.4 *A net $\{x_i\}_{i\in I}$ in a topological space $\mathcal{S}$* converges to $x \in \mathcal{S}$ *if for any neighborhood $\mathcal{N}$ of x there exists $i \in I$ such that if $i \leq j$ then $x_j \in \mathcal{N}$. We will write $x_i \to x$. If $\mathcal{S}$ is Hausdorff, then a net in $\mathcal{S}$ can have at most one limit and one can also write $\lim x_i = x$.*

Definition 2.5 *Let $\mathcal{X}$ be a topological space and $\mathcal{U} \subset \mathcal{X}$. Then $\mathcal{U}^{\rm cl}$ will denote the* closure of $\mathcal{U}$*, which is equal to the set of limits of all convergent nets in $\mathcal{U}$.*

2.1.2 Functions

Definition 2.6 *Let $\mathcal{X}, \mathcal{Y}$ be sets. Then $c(\mathcal{X}, \mathcal{Y})$ is the set of all functions from $\mathcal{X}$ to $\mathcal{Y}$. Clearly, $c(\mathcal{X}, \mathbb{K})$, is a vector space over $\mathbb{K}$. We often write $c(\mathcal{X})$ for $c(\mathcal{X}, \mathbb{C})$. $f \in c(\mathcal{X}, \mathbb{K})$ is called* finitely supported *if $f^{-1}\big(\mathbb{K}\backslash\{0\}\big)$ is finite. $c_{\rm c}(\mathcal{X}, \mathbb{K})$*

https://doi.org/10.1017/9781009290876.003 Published online by Cambridge University Press

denotes the space of finitely supported functions in $c(\mathcal{X},\mathbb{K})$*. If* $x\in\mathcal{X}$*, define* $\delta_x\in c_{\rm c}(\mathcal{X},\mathbb{K})$ *by* $\delta_x(y):=\begin{cases}1, & x=y,\\ 0, & x\neq y.\end{cases}$*. Clearly, each element of* $c_{\rm c}(\mathcal{X},\mathbb{K})$ *can be written as a unique finite linear combination of* $\{\delta_x\ :\ x\in\mathcal{X}\}$*. Sometimes, it will be convenient to write* x *instead of* δ_x*.*

Definition 2.7 *Let* $\mathcal{X},\mathcal{Y}$ *be topological spaces. Then* $C(\mathcal{X},\mathcal{Y})$ *is the set of all continuous functions from* $\mathcal{X}$ *to* $\mathcal{Y}$*. Clearly,* $C(\mathcal{X},\mathbb{K})$ *is a vector space over* $\mathbb{K}$*. We often write* $C(\mathcal{X})$ *for* $C(\mathcal{X},\mathbb{C})$*.* $C_{\rm c}(\mathcal{X},\mathbb{K})$ *denotes the set of* compactly supported functions *in* $C(\mathcal{X},\mathbb{K})$*.*

We will use various styles of notation to introduce a function f with domain $\mathcal{X}$, such as $\mathcal{X}\ni x\mapsto f(x)$ or $\{f(x)\}_{x\in\mathcal{X}}$. Sometimes, we will simply write that *we are given a function* $f(x)$. This is possible, if we declared before that x is the *generic variable in* $\mathcal{X}$, or at least if it is clear from the context that x should be understood this way. Thus x is not a concrete element of $\mathcal{X}$, it is just a symbol for which we can substitute an arbitrary element of $\mathcal{X}$.

For example, the notation $[a_{ij}]$ is sometimes used for a matrix. Here, i is understood as the generic variable in $\{1,\dots,n\}$ and j as the generic variable in $\{1,\dots,m\}$, where n, m should be clear from the context. Thus $[a_{ij}]$ is an abbreviation for $\{1,\dots,n\}\times\{1,\dots,m\}\ni(i,j)\mapsto a_{i,j}$.

Generic variables are also used in some other situations, e.g. as a part of the notation for integration or differentiation.

2.1.3 Topological vector spaces

Let $\mathcal{E}$ be a topological vector space.

Definition 2.8 *If* $\mathcal{U}\subset\mathcal{E}$*, we will use the shorthand* $\mathrm{Span}^{\rm cl}(\mathcal{U})$ *for* $(\mathrm{Span}(\mathcal{U}))^{\rm cl}$*.*

Definition 2.9 *A net* $\{x_i\}_{i\in I}$ *in a topological vector space* $\mathcal{E}$ *is* Cauchy *if, for any neighborhood* $\mathcal{N}$ *of* 0*, there exists* $i\in I$ *such that if* $i\leq j,k$*, then* $x_j-x_k\in\mathcal{N}$*.* $\mathcal{E}$ *is* complete *if every Cauchy net is convergent.*

Proposition 2.10 *There exists a complete topological vector space containing* $\mathcal{E}$ *as a dense subspace. If* $\mathcal{E}_1$ *and* $\mathcal{E}_2$ *are two such complete spaces, then there exists a unique linear homeomorphism* $T:\mathcal{E}_1\to\mathcal{E}_2$ *such that* $T\big|_{\mathcal{E}}=1\!\!1_{\mathcal{E}}$*.*

Definition 2.11 *The complete vector space, described in Prop. 2.10 uniquely up to isomorphism, is called the* completion of $\mathcal{E}$ *and denoted* $\mathcal{E}^{\rm cpl}$*.*

2.1.4 Infinite sums

Let $\mathcal{E}$ be a topological vector space and $\{x_i\}_{i\in I}$ a family of elements of $\mathcal{E}$.

https://doi.org/10.1017/9781009290876.003 Published online by Cambridge University Press

Definition 2.12 *We say that the* series $\sum\limits_{i\in I} x_i$ *is* convergent *if the net* $\left\{\sum_{i\in J} x_i\right\}_{J\in 2^I_{\mathrm{fin}}}$*, is convergent. The limit of the above net will be denoted by*

$$\sum_{i\in I} x_i.$$

Assume that $\mathcal{E}$ is a normed space.

Definition 2.13 *We say that the series* $\sum\limits_{i\in I} x_i$ *is* absolutely convergent *if the numerical series* $\sum\limits_{i\in I} \|x_i\|$ *is convergent.*

Proposition 2.14 (1) *For every absolutely convergent series, the set* $\{i \;:\; x_i \neq 0\}$ *is at most countable.*

(2) *Every absolutely convergent series in a Banach space is convergent.*

(3) *In a finite-dimensional space, a series is convergent iff it is absolutely convergent.*

2.1.5 Infinite products

Let $\{x_i\}_{i\in I}$ be a family in $\mathbb{C}$.

Definition 2.15 *First assume that* $x_i \neq 0$ *for all* $i \in I$*. In this case, the* infinite product $\prod\limits_{i\in I} x_i$ *is called* convergent *if the net* $\left\{\prod\limits_{i\in J} x_i\right\}_{J\in 2^I_{\mathrm{fin}}}$ *converges to a non-zero limit in* $\mathbb{C}$*. The limit will be denoted by*

$$\prod_{i\in I} x_i.$$

In the general case, one says that $\prod\limits_{i\in I} x_i$ *is convergent if* $I_0 = \{i \in I \;:\; x_i = 0\}$ *is finite and the infinite product* $\prod\limits_{i\in I\setminus I_0} x_i$ *is convergent in the above sense. If* $I_0 \neq \emptyset$*, one sets*

$$\prod_{i\in I} x_i := 0.$$

It is easy to see that the convergence of $\prod\limits_{i\in I} x_i$ is equivalent to the convergence of $\sum\limits_{i\in I} |x_i - 1|$. Therefore, if $\prod\limits_{i\in I} x_i$ converges, then the set $\{i \in I \;:\; x_i \neq 1\}$ is at most countable and $x_i \to 1$.

2.2 Bounded and unbounded operators

2.2.1 Normed vector spaces

Let $\mathcal{H}, \mathcal{K}$ be normed spaces over $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$.

Definition 2.16 *We equip the complex conjugate space $\overline{\mathcal{H}}$ with the norm $\|\overline{\Phi}\| := \|\Phi\|$, $\Phi \in \mathcal{H}$.*

Definition 2.17 *$B(\mathcal{H}, \mathcal{K})$ denotes the space of bounded linear operators from $\mathcal{H}$ to $\mathcal{K}$. We set $B(\mathcal{H}) := B(\mathcal{H}, \mathcal{H})$. $\mathcal{H}^{\#} := B(\mathcal{H}, \mathbb{K})$ is the* topological dual of $\mathcal{H}$ *and $\mathcal{H}^* := B(\overline{\mathcal{H}}, \mathbb{K}) = \overline{\mathcal{H}^{\#}} = \overline{\mathcal{H}}^{\#}$ is the* topological anti-dual of $\mathcal{H}$.

Remark 2.18 *Note that the meaning of the symbol $\mathcal{H}^{\#}$, resp. $\mathcal{H}^*$ depends on the context: if we consider $\mathcal{H}$ as a vector space without a topology, it will denote the algebraic dual, resp. anti-dual. see Defs; 1.8, resp. 1.54. If $\mathcal{H}$ is considered together with its topology, it will denote the topological dual, resp. anti-dual.*

Definition 2.19 *By saying that A is a* linear operator from $\mathcal{H}$ to $\mathcal{K}$, *we will not necessarily mean that it is defined on the whole $\mathcal{H}$. We will just mean that there exists a subspace $\mathcal{D}$ of $\mathcal{H}$ such that $A \in L(\mathcal{D}, \mathcal{K})$. The space $\mathcal{D}$ will be called the* domain of A *and denoted* $\mathrm{Dom}\, A$. *The subspace* $\mathrm{Gr}\, A := \big\{(\Phi, A\Phi) \ : \ \Phi \in \mathrm{Dom}\, A\big\} \subset \mathcal{H} \oplus \mathcal{K}$ *is called the* graph of A.

Definition 2.20 *A linear operator A from $\mathcal{H}$ to $\mathcal{K}$ is* closed *if* $\mathrm{Gr}\, A$ *is closed in $\mathcal{H} \oplus \mathcal{K}$. It is called* closable *if it has a closed extension. Its minimal closed extension is called the* closure of A *and denoted by A^{cl}. $Cl(\mathcal{H}, \mathcal{K})$ will denote the set of closed, densely defined operators from $\mathcal{H}$ to $\mathcal{K}$.*

Proposition 2.21 *Let $A \in B(\mathcal{H}, \mathcal{K})$. Then A is closable as an operator from $\mathcal{H}^{\mathrm{cpl}}$ to $\mathcal{K}^{\mathrm{cpl}}$ and $A^{\mathrm{cl}} \in B(\mathcal{H}^{\mathrm{cpl}}, \mathcal{K}^{\mathrm{cpl}})$.*

Definition 2.22 *Let A be an operator on $\mathcal{H}$. We say that $z \in \mathbb{C}$ belongs to the* resolvent set of A *if $A - z\mathbb{1} : \mathrm{Dom}\, A \to \mathcal{H}$ is bijective and $(A - z\mathbb{1})^{-1} \in B(\mathcal{H})$. The resolvent set of A is denoted by* $\mathrm{res}\, A$. *The set* $\mathrm{spec}\, A = \mathbb{C} \backslash \mathrm{res}\, A$ *is called the* spectrum of A.

Definition 2.23 (1) *If A is an injective linear operator, then we set* $\mathrm{Dom}\, A^{-1} := \mathrm{Ran}\, A$.

(2) *If A, B are two linear operators, we set*

$$\mathrm{Dom}\, AB := \{\Phi \in \mathrm{Dom}\, B \ : \ B\Phi \in \mathrm{Dom}\, A\}.$$

(3) *If A, B are two linear operators on $\mathcal{H}$, their* commutator *and* anti-commutator *are the operators given by*

$$[A, B] := AB - BA, \ \ [A, B]_+ := AB + BA, \ \ \textit{on } \mathrm{Dom}\, AB \cap \mathrm{Dom}\, BA.$$

In the case that $\mathcal{H}$ is a Hilbert space, sometimes we will consider $[A, B]$, $[A, B]_+$ as quadratic forms on $\mathrm{Dom}\, A \cap \mathrm{Dom}\, A^* \cap \mathrm{Dom}\, B \cap \mathrm{Dom}\, B^*$. For example,

$$(\Phi|[A, B]\Psi) := (A^*\Phi|B\Psi) - (B^*\Phi|A\Psi).$$

2.2.2 Scalar product spaces

Let $\mathcal{H}$ be a unitary space (a complex space equipped with a scalar product). The scalar product of $\Phi, \Psi \in \mathcal{H}$ will be denoted by $(\Phi|\Psi)$ or $\overline{\Phi} \cdot \Psi$. Recall that a complete unitary space is called a complex Hilbert space, where one usually omits the word "complex". Note that if $\mathcal{H}$ is a Hilbert space, then $\overline{\mathcal{H}}$ equipped with the scalar product $(\overline{\Psi}|\overline{\Phi}) := \overline{(\Psi|\Phi)}$ is a Hilbert space as well, and the map $\mathcal{H} \ni \Phi \mapsto \overline{\Phi} \in \overline{\mathcal{H}}$ is anti-unitary (see Subsect. 1.2.10). The Riesz lemma says that $\mathcal{H}^* = \overline{\mathcal{H}}^{\#}$ is naturally isomorphic to $\mathcal{H}$. Sometimes, however, other identifications are convenient; see Subsect. 2.3.4.

In a Euclidean space (a real space equipped with a scalar product) we prefer to denote the scalar product by $\langle\Phi|\Psi\rangle$ or $\Phi \cdot \Psi$. Recall that a complete Euclidean space is called a real Hilbert space. If $\mathcal{H}$ is a real Hilbert space, the Riesz lemma says that $\mathcal{H}^{\#}$ is naturally isomorphic to $\mathcal{H}$.

Remark 2.24 *If we compare Def. 1.54 with this subsection, we see that* $\overline{z}\cdot w$ *or* $(w|z)$ *may stand for the pairing between vectors in two distinct spaces in an anti-dual pair, or for the scalar product of two vectors in the same Hilbert space.*

Analogously, if we compare Def. 1.8 with this subsection, we see that $v\cdot y$ *or* $\langle v|y\rangle$ *may stand for the pairing within a dual pair, or for the scalar product in the same real Hilbert space.*

There are more such ambiguous notations, whose exact meaning depends on the context; see e.g. Remark 2.18. These ambiguities should not cause any difficulties.

Remark 2.25 *As we see above, there are minor differences in the notation and terminology between real and complex Hilbert spaces. In what follows, we often discuss both cases at once. We then use the notation and terminology of complex Hilbert spaces, their modification to the real case being obvious.*

Definition 2.26 *Let* $\mathcal{H}$ *be a real or complex Hilbert space. A family of vectors* $\{e_i\}_{i\in I}$ *is called an* orthonormal system *if* $(e_i|e_j) = \delta_{ij}$. *If in addition* $\mathrm{Span}^{\mathrm{cl}}\{e_i : i \in I\} = \mathcal{H}$, *we say that it is an* orthonormal basis, *or an* o.n. basis *for brevity.*

Definition 2.27 *Let* $\mathcal{H}$ *be a topological vector space. We say that it is a* Hilbertizable space *if there exists a scalar product on* $\mathcal{H}$ *that generates its topology and* $\mathcal{H}$ *is complete in the corresponding norm.*

2.2.3 Operators on Hilbert spaces

In this subsection we discuss basic definitions concerning operators on complex and real Hilbert spaces. We try to be as close as possible to the usual terminology, fixing, however, some of its obvious flaws (see Remark 2.30).

We start with the complex case. Let $\mathcal{H}_1, \mathcal{H}_2, \mathcal{H}$ be complex Hilbert spaces. Let A be a densely defined operator from $\mathcal{H}_1$ to $\mathcal{H}_2$.

https://doi.org/10.1017/9781009290876.003 Published online by Cambridge University Press

Definition 2.28 *The operator* A^* *from* $\mathcal{H}_2$ *to* $\mathcal{H}_1$ *defined by*

$$(\Phi_2, \Psi_1) \in \operatorname{Gr} A^* \Leftrightarrow (\Phi_2|A\Phi_1) = (\Psi_1|\Phi_1), \;\; \Phi_1 \in \operatorname{Dom} A,$$

is called the adjoint of A. *We set* $A^\# := \overline{A}^* = \overline{A^*}$, *which is an operator from* $\overline{\mathcal{H}}_2$ *to* $\overline{\mathcal{H}}_1$.

Note that A^* and $A^\#$ are automatically closed. Moreover, A is closable iff $\operatorname{Dom} A^*$, or $\operatorname{Dom} A^\#$ is dense. We then have $A^{**} = A^{\#\#} = A^{\mathrm{cl}}$.

If A is bounded, then so are A^* and $A^\#$. As an example of adjoints, consider $\Phi \in \mathcal{H}$ and let us note the identities $|\Phi)^* = (\Phi|$ (see Def. 1.70).

Definition 2.29 (1) *Densely defined operators on* $\mathcal{H}$ *satisfying* $A \subset A^*$ *are called* Hermitian.

(2) *Densely defined operators from* $\overline{\mathcal{H}}$ *to* $\mathcal{H}$ *satisfying* $A \subset A^\#$ *are called* symmetric.

Remark 2.30 *Note that, unfortunately, in a part of the literature the word "symmetric" is often used to denote Hermitian operators. This is an incorrect usage.*

Definition 2.31 (1) *Densely defined operators on* $\mathcal{H}$ *satisfying* $A^* = A$ *are called* self-adjoint *and those satisfying* $A^* = -A$ anti-self-adjoint. *The set of bounded self-adjoint operators on* $\mathcal{H}$ *is denoted by* $B_{\mathrm{h}}(\mathcal{H})$, *and the set of all self-adjoint operators on* $\mathcal{H}$ *by* $Cl_{\mathrm{h}}(\mathcal{H})$.

(2) *The set of bounded symmetric, resp. anti-symmetric operators from* $\overline{\mathcal{H}}$ *to* $\mathcal{H}$ *is denoted* $B_{\mathrm{s}}(\overline{\mathcal{H}}, \mathcal{H})$, *resp.* $B_{\mathrm{a}}(\overline{\mathcal{H}}, \mathcal{H})$. *The set of all operators from* $\overline{\mathcal{H}}$ *to* $\mathcal{H}$ *satisfying* $A = A^\#$, *resp.* $A = -A^\#$ *is denoted* $Cl_{\mathrm{s}}(\overline{\mathcal{H}}, \mathcal{H})$, *resp.* $Cl_{\mathrm{a}}(\overline{\mathcal{H}}, \mathcal{H})$.

Self-adjoint and anti-self-adjoint operators are automatically closed. Likewise, operators in $Cl_{\mathrm{s}}(\overline{\mathcal{H}}, \mathcal{H})$ and $Cl_{\mathrm{a}}(\overline{\mathcal{H}}, \mathcal{H})$ are automatically closed.

A is anti-self-adjoint iff $\mathrm{i}A$ is self-adjoint.

Let us now consider the real case. Let $\mathcal{H}_1, \mathcal{H}_2, \mathcal{H}$ be real Hilbert spaces. Let A be a densely defined operator from $\mathcal{H}_1$ to $\mathcal{H}_2$.

Definition 2.32 *The operator* $A^\#$ *from* $\mathcal{H}_2$ *to* $\mathcal{H}_1$ *defined by*

$$(\Phi_2, \Psi_1) \in \operatorname{Gr} A^\# \Leftrightarrow \langle \Phi_2|A\Phi_1\rangle = \langle \Psi_1|\Phi_1\rangle, \;\; \Phi_1 \in \operatorname{Dom} A,$$

is called the adjoint of A.

Note that $A^\#$ is automatically closed. Moreover, A is closable iff $\operatorname{Dom} A^\#$ is dense and we then have $A^{\#\#} = A^{\mathrm{cl}}$.

If A is bounded, then so is $A^\#$. As an example of adjoints, consider $\Phi \in \mathcal{H}$ and let us note the identity $|\Phi\rangle^\# = \langle\Phi|$ (see Def. 1.24).

Definition 2.33 *Densely defined operators on* $\mathcal{H}$ *satisfying* $A \subset A^\#$ *are called* symmetric.

https://doi.org/10.1017/9781009290876.003 Published online by Cambridge University Press

Definition 2.34 *Densely defined operators on $\mathcal{H}$ satisfying $A^{\#} = A$, resp. $A^{\#} = -A$ are called* self-adjoint, *resp.* anti-self-adjoint. *The set of bounded self-adjoint, resp. anti-self-adjoint operators on $\mathcal{H}$ is denoted by $B_{\mathrm{s}}(\mathcal{H})$, resp. $B_{\mathrm{a}}(\mathcal{H})$. The set of all self-adjoint, resp. anti-self-adjoint operators on $\mathcal{H}$ is denoted by $Cl_{\mathrm{s}}(\mathcal{H})$, resp. $Cl_{\mathrm{a}}(\mathcal{H})$.*

Self-adjoint and anti-self-adjoint operators are automatically closed.

2.2.4 Product of a closed and a bounded operator

Proposition 2.35 *Let $G \in Cl(\mathcal{H}_1, \mathcal{H}_2)$, $H \in B(\mathcal{H}_2, \mathcal{H}_3)$. We define HG and G^*H^* with their natural domains, as in Def. 2.23. Then HG is densely defined, so that we can define its adjoint, and we have*

$$(HG)^* = G^*H^*. \tag{2.1}$$

Besides, G^*H^* is closed.

Proof By Def. 2.23,

$$\mathrm{Dom}\, HG = \mathrm{Dom}\, G, \tag{2.2}$$
$$\mathrm{Dom}\, G^*H^* = \{\Phi \in \mathcal{H}_3 : \ H^*\Phi \in \mathrm{Dom}\, G^*\}. \tag{2.3}$$

G is densely defined. By (2.2), so is HG. It immediately follows that

$$(HG)^* \supset G^*H^*.$$

Suppose that $\Psi \in \mathrm{Dom}(HG)^*$. This means that for some C

$$|(\Psi|HG\Phi)| \leq C\|\Phi\|, \quad \Phi \in \mathrm{Dom}\, G.$$

Thus

$$|(H^*\Psi|G\Phi)| \leq C\|\Phi\|, \quad \Phi \in \mathrm{Dom}\, G.$$

Hence, $H^*\Psi \in \mathrm{Dom}\, G^*$. Thus

$$(HG)^* \subset G^*H^*.$$

This ends the proof of (2.1). G^*H^* is closed as the adjoint of a densely defined operator. □

2.2.5 Compact operators

Let $\mathcal{H}_1, \mathcal{H}_2, \mathcal{H}$ be real or complex Hilbert spaces.

Definition 2.36 *We denote by $B_\infty(\mathcal{H}_1, \mathcal{H}_2)$ the space of* compact *operators from $\mathcal{H}_1$ to $\mathcal{H}_2$ and set $B_\infty(\mathcal{H}) := B_\infty(\mathcal{H}, \mathcal{H})$.*

https://doi.org/10.1017/9781009290876.003 Published online by Cambridge University Press

Proposition 2.37 *If $A \in B_\infty(\mathcal{H})$ is self-adjoint, then $\mathcal{H}$ has an o.n. basis $\{e_j\}_{j\in I}$ of eigenvectors of A for a family $\{\lambda_j\}_{j\in I}$ of real eigenvalues having 0 as its only possible accumulation point.*

2.2.6 Hilbert–Schmidt and trace-class operators

Let $\mathcal{H}_1, \mathcal{H}_2, \mathcal{H}$ be real or complex Hilbert spaces.

Definition 2.38 *$A \in B(\mathcal{H}_1, \mathcal{H}_2)$ is called* Hilbert–Schmidt *if* $\mathrm{Tr} A^* A < \infty$. *The space of Hilbert–Schmidt operators is denoted $B^2(\mathcal{H}_1, \mathcal{H}_2)$ and is a Hilbert space for the scalar product* $\mathrm{Tr}\, B^* A$.

Definition 2.39 *If $A \in B(\mathcal{H}_1, \mathcal{H}_2)$, then $|A| := \sqrt{A^* A}$ is called the* absolute value of A. *We say that A is* trace class *if* $\mathrm{Tr}|A| < \infty$. *The space of trace-class operators is denoted $B^1(\mathcal{H}_1, \mathcal{H}_2)$.*

Note the following proposition:

Proposition 2.40 *Let $A \in B^1(\mathcal{H})$ and $B_n \in B(\mathcal{H})$, with $B_n \to B$ weakly. Then* $\mathrm{Tr}\, B_n A \to \mathrm{Tr}\, BA$.

Definition 2.41 *Positive elements of $B^1(\mathcal{H})$ having trace 1 are called* density matrices.

Definition 2.42 *If $\beta > 0$ is a number, H a self-adjoint operator and $\mathrm{Tr}\, \mathrm{e}^{-\beta H} < \infty$, then the density matrix*

$$\mathrm{e}^{-\beta H} / \mathrm{Tr}\, \mathrm{e}^{-\beta H}$$

is called the Gibbs density matrix for the Hamiltonian H and inverse temperature β.

Definition 2.43 *For $1 \le p < \infty$, the* p-th Schatten ideal *is*

$$B^p(\mathcal{H}_1, \mathcal{H}_2) := \big\{ A \in B(\mathcal{H}_1, \mathcal{H}_2) \ : \ \mathrm{Tr}|A|^p < \infty \big\}.$$

2.2.7 Fredholm determinant

Let $\mathcal{H}$ be a real or complex Hilbert space.

Definition 2.44 *We denote by $1\!\!1 + B^1(\mathcal{H})$ the set of operators of the form $1\!\!1 + A$ with $A \in B^1(\mathcal{H})$. If $\mathcal{H}$ is a complex, resp. real Hilbert space, we set*

$$U_1(\mathcal{H}) := U(\mathcal{H}) \cap (1\!\!1 + B^1(\mathcal{H})), \ \ \textit{resp.}\ O_1(\mathcal{H}) := O(\mathcal{H}) \cap (1\!\!1 + B^1(\mathcal{H})).$$

Theorem 2.45 *There exists a unique function $1\!\!1 + B^1(\mathcal{H}) \ni R \mapsto \det R \in \mathbb{C}$ satisfying the following properties:*

https://doi.org/10.1017/9781009290876.003 Published online by Cambridge University Press

(1) *If* $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$ *with* $\dim \mathcal{H}_1 < \infty$ *and* $R = R_1 \oplus 1\!\!1$, *then* $\det R = \det R_1$, *where* $\det R_1$ *is the usual determinant of the finite-dimensional operator* R_1.
(2) $B^1(\mathcal{H}) \ni A \mapsto \det(1\!\!1 + A)$ *is continuous in the trace norm.*

Definition 2.46 $\det R$ *is called the* Fredholm determinant of R.

The following properties follow easily from Thm. 2.45:

Proposition 2.47 (1) $\det R_1 R_2 = \det R_1 \det R_2$, $\det R^* = \overline{\det R}$.
(2) *Let* $A \in B^1(\mathcal{H})$. *Then* $1\!\!1 + A$ *is invertible iff* $\det(1\!\!1 + A) \neq 0$.
(3) *If* $\mathcal{H}$ *is a complex, resp. real Hilbert space, then*

$$|\det R| = 1, \ \textit{for } R \in U_1(\mathcal{H}), \ \textit{resp. } \det R = \pm 1, \ \textit{for } R \in O_1(\mathcal{H}).$$

Definition 2.48 *Let* $A \in B^2(\mathcal{H})$. *The* regularized determinant of $1\!\!1 + A$ *is*

$$\det{}_2(1\!\!1 + A) := \det\big((1\!\!1 + A)\mathrm{e}^{-A}\big). \tag{2.4}$$

The regularized determinant can sometimes be used instead of the usual determinant.

Proposition 2.49 *Let* $A \in B^2(\mathcal{H})$. *Then* $1\!\!1 + A$ *is invertible iff* $\det_2(1\!\!1 + A) \neq 0$.

2.2.8 Derivatives

For functions on a vector space, one can distinguish several kinds of derivatives. In the following definition we recall the directional derivative, the Gâteaux derivative and the (most commonly used) Fréchet derivative.

Let $\mathcal{Y}$ be a real or complex vector space and G be a complex-valued function defined on a subset U of $\mathcal{Y}$. To define the directional derivative of G at a point $y_0 \in U$, U has to be *finitely open*, i.e. the intersection of U with any finite-dimensional subspace of $\mathcal{Y}$ should be open (for its canonical topology).

Definition 2.50 *Let* $\mathcal{Y}$ *be a real or complex normed space and* G *be a complex-valued function defined on a subset* U *of* $\mathcal{Y}$.

(1) *Assume that* U *is finitely open. We say that the* derivative of G in the direction of $y \in \mathcal{Y}$ at y_0 *exists if*

$$y \cdot \nabla G(y_0) := \frac{\mathrm{d}}{\mathrm{d}t} G(y_0 + ty)\big|_{t=0} \ \textit{exists.}$$

(Here t *is a real parameter if* $\mathcal{Y}$ *is real, and complex if* $\mathcal{Y}$ *is complex.)*
We say that G *is* Gâteaux differentiable at y_0 *if*

$$\mathcal{D} := \big\{ y \in \mathcal{Y} \ : \ y \cdot \nabla G(y_0) \ \textit{exists} \big\}$$

is a dense linear subspace of $\mathcal{Y}$ *and the map*

$$\mathcal{D} \ni y \mapsto y \cdot \nabla G(y_0) \in \mathbb{C}$$

is a bounded linear functional.

https://doi.org/10.1017/9781009290876.003 Published online by Cambridge University Press

(2) *Assume that* U *is open. We say that* G *is* Fréchet differentiable at y_0 *if there exists a bounded linear functional* v *such that*

$$\lim_{y \to 0} \frac{G(y_0 + y) - G(y_0) - v \cdot y}{\|y\|} = 0.$$

If such a functional exists, it is necessarily unique and is denoted $\nabla G(y_0)$*.*

Note that if the Fréchet derivative exists, then so does the Gâteaux derivative, and they are equal.

For example, consider the function $\mathrm{Dom}\, H^{\frac{1}{2}} \ni y \mapsto G(y) = (y|Hy)$, where H is a positive self-adjoint operator. The set $\mathrm{Dom}\, H^{\frac{1}{2}}$ is obviously finitely open. G is Gâteaux differentiable at y_0 iff $y_0 \in \mathrm{Dom}\, H$. It is Fréchet differentiable iff H is bounded.

2.3 Functional calculus

2.3.1 Holomorphic functional calculus

Let $\mathcal{H}$ be a Banach space and $A \in B(\mathcal{H})$. The basic construction of the *holomorphic functional calculus* is described in the following definition:

Definition 2.51 *Let* f *be a function on* $\mathrm{spec}\, A$ *that extends to a function holomorphic on an open neighborhood of* $\mathrm{spec}\, A$*. Let* γ *be a closed curve encircling* $\mathrm{spec}\, A$ *counterclockwise and contained in the domain of* f*. We set*

$$f(A) := \frac{1}{2\pi \mathrm{i}} \oint_\gamma f(z)(z\mathbb{1} - A)^{-1} \mathrm{d}z. \tag{2.5}$$

It is easy to see that (2.5) does not depend on the choice of the curve γ.

Let Θ be a subset of $\mathrm{spec}\, A$.

Definition 2.52 *The* characteristic function *of the set* Θ *is defined as*

$$\mathbb{1}_\Theta(z) := \begin{cases} 1, & z \in \Theta, \\ 0, & z \in \mathrm{spec}\, A \backslash \Theta. \end{cases}$$

Suppose that Θ is a relatively open and closed subset of $\mathrm{spec}\, A$. Then the function $\mathbb{1}_\Theta$ satisfies the assumptions of the holomorphic spectral calculus.

Definition 2.53 $\mathbb{1}_\Theta(A)$ *is called the* (Riesz) spectral projection of A onto Θ.

Clearly, if γ encircles Θ, staying outside of $\mathrm{spec}\, A \backslash \Theta$, then

$$\mathbb{1}_\Theta(A) = \frac{1}{2\pi \mathrm{i}} \oint_\gamma (z\mathbb{1} - A)^{-1} \mathrm{d}z. \tag{2.6}$$

https://doi.org/10.1017/9781009290876.003 Published online by Cambridge University Press

2.3.2 Functional calculus for normal operators

In the case of Hilbert spaces, besides the holomorphic calculus, we have another functional calculus based on the spectral theorem, which applies to normal operators.

Let us be more precise. Let $\mathcal{H}$ be a real or complex Hilbert space.

Definition 2.54 *An operator A on $\mathcal{H}$ is called* normal *if* $\mathrm{Dom}\,A = \mathrm{Dom}\,A^*$ *and* $(A\Phi|A\Psi) = (A^*\Phi|A^*\Psi)$, $\Phi, \Psi \in \mathrm{Dom}\,A$.

Self-adjoint and unitary operators are normal. In the case of normal operators the spectral theorem can be used to extend the functional calculus to a much larger class of functions.

Let A be a normal operator on a complex Hilbert space.

Definition 2.55 *If $f : \mathrm{spec}\,A \to \mathbb{C}$ is Borel, we define $f(A)$ by the* functional calculus for normal operators.

For normal operators we can extend the definition of spectral projections to a much larger class of sets.

Definition 2.56 *Let Θ be a Borel subset of* $\mathrm{spec}\,A$. *The operator $1\!\!1_\Theta(A)$ is called the* spectral projection of A onto Θ.

Let us now consider the functional calculus on real Hilbert spaces. Let $\mathcal{H}$ be a real Hilbert space and A a normal operator on $\mathcal{H}$. Then we can apply the functional calculus to the operator $A_{\mathbb{C}}$ on $\mathbb{C}\mathcal{H}$. Note that $\mathrm{spec}\,A_{\mathbb{C}}$ satisfies $\overline{\mathrm{spec}\,A_{\mathbb{C}}} = \mathrm{spec}\,A_{\mathbb{C}}$. If a Borel function f on $\mathrm{spec}\,A$ satisfies

$$f(\overline{z}) = \overline{f(z)}, \tag{2.7}$$

then $f(A_{\mathbb{C}})$ preserves $\mathcal{H}$, and the formula $f(A) := f(A_{\mathbb{C}})\big|_{\mathcal{H}}$ defines an operator on $\mathcal{H}$.

These conditions are satisfied, for instance, if A is a self-adjoint operator on $\mathcal{H}$ and f is a real Borel function. Note that in this case $f(A)$ is a self-adjoint operator on $\mathcal{H}$.

Let us describe another application of functional calculus on real Hilbert spaces that we will need. Let $R \in O(\mathcal{H})$ be such that $\mathrm{Ker}(R + 1\!\!1) = \{0\}$. Consider the function $f(z) = z^t$ for $t \in \mathbb{R}$, where if $t \notin \mathbb{Z}$ we take the principal branch of z^t, with a cut along the negative semi-axis. Note that z^t is not defined for $z = -1$. However, $1\!\!1_{\{-1\}}(R_{\mathbb{C}}) = 0$; therefore $R_{\mathbb{C}}^t$ is well defined. Moreover z^t satisfies (2.7), so we can define R^t. Note that $R^t \in O(\mathcal{H})$ and $R^t R^s = R^{t+s}$. For $|t| \leq 1$, we have $\mathrm{Ker}(R^t + 1\!\!1) = \{0\}$ and $(R^t)^s = R^{ts}$.

2.3.3 Spectrum of the product of operators

It is well known that if $A, B \in B(\mathcal{H})$, then

$$\mathrm{spec}\,(AB)\backslash\{0\} = \mathrm{spec}\,(BA)\backslash\{0\}.$$

This is also true if AB and BA are closed with $\operatorname{spec}(AB)$, $\operatorname{spec}(BA) \neq \mathbb{C}$; see Hardt–Konstantinov–Mennicken (2000). We will need the following related facts:

Proposition 2.57 (1) *Let A, B be two linear operators on a Hilbert space $\mathcal{H}$ such that AB and BA are closed. Let $z \in \mathbb{C}$ such that $z \notin \operatorname{spec}(AB) \cup \operatorname{spec}(BA)$. Then*

$$A(z\mathbb{1} - BA)^{-1} = (z\mathbb{1} - AB)^{-1}A.$$

Moreover, if $A, B \in B(\mathcal{H})$ and f is holomorphic near $\operatorname{spec}(AB) \cup \operatorname{spec}(BA)$, then

$$Af(BA) = f(AB)A.$$

(2) *If $A \in Cl(\mathcal{H})$ and f is a bounded Borel function, then*

$$Af(A^*A) = f(AA^*)A.$$

Proof Let $\Phi \in \operatorname{Dom} A$ and $(z\mathbb{1} - BA)\Psi = \Phi$. Then $BA\Psi = z\Psi - \Phi \in \operatorname{Dom} A$ and $ABA\Psi = zA\Psi - A\Phi$ hence $A\Psi \in \operatorname{Dom} AB$ and $(z\mathbb{1} - AB)A\Psi = A\Phi$. This proves (1).

To prove (2) we note that A^*A and AA^* are self-adjoint, so the identity $A(z\mathbb{1} - A^*A)^{-1} = (z\mathbb{1} - AA^*)^{-1}A$ for $z \in \mathbb{C}\backslash\mathbb{R}$ is true by (1). It extends by the usual argument to all bounded Borel functions. □

2.3.4 Scale of Hilbert spaces associated with a positive operator

Let $\mathcal{H}$ be a real or complex Hilbert space.

Definition 2.58 *For an operator B on $\mathcal{H}$ we will write $B \geq 0$ if it is positive self-adjoint. If in addition 0 is not an eigenvalue of B, then we will write $B > 0$.*

Let $B > 0$. Let us introduce the scale of Hilbert spaces associated with B. The Hilbert space $\mathcal{H}$ will play the role of a "pivot" space.

If $\mathcal{H}$ is real, we will identify $\mathcal{H}^{\#}$ with $\mathcal{H}$, and if $\mathcal{H}$ is complex, we identify $\mathcal{H}^*$ with $\mathcal{H}$, using the scalar product.

Definition 2.59 *We equip $\operatorname{Dom} B^{-s}$ with the scalar product $(\Phi|\Psi)_{-s} := (B^{-s}\Phi|B^{-s}\Psi)$ and the norm $\|B^{-s}\Phi\|$. We set*

$$B^s\mathcal{H} := \left(\operatorname{Dom} B^{-s}\right)^{\mathrm{cpl}}.$$

Proposition 2.60 (1) *$B^{-s}\mathcal{H} = \operatorname{Dom} B^s$ if $s \geq 0$ and $0 \notin \operatorname{spec} B$.*

(2) *$B^t : \operatorname{Dom} B^{-s} \cap \operatorname{Dom} B^t \to \operatorname{Dom} B^{-s-t}$ extends continuously to a unitary map from $B^s\mathcal{H}$ to $B^{s+t}\mathcal{H}$.*

(3) *$(B^t)^s\mathcal{H} = B^{st}\mathcal{H}$.*

https://doi.org/10.1017/9781009290876.003 Published online by Cambridge University Press

(4) *If $\mathcal{H}$ is complex, the sesquilinear product $(\Psi|\Phi)$ on $\mathrm{Dom}\, B^s \times \mathrm{Dom}\, B^{-s}$ extends continuously to $B^{-s}\mathcal{H} \times B^s\mathcal{H}$ and one can unitarily identify $(B^s\mathcal{H})^*$ with $B^{-s}\mathcal{H}$.*

(5) *If $\mathcal{H}$ is real, the bilinear product $\langle\Psi|\Phi\rangle$ on $\mathrm{Dom}\, B^s \times \mathrm{Dom}\, B^{-s}$ extends continuously to $B^{-s}\mathcal{H} \times B^s\mathcal{H}$ and one can isometrically identify $(B^s\mathcal{H})^\#$ with $B^{-s}\mathcal{H}$.*

Definition 2.61 *If B_1, B_2 are two positive self-adjoint operators, we write $B_1 \leq B_2$ if $\mathrm{Dom}\, B_2^{\frac{1}{2}} \subset \mathrm{Dom}\, B_1^{\frac{1}{2}}$ and*

$$\|B_1^{\frac{1}{2}}\Phi\|^2 \leq \|B_2^{\frac{1}{2}}\Phi\|^2,\ \Phi \in \mathrm{Dom}\, B_2^{\frac{1}{2}}.$$

If $0 \leq B_1 \leq B_2$, then the Kato–Heinz theorem says that $0 \leq B_1^\alpha \leq B_2^\alpha$ for $\alpha \in [0,1]$. If $0 < B_1 \leq B_2$, then also $0 \leq B_2^{-\alpha} \leq B_1^{-\alpha}$, for $\alpha \in [0,1]$. This implies the following fact:

Proposition 2.62 *Let $0 < B_1 \leq B_2$ and $-\frac{1}{2} \leq \alpha \leq \frac{1}{2}$. Then the natural embeddings*

$$I_\alpha : B_1^\alpha\mathcal{H} \to B_2^\alpha\mathcal{H}$$

are contractive and $I_\alpha^* = I_{-\alpha}$.

Note also the following useful fact, which follows from the three lines theorem.

Proposition 2.63 *Let $B > 0$ be a self-adjoint operator. Let $\Psi \in \mathrm{Dom}\, B$. Then*

$$\{z \ :\ 0 \leq \mathrm{Re}\, z \leq 1\} \ni z \mapsto B^z\Psi$$

is a continuous function holomorphic in the interior of the domain and satisfying the bound

$$\|B^z\Psi\| \leq \|\Psi\|^{1-\mathrm{Re}\, z}\|B\Psi\|^{\mathrm{Re}\, z}.$$

2.3.5 C_0-semi-groups

Let $\mathcal{H}$ be a real or complex Hilbert space.

Definition 2.64 *A* C_0-semi-group *is a one-parameter semi-group $[0,\infty[\ni t \mapsto U(t) \in B(\mathcal{H})$ continuous in the strong topology. Every C_0-semi-group $U(t)$ has the* generator *A defined by*

$$\mathrm{Dom}\, A := \left\{\Phi \in \mathcal{H} \ :\ \mathrm{s}-\lim_{t\to 0} t^{-1}(U(t)\Phi - \Phi) =: A\Phi \ \textit{exists}\right\}.$$

In such a case we will write $U(t) =: \mathrm{e}^{tA}$.

The generator of a C_0-semi-group is always closed and densely defined.

The set of generators of C_0-groups in $O(\mathcal{H})$ and $U(\mathcal{H})$ coincides with the set of anti-self-adjoint operators. This fact is known as *Stone's theorem.*

https://doi.org/10.1017/9781009290876.003 Published online by Cambridge University Press

Definition 2.65 *If* $\mathbb{R} \ni t \mapsto U(t)$ *is a unitary* C_0*-group, then the* self-adjoint generator of $U(t)$ *is the operator* B *defined as* $U(t) = \mathrm{e}^{\mathrm{i}tB}$.

Definition 2.66 A *is a* maximal dissipative operator *if it is a closed densely defined operator such that* $\mathrm{Re}(\Phi|A\Phi) \leq 0$ *for* $\Phi \in \mathrm{Dom}\, A$ *and* $\mathrm{Ran}(-A + \lambda \mathbb{1}) = \mathcal{H}$ *for some* $\lambda > 0$.

The Hille–Yosida theorem says that the set of generators of C_0-semi-groups of contractions coincides with the set of maximal dissipative operators. A is maximal accretive if $-A$ is maximal dissipative.

2.3.6 Local Hermitian semi-groups

Let $\mathcal{H}$ be a real or complex Hilbert space. Clearly, if $[0, \infty[\ni t \mapsto U(t) \in B(\mathcal{H})$ is a C_0-semi-group of self-adjoint contractions, then $U(t) = \mathrm{e}^{-tA}$ for A positive self-adjoint.

The notion of local Hermitian semi-groups, due to Klein–Landau (1981a) and Fröhlich (1980), allows us to extend this construction to the case of semi-groups of *unbounded* Hermitian operators. It is particularly important in the Euclidean approach to quantum field theory, especially at positive temperatures.

Definition 2.67 *Let* $T > 0$. *A* local Hermitian semi-group $\{P(t), \mathcal{D}_t\}_{t\in[0,T]}$ *is a family of linear operators* $P(t)$ *on* $\mathcal{H}$ *and subspaces* $\mathcal{D}_t$ *of* $\mathcal{H}$ *such that*

(1) $\mathcal{D}_0 = \mathcal{H}$, $\mathcal{D}_t \supset \mathcal{D}_s$ *if* $0 \leq t \leq s \leq T$ *and* $\mathcal{D} = \underset{0<t\leq T}{\cup} D_t$ *is dense in* $\mathcal{H}$*;*
(2) $P(t)$ *is a Hermitian linear operator with* $\mathrm{Dom}\, P(t) = \mathcal{D}_t$ *such that* $P(0) = \mathbb{1}$, $P(s)\mathcal{D}_t \subset \mathcal{D}_{t-s}$ *for* $0 \leq s \leq t \leq T$, *and* $P(t)P(s) = P(t+s)$ *on* $\mathcal{D}_{t+s}$ *for* $t, s, t+s \in [0, T]$*;*
(3) $t \mapsto P(t)$ *is weakly continuous, i.e. for* $\Phi \in \mathcal{D}_s$ *the map* $[0, s] \ni t \mapsto (\Phi, P(t)\Phi)$ *is continuous.*

Remark 2.68 *In the literature, local Hermitian semi-groups are often called* local symmetric semi-groups.

An example of a local Hermitian semi-group is $P(t) = \mathrm{e}^{-tH}$, $\mathcal{D}_t = \mathrm{Dom}\, \mathrm{e}^{-tH}$, with $T = \infty$, if H is a self-adjoint operator on $\mathcal{H}$. The following theorem shows that all local Hermitian semi-groups are restrictions of groups of unbounded self-adjoint operators of this form.

Theorem 2.69 *Let* $\{P(t), \mathcal{D}_t\}_{t\in[0,T]}$ *be a local Hermitian semi-group on* $\mathcal{H}$. *Then there exists a unique self-adjoint operator* H *on* $\mathcal{H}$ *such that*

(1) $\mathcal{D}_t \subset \mathrm{Dom}\, \mathrm{e}^{-tH}$, $\mathrm{e}^{-tH}\big|_{\mathcal{D}_t} = P(t)$ *for* $0 \leq t \leq T$*;*
(2) *For any* $0 < T' \leq T$, $\underset{0<t\leq T'}{\cup}\ \underset{0<s<t}{\cup} P(s)\mathcal{D}_t$ *is a core for* H.

(The core of a Hermitian operator is defined in Subsect. 2.3.7). For the proof one needs a definition and a lemma due to Widder (1934).

https://doi.org/10.1017/9781009290876.003 Published online by Cambridge University Press

Definition 2.70 *A continuous function* $r : [T_1, T_2] \to \mathbb{R}$ *is* OS positive *if for any* $n \in \mathbb{N}$ *and* $t_1, \ldots, t_n \in \mathbb{R}$ *such that* $T_1 \leq t_i + t_j \leq T_2$ *the matrix* $[r(t_i + t_j)]_{1 \leq i,j \leq n}$ *is positive.*

Lemma 2.71 *The continuous function* $r : [T_1, T_2] \to \mathbb{R}$ *is OS positive iff there exists a positive measure* ν *such that* $\lambda \mapsto \mathrm{e}^{-t\lambda}$ *belongs to* $L^1(\mathbb{R}, \mathrm{d}\nu)$ *for each* $t \in [T_1, T_2]$ *and*

$$r(t) = \int_{\mathbb{R}} \mathrm{e}^{-t\lambda} \mathrm{d}\nu(\lambda).$$

Proof of Thm. 2.69. We fix $0 < t < T$ and $\Phi \in \mathcal{D}_t$ and set $r(s) = \|P(s/2)\Phi\|^2$ for $s \in [0, 2t]$. The function r is continuous by the weak continuity of $P(s)$. Using the symmetry and semi-group property we see that r is OS positive on $[0, 2t]$. By Lemma 2.71, there exists a measure ν on $\mathbb{R}$ such that $r(s) = \int_{\mathbb{R}} \mathrm{e}^{-s\lambda} \mathrm{d}\nu(\lambda)$, $s \in [0, 2t]$. We note that

$$(P(s_1)\Phi | P(s_2)\Phi) = r(s_1 + s_2) = \int_{\mathbb{R}} \mathrm{e}^{-s_1\lambda} \mathrm{e}^{-s_2\lambda} \mathrm{d}\nu(\lambda), \ 0 \leq s_1, s_2 \leq t. \quad (2.8)$$

For $z \in \mathbb{C}$, set $g_z(\lambda) := \mathrm{e}^{-z\lambda}$. Since the span of $\{g_s \ : \ 0 \leq s \leq t\}$ is dense in the Hilbert space $L^2(\mathbb{R}, \mathrm{d}\nu)$, we see that the map

$$J : L^2(\mathbb{R}, \mathrm{d}\nu) \ni g_s \mapsto P(s)\Phi \in \mathcal{H}$$

extends by linearity and density to a unitary map between $L^2(\mathbb{R}, \mathrm{d}\nu)$ and the closed span of $\{P(s)\Phi \ : \ s \in [0, t]\}$. The map

$$z \mapsto g_z(\lambda) \in L^2(\mathbb{R}, \mathrm{d}\nu)$$

is clearly holomorphic in the strip $\{0 < \mathrm{Re}\, z < t\}$ and continuous up to the boundary. Applying J, we obtain that the map $s \to P(s)\Phi$ is the restriction to $[0, t]$ of a map $z \mapsto \Phi(z)$ with the same properties. We define now

$$U(y)\Phi := \Phi(\mathrm{i}y), \quad y \in \mathbb{R}. \quad (2.9)$$

Clearly, $U(y)$ is defined on $\mathcal{D}$. We claim that $U(y)$ extends to $\mathcal{H}$ as a strongly continuous unitary group. To prove that $U(y)$ is isometric, we use the identity

$$(\Phi(z_1)|\Phi(z_2)) = \int_{\mathbb{R}} \mathrm{e}^{-(\overline{z_1} - z_2)\lambda} \mathrm{d}\nu(\lambda),$$

which follows from (2.8) by analytic continuation. The map $U(y)$ is clearly linear on $\mathcal{D}$, if we note that $U(y)\Phi$ is independent of the space $\mathcal{D}_t$ to which Φ belongs and use that two vectors $\Phi, \Psi \in \mathcal{D}$ always belong to a common space $\mathcal{D}_t$. The strong continuity of $y \mapsto U(y)$ follows from the norm continuity of $\Phi(z)$.

To prove the group property, we pick $\Phi \in \mathcal{D}_t$ and set $\Phi(s_1, s_2) = P(s_1)P(s_2)\Phi = P(s_1 + s_2)\Phi$ for $s_1, s_2, s_1 + s_2 \in [0, t]$. We first analytically continue $\Phi(s_1, s_2)$ in s_1 to $\Phi(\mathrm{i}y_1, s_2) = U(y_1)P(s_1)\Phi$ and then in s_2 to $\Phi(\mathrm{i}y_1, \mathrm{i}y_2) = U(y_1)U(y_2)\Phi$. Since $P(s)\Phi$ analytically continues to $\Phi(z)$, we see that

$P(s_1+s_2)f$ analytically continues in (s_1,s_2) to $\Phi(\mathrm{i}y_1+\mathrm{i}y_2)=U(y_1+y_2)f$. Therefore, $U(y_1)U(y_2)\Phi=U(y_1+y_2)\Phi$.

We now uniquely define a self-adjoint operator H by $U(y)=:\mathrm{e}^{-\mathrm{i}yH}$. We note that if $\Phi\in\mathcal{D}_t$, then

$$(\Phi|U(y)\Phi)=\int_{\mathbb{R}}\mathrm{e}^{-\mathrm{i}y\lambda}\mathrm{d}\nu(\lambda),$$

hence $\mathrm{d}\nu(\lambda)=\mathrm{d}(\Phi|\mathbb{1}_{]-\infty,\lambda]}(H)\Phi)$, which implies that $\Phi\in\operatorname{Dom}\mathrm{e}^{-tH}$. The two functions $\mathrm{e}^{-\mathrm{i}yH}\Phi$ and $\Phi(\mathrm{i}y)$ coincide and are the boundary values of the functions $\mathrm{e}^{-zH}\Phi$ and $\Phi(z)$, both holomorphic in the strip $\{0<\operatorname{Re}z<t\}$ and continuous up to the boundary. It follows that these two holomorphic functions coincide everywhere and hence in particular

$$\Phi(t)=P(t)\Phi=\mathrm{e}^{-tH}\Phi.$$

This shows the existence of a self-adjoint operator H satisfying (1). If H_1, H_2 are two such operators, then the same analytic continuation argument shows that $\mathrm{e}^{-\mathrm{i}yH_1}\Phi=\mathrm{e}^{-\mathrm{i}yH_2}\Phi$ for $\Phi\in\mathcal{D}$, which implies that $H_1=H_2$. We refer to Klein–Landau (1981a) for the proof of (2). □

2.3.7 Essential self-adjointness

Let A be a Hermitian linear operator on a Hilbert space $\mathcal{H}$, i.e. such that $A\subset A^*$.

Definition 2.72 *A is called* essentially self-adjoint *if A^{cl} is self-adjoint. If the domain $\mathcal{D}$ of A needs to be specified, one says that A is* essentially self-adjoint on $\mathcal{D}$. *If a self-adjoint operator A is the closure of $A\big|_{\mathcal{D}}$, one says that $\mathcal{D}$ is a* core for *A.*

Definition 2.73 *If A is any operator, vectors $\Phi\in\bigcap_n\operatorname{Dom}A^n$ satisfying for some $t>0$*

$$\sum_{n=0}^{\infty}\frac{t^n\|A^n\Phi\|}{n!}<\infty$$

are called analytic vectors *of A.*

Let us give three criteria for essential self-adjointness, all due to Nelson.

Theorem 2.74 (1) (Nelson's commutator theorem) *Let A be Hermitian and B self-adjoint positive on $\mathcal{H}$ with* $\operatorname{Dom}B\subset\operatorname{Dom}A$. *Assume that*

$$\|A\Phi\|^2\leq C\|(B+\mathbb{1})\Phi\|^2,\quad |(A\Phi|B\Phi)-(B\Phi|A\Phi)|\leq C(\Phi|(B+\mathbb{1})\Phi),$$
$$\Phi\in\operatorname{Dom}B.$$

Then A is essentially self-adjoint on $\operatorname{Dom}B$.

(2) (Nelson's invariant domain theorem) *Consider $U_t=\mathrm{e}^{\mathrm{i}tA}$, a strongly continuous unitary group on $\mathcal{H}$. Let $\mathcal{D}$ be a dense subspace of $\mathcal{H}$ such that*

https://doi.org/10.1017/9781009290876.003 Published online by Cambridge University Press

$\mathcal{D} \subset \mathrm{Dom}\, A$ and $\mathcal{D}$ is invariant under U_t. Then A is essentially self-adjoint on $\mathcal{D}$.

(3) (Nelson's analytic vectors theorem) *Let A be a Hermitian operator possessing a dense space of analytic vectors. Then it is essentially self-adjoint on this space.*

A useful application of the notion of essential self-adjointness are the following two versions of Trotter's product formula:

Theorem 2.75 (1) *Let A, B be two self-adjoint operators on $\mathcal{H}$ such that $A+B$ with domain $\mathrm{Dom}\, A \cap \mathrm{Dom}\, B$ is essentially self-adjoint. Then*

$$\mathrm{e}^{\mathrm{i}t(A+B)^{\mathrm{cl}}} = \mathrm{s} - \lim_{n\to\infty} \left(\mathrm{e}^{\mathrm{i}tA/n}\mathrm{e}^{\mathrm{i}tB/n}\right)^n .$$

(2) *Suppose in addition that A, B are bounded below. Then*

$$\mathrm{e}^{-t(A+B)^{\mathrm{cl}}} = \mathrm{s} - \lim_{n\to\infty} \left(\mathrm{e}^{-tA/n}\mathrm{e}^{-tB/n}\right)^n , \ t \geq 0.$$

2.3.8 Commuting self-adjoint operators

Let A_1, A_2 be self-adjoint operators on $\mathcal{H}$.

Definition 2.76 *We say that A_1 and A_2* commute *if all their bounded Borel functions commute in the usual sense. (It is enough to demand e.g. that $\mathrm{e}^{\mathrm{i}t_1A_1}$ commutes with $\mathrm{e}^{\mathrm{i}t_2A_2}$ for any $t_1, t_2 \in \mathbb{R}$.)*

If $A_1, \dots, A_n$ are commuting self-adjoint operators, then for any Borel function F on $\mathbb{R}^n$ we can define $F(A_1, \dots, A_n)$ by the self-adjoint calculus.

One can generalize this as follows. Let $\mathcal{X}$ be a real vector space.

Definition 2.77 *We will say that*

$$\mathcal{X} \ni x \mapsto \langle x|A\rangle \in Cl_{\mathrm{h}}(\mathcal{H}) \tag{2.10}$$

is an $\mathcal{X}^{\#}$-vector of commuting self-adjoint operators *if there exists a unitary representation $\mathcal{X} \ni x \mapsto U(x) \in U(\mathcal{H})$ such that, for all $x \in \mathcal{X}$, $\mathbb{R} \ni t \mapsto U(tx)$ is strongly continuous and $U(tx) = \mathrm{e}^{\mathrm{i}t\langle x|A\rangle}$.*

Consider a vector of commuting self-adjoint operators (2.10). Clearly, $\langle x_1|A\rangle$, $\langle x_2|A\rangle$ commute for any $x_1, x_2 \in \mathcal{X}$. If F is a Borel function that depends on a finite-dimensional subspace of $\mathcal{X}^{\#}$, we can define $F(A)$ by the self-adjoint functional calculus.

Definition 2.78 *C^∞* vectors for (2.10) *are elements of*

$$\bigcap_{n=1}^{\infty} \bigcap_{x_1,\dots,x_n \in \mathcal{X}} \mathrm{Dom}\left(\langle x_1|A\rangle \cdots \langle x_n|A\rangle\right).$$

https://doi.org/10.1017/9781009290876.003 Published online by Cambridge University Press

2.3.9 Conjugations adapted to a self-adjoint operator

Let $\mathcal{H}$ be a complex Hilbert space. Recall that τ is a conjugation on $\mathcal{H}$ if it is an anti-unitary involution.

Proposition 2.79 *Let A be a self-adjoint operator on a (complex) Hilbert space $\mathcal{H}$. Then there exists a conjugation τ such that $\tau A \tau = A$. We then say that τ* is adapted to *A.*

Proof By the spectral theorem, there exists a collection $\{Q_i, \mu_i\}_{i \in I}$ of measure spaces such that $\mathcal{H} = \bigoplus_{i \in I} L^2(Q_i, \mu_i)$ and A is unitarily equivalent to the multiplication by a real measurable function. Then we take the standard conjugation on $\bigoplus_{i \in I} L^2(Q_i, \mu_i)$. □

2.4 Polar decomposition

Every operator on a Hilbert space possesses a canonical decomposition into the product of a positive operator and a partial isometry. It is called the *polar decomposition*. In this section we discuss various forms and consequences of the polar decomposition of an operator on a complex or real Hilbert space.

We will mostly consider the polar decomposition for operators that have a trivial kernel and co-kernel. In this case the decomposition into a positive operator and a partial isometry (which in this case is a unitary, resp. orthogonal operator) is unique, and not only canonical.

2.4.1 Polar decomposition

Let $\mathcal{H}, \mathcal{K}$ be real or complex Hilbert spaces and $A \in Cl(\mathcal{H}, \mathcal{K})$.

Theorem 2.80 *There exist a unique positive operator $|A| \in Cl(\mathcal{H})$ and a unique partial isometry $U \in B(\mathcal{H}, \mathcal{K})$ such that $A = U|A|$ and $\mathrm{Ker}\,|A| = (\mathrm{Ran}\,U)^{\perp}$. We have $|A| := (A^*A)^{\frac{1}{2}}$. Moreover one has $A = |A^*|U$ for $|A^*| = (AA^*)^{\frac{1}{2}}$.*

Definition 2.81 *The decomposition $A = |U|A$ described in Thm. 2.80 is called the* polar decomposition of *A.*

We will actually mostly need a special case of the polar decomposition, described in the following proposition:

Proposition 2.82 *Assume that $\mathrm{Ker}\,A = \{0\}$ and $\mathrm{Ran}\,A$ is dense in $\mathcal{K}$. Then there exists a unique positive operator $|A|$ and a unique orthogonal, resp. unitary operator U such that*

$$A = U|A| = |A^*|U. \tag{2.11}$$

https://doi.org/10.1017/9781009290876.003 Published online by Cambridge University Press

2.4.2 Polar decomposition of self-adjoint and anti-self-adjoint operators

In the self-adjoint case the polar decomposition has additional properties:

Proposition 2.83 *Let A be a self-adjoint operator on a real or complex Hilbert space. Assume that* $\operatorname{Ker} A = \{0\}$. *Let $A = U|A|$ be the polar decomposition of A. Then $|A|U = U|A|$ and $U^2 = \mathbb{1}$.*

Next let us consider anti-self-adjoint operators. Only the real case is interesting, because in the complex case the multiplication of anti-self-adjoint operators by the imaginary unit makes them self-adjoint. Therefore, until the end of this subsection $\mathcal{H}$ will be a real Hilbert space.

Proposition 2.84 (1) *Let A be an anti-self-adjoint operator on $\mathcal{H}$ such that* $\operatorname{Ker} A = \{0\}$. *Let $A = U|A|$ be its polar decomposition. Then $U \in O(\mathcal{H})$, $U^2 = -\mathbb{1}$ (U is a Kähler anti-involution) and $U|A| = |A|U$.*

(2) *Let $R \in O(\mathcal{H})$ such that* $\operatorname{Ker}(R^2 - \mathbb{1}) = \{0\}$. *Define $C = \frac{1}{2}(R + R^*)$. Then $-\mathbb{1} \le C \le \mathbb{1}$. Moreover, we have the polar decomposition $\frac{1}{2}(R - R^*) = V\sqrt{\mathbb{1} - C^2}$, where $V \in O(\mathcal{H})$, $V^2 = -\mathbb{1}$ and $[V, C] = 0$. Finally, we have $R = C + V\sqrt{\mathbb{1} - C^2}$.*

Proof (1) The identity $A = U|A| = |A^*|U$ implies that $U = -U^*$ since $A = -A^*$. Since $U \in O(\mathcal{H})$, we have $U^2 = -\mathbb{1}$.

(2) Since $R \in O(\mathcal{H})$, we get that $-\mathbb{1} \le C \le \mathbb{1}$. The operator $\frac{1}{2}(R - R^*)$ is anti-self-adjoint and has a zero kernel since $\operatorname{Ker}(R^2 - \mathbb{1}) = \{0\}$. Moreover,

$$\frac{1}{2}(R - R^*)^* \frac{1}{2}(R - R^*) = \frac{1}{4}(2\mathbb{1} - R^2 - R^{*2}) = \mathbb{1} - C^2. \tag{2.12}$$

Applying (1), we get that $V^2 = -\mathbb{1}$ and $[V, \sqrt{\mathbb{1} - C^2}] = 0$. Also $\sqrt{\mathbb{1} - C^2}[V, C] = [\sqrt{\mathbb{1} - C^2}V, C] = [R, C] = 0$. Since by (2.12) we know that $\operatorname{Ker}(\mathbb{1} - C^2) = 0$, this implies that $[V, C] = 0$. □

Let $R \in O(\mathcal{H})$. Set $\mathcal{H}_\pm := \operatorname{Ker}(R \mp \mathbb{1})$ and $\mathcal{H}_1 := (\mathcal{H}_- + \mathcal{H}_+)^\perp$. Then $\mathcal{H}_1$ is a subspace invariant w.r.t. R and $(R^2 - \mathbb{1})\big|_{\mathcal{H}_1}$ has a trivial kernel. Thus Prop. 2.84 can be applied also in situations when $\operatorname{Ker} A$ and $\operatorname{Ker}(R^2 - \mathbb{1})$ are non-trivial.

Corollary 2.85 (1) *Let A be an anti-self-adjoint compact operator. Then there exists an o.n. basis $\{e_{i\pm}, e_j\}_{i\in I, j\in J}$ and real numbers $\{\lambda_i\}_{i\in I}$ with $\lambda_i > 0$ such that*

$$Ae_{i+} = \lambda_i e_{i-}, \quad Ae_{i-} = -\lambda_i e_{i+}, \quad Ae_j = 0.$$

(2) *Let $R \in O(\mathcal{H}) \cap (\mathbb{1} + B_\infty(\mathcal{H}))$. Then there exist an o.n. basis $\{e_{i\pm}, f_j, g_k\}_{i\in I, j\in J, k\in K}$ and numbers $\{\theta_i\}_{i\in I}$ with $\operatorname{Im}\theta_i > 0$ such that*

$$Re_{i+} = \theta_i e_{i-}, \quad Re_{i-} = \overline{\theta}_i e_{i+}, \quad Rf_j = f_j, \quad Rg_k = -g_k.$$

https://doi.org/10.1017/9781009290876.003 Published online by Cambridge University Press

Proof Since A preserves $(\operatorname{Ker} A)^{\perp}$, we can assume that $\operatorname{Ker} A = \{0\}$. Let $A = V|A|$ the polar decomposition of A. Let $\{\lambda_i\}_{i\in I}$ be the eigenvalues of $|A|$ and $\mathcal{H}_i = \operatorname{Ker}(|A| - \lambda_i)$. Then $\mathcal{H}_i$ is invariant under V, so V is a Kähler anti-involution of $\mathcal{H}_i$. Let $(e_1, \cdots, e_n)$ be an o.n. basis of the complex Hilbert space $\mathbb{C}\mathcal{H}_i$. We set $e_{j+} = e_j$, $e_{j-} = Ve_j$, so that $(e_{1+}, \cdots e_{n+}, e_{1-}, \dots, e_{n-})$ is an o.n. basis of the real Hilbert space $\mathcal{H}_i$ and $Ae_{j+} = \lambda_i e_{j-}$, $Ae_{j-} = -\lambda_i e_{j+}$. Collecting the above bases of $\mathcal{H}_i$ we obtain the first statement of the corollary. □

Proposition 2.86 *Let $(\mathcal{Y}, \nu, \omega, \mathrm{j})$ be a complete Kähler space.*

(1) *Let A be a self-adjoint or anti-self-adjoint operator on $(\mathcal{Y}, \nu)$ such that* $\operatorname{Ker} A = \{0\}$ *and* $A\mathrm{j} = \mathrm{j}A$. *Let $|A|$, U be as in Prop. 2.83 or Prop. 2.84 (1). Then* $\mathrm{j}|A| = |A|\mathrm{j}$, $U\mathrm{j} = \mathrm{j}U$.
(2) *Let $R \in O(\mathcal{Y})$ such that* $\operatorname{Ker}(R^2 - 1\!\!1) = \{0\}$ *and* $R\mathrm{j} = \mathrm{j}R$. *Let C, V be as in Prop. 2.84 (2). Then* $V\mathrm{j} = \mathrm{j}V$ *and* $\mathrm{j}C = C\mathrm{j}$.

Proof To prove (1) we use that $\mathrm{j}^* = -\mathrm{j}$, since (ν, j) is Kähler, and hence $[A^*, \mathrm{j}] = 0$. This implies that $[A^*A, \mathrm{j}] = 0$ and hence $[|A|, \mathrm{j}] = 0$, $[V, \mathrm{j}] = 0$. The proof of (2) is similar. □

2.4.3 Polar decomposition of symmetric and anti-symmetric operators

In this subsection $\mathcal{H}$ is a complex Hilbert space. We use the notation $A^{\#} = \overline{A}^*$ defined in Subsect. 2.2.3. Recall that $Cl_{\mathrm{s/a}}(\overline{\mathcal{H}}, \mathcal{H})$ stands for the set of operators A from $\overline{\mathcal{H}}$ to $\mathcal{H}$ satisfying $A = A^{\#}$, resp. $A = -A^{\#}$.

Proposition 2.87 *Let $A \in Cl_{\mathrm{s/a}}(\overline{\mathcal{H}}, \mathcal{H})$ such that* $\operatorname{Ker} A = \{0\}$. *Consider the polar decomposition $A = U|A|$. Then we have*

$$U \in U(\overline{\mathcal{H}}, \mathcal{H}), \quad U|A| = |\overline{A}|U, \quad \overline{U}U = \pm 1\!\!1. \tag{2.13}$$

Proof Consider the real Hilbert space $\mathcal{H}_{\mathbb{R}}$, that is, the realification of $\mathcal{H}$. It can be identified with the realification of $\overline{\mathcal{H}}$. Let $A_{\mathbb{R}}$ denote the operator A understood as an operator on $\mathcal{H}_{\mathbb{R}}$. It is easy to see that

$$(A^{\#})_{\mathbb{R}} = (A_{\mathbb{R}})^{\#},$$

where the superscript $^{\#}$ is defined in the complex sense on the left and in the real sense on the right. Therefore, $A_{\mathbb{R}}^{\#} = \pm A_{\mathbb{R}}$. By the real case of Prop. 2.83, resp. Prop. 2.84 (1), we obtain $A_{\mathbb{R}} = U_{\mathbb{R}}|A_{\mathbb{R}}|$ with $U_{\mathbb{R}} \in O(\mathcal{H}_{\mathbb{R}})$:

$$U_{\mathbb{R}} \in O(\mathcal{H}_{\mathbb{R}}), \quad U_{\mathbb{R}}|A_{\mathbb{R}}| = |A_{\mathbb{R}}|U_{\mathbb{R}}, \quad U_{\mathbb{R}}^2 = \pm 1\!\!1. \tag{2.14}$$

Then we go back from $\mathcal{H}_{\mathbb{R}}$ to $\mathcal{H}$ and $\overline{\mathcal{H}}$, and (2.14) becomes (2.13). □

Corollary 2.88 (1) *Let* $A \in B_{\mathrm{s}}(\overline{\mathcal{H}}, \mathcal{H})$ *be compact. Then there exists an o.n. basis of* $(\operatorname{Ker} A)^{\perp}$, $\{e_i\}_{i\in I}$, *and positive numbers* $\{\lambda_i\}_{i\in I}$ *such that* $A\overline{e}_i = \lambda_i e_i$.

(2) *Let* $A \in B_{\mathrm{a}}(\overline{\mathcal{H}}, \mathcal{H})$ *be compact. Then there exists an o.n. basis of* $(\operatorname{Ker} A)^{\perp}$, $\{e_{i+},\ e_{i-}\}_{i\in I}$, *and positive numbers* $\{\lambda_i\}_{i\in I}$ *such that* $A\overline{e}_{i+} = \lambda_i e_{i-}$, $A\overline{e}_{i-} = -\lambda_i e_{i+}$.

2.5 Notes

The standard reference for operators on Hilbert spaces is the four-volume monograph by Reed–Simon (1975, 1978a,b, 1980), and also the books by Kato (1976) and by Davies (1980).

The Fredholm and regularized determinants are discussed e.g. in Simon (1979).

Thm. 2.69 about local Hermitian semi-groups is shown in Klein–Landau (1981a) and Fröhlich (1980).