A GEOMETRICAL APPROACH TO THE SECOND-ORDER LINEAR DIFFERENTIAL EQUATION

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1. Introduction. In this paper various concepts intrinsically defined by the differential equation

$$u'' + R(t)u = 0 \tag{1.1}$$

are interpreted geometrically by concepts analogous to those in the Minkowski plane. This is carried out in § 2. The point of such a development is that one may apply the techniques or transfer known results in the theory of curves (in particular, convex curves) to (1.1), thereby gaining an additional tool in the investigation of this equation. For an application of a result obtained in this way, namely (3.12), see **(4)**.

Throughout this paper, $R(t)$ is a real-valued, continuous function of t on the real line $(-\infty < t < +\infty)$ and only the real solutions of (1.1) are considered.

2. Analogies to Minkowskian geometry. Let u_1, u_2 be solutions of (1.1) with Wronskian $W(u_1, u_2) = u_1'u_2 - u_1u_2' = 1$ and consider the curve in the Euclidean plane with parametric representation:

$$\begin{cases} x = u_2(t), \\ y = u_1(t). \end{cases} \tag{2.1}$$

If u^*_1, u^*_2 are any two other solutions with Wronskian $W(u^*_1, u^*_2) = 1$, then $u^*_1 = c_1u_1 + c_2u_2$, $u^*_2 = c_3u_1 + c_4u_2$, where $c_1c_4 - c_2c_3 = 1$. Consequently, a geometrical interpretation of a quantity intrinsically determined by $R(t)$ must be invariant under central orientation and area-preserving affine transformations. The curve (2.1) will be referred to as the indicatrix. Since $W(u_1, u_2) = +1$, it follows that as t increases, the radius vector moves counter-clockwise and twice the area swept out in moving from P_1 (corresponding to t_1) to P_2 (corresponding to t_2) is simply $t_2 - t_1$.

Analogous to the Busemann sine function in Minkowski geometry **(1)**, we define

$$F(t_1, t_2) = u_1(t_2)u_2(t_1) - u_1(t_1)u_2(t_2) \tag{2.2}$$

which is twice the signed area of the triangle with vertices z (the origin), P_1, P_2 where we interpret the sign in the usual way. Also, analogous to the Finsler cosine function **(3)**, we define

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(2.3) $$G(t_1, t_2) = u_1'(t_1)\, u_2(t_2) - u_1(t_2)\, u_2'(t_1).$$

Let the line through z and P_2 intersect the tangent to the indicatrix at P_1 in $P^*{}_2$, then

(2.4) $$G(t_1, t_2) = \begin{cases} \dfrac{zP_2}{zP_2^*} \text{ if } z \text{ does not separate } P_2, P_2^* \\ -\dfrac{zP_2}{zP_2^*} \text{ if } z \text{ separates } P_2 \text{ and } P_2^* \\ 0 \quad \text{if the line } zP_2 \text{ is parallel to the tangent at } P_1. \end{cases}$$

To prove (2.4), note that the line through z and P_2 may be written $x = su_2(t_2)$, $y = su_1(t_2)$ and the tangent at P_1 may be written $x = u_2(t_1) + tu_2'(t_1)$, $y = u_1(t_1) + tu_1'(t_1)$. Now solve this linear system in s and t for s and use (2.3).

The functions $F(t_1, t_2)$ and $G(t_1, t_2)$ are independent of which pair of solutions u_1, u_2 with $W(u_1, u_2) = +1$ is chosen. In particular, for any t_0, we may choose u_1, u_2 with initial conditions

(2.5) $$\begin{cases} u_1(t_0) = 0, \ u_1'(t_0) = 1, \\ u_2(t_0) = 1, \ u_2'(t_0) = 0. \end{cases}$$

With reference to (2.5), a zero of $u_2(t)$ is called a focal point of t_0 and a zero of $u_1(t)$ is called a conjugate point of t_0. If every point has a focal point which follows it we define $m(t)$ and $\alpha(t)$. Let $t^*{}_0$ be the smallest focal point of t_0 following t_0, then

(2.6) $$\begin{cases} m(t_0) = \overset{*}{t_0} - t_0 \\ \alpha(t_0) = u_1(\overset{*}{t_0}) \end{cases}$$

where $u_1(t)$ has initial conditions (2.5). It follows that

(2.7) $$G(t, t + m(t)) \equiv 0,$$

(2.8) $$\alpha(t) \equiv F(t, t + m(t)).$$

Now, from (2.4), (2.7), (2.8) we see that the tangent to the indicatrix at $P(t)$ is parallel to the line through z and $P(t + m(t))$ and $m(t)$ is twice the area swept out by the radius vector as the parameter increases from t to $t + m(t)$. Also $\alpha(t)$ is twice the area of the triangle determined by z, $P(t)$, and $P(t + m(t))$.

If (1.1) is oscillatory as $t \to +\infty$, we define $\lambda(t)$ by

(2.9) $$\lambda(t_0) = \overset{*}{t_0} - t_0$$

where $t^*{}_0$ is the smallest conjugate point to t_0 exceeding t_0. It follows that

(2.10) $$F(t, t + \lambda(t)) \equiv 0$$

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and the points on the indicatrix corresponding to t and $t + \lambda(t)$ lie on a line through z and are separated by z. The quantity $\lambda(t)$ is twice the area swept out by the radius vector as it moves counter-clockwise through the corresponding straight angle.

Consider the curve with parametric representation:

$$\begin{cases} x = u_2'(t) \\ y = u_1'(t) \end{cases}. \tag{2.11}$$

From the fact that $W(u_1, u_2) = 1$, it follows that (2.11) may be constructed from the indicatrix by a polar reciprocation with respect to z followed by a 90° rotation counter-clockwise. This is the same manner that a solution to the isoperimetric problem is found from the Minkowski unit indicatrix **(2)**; however, we shall refer to (2.11) as the hodograph. Now if P is the point of the indicatrix corresponding to $t + m(t)$ and Q is the point of the hodograph corresponding to t, then these two points lie on a common ray from the origin z and

$$\alpha^{-1}(t) = \frac{zQ}{zP}. \tag{2.12}$$

Geometrically, (2.12) follows from the triangle area interpretation of $\alpha(t)$ and the parallelogram area interpretation of the Wronskian.

Since $u_1''(t)u_2'(t) - u_1'(t)u_2''(t) = R(t)$ we conclude that the radius vector of the hodograph with increasing t moves counter-clockwise if $R(t) > 0$, clockwise if $R(t) < 0$, and is stationary if $R(t) = 0$. The area swept out will be considered positive or negative as the radius vector moves counter-clockwise or clockwise respectively. It follows, then, that the integral

$$\int_{t_1}^{t_2} R(t)\, dt \tag{2.13}$$

is twice the signed area swept out by the radius vector of (2.11) as t runs from t_1 to t_2.

We may introduce concepts whose definitions are analogous to those of arc length and curvature in Minkowski geometry **(6)**. The "relative" arc length element ds on the indicatrix is obtained from the ordinary (euclidean) arc length element $d\bar{s} = \{[u_1'(t)]^2 + [u_2'(t)]^2\}^{1/2}dt$ by dividing by $[u_1^2(t + m(t)) + u_2^2(t + m(t))]^{1/2}$. Using (2.12) we have $ds = \alpha^{-1}(t)dt$. The relative length $s(t_1, t_2)$ of the indicatrix from t_1 to t_2 is then

$$s(t_1, t_2) = \int_{t_1}^{t_2} \alpha^{-1}(t)\, dt. \tag{2.14}$$

The concept of relative curvature may be introduced by defining angle $\theta(t_1, t_2) = t_2 - t_1$ as twice the signed area swept out by the radius vector of the indicatrix as t goes from t_1 to t_2. On the indicatrix, the curvature $\kappa(t)$ is then defined by

$$\kappa(t) = \lim_{\Delta t \to 0} \frac{\theta(t + m(t), t + \Delta t + m(t + \Delta t))}{s(t, t + \Delta t)} = R(t)\alpha^3(t). \tag{2.15}$$

In a similar manner $R(t)$ may be interpreted as the relative radius of curvature of the hodograph at a point corresponding to t.

3. Properties of the solutions. Here we seek those properties of the solutions which are equivalent to either $\lambda(t)$ or $\alpha(t)$ being constant. This is closely associated with characterizing F and G as Minkowskian sine and cosine functions respectively, that is, the indicatrix is a closed convex curve with centre z. Geometrical techniques are then further applied to delineate the properties of the corresponding families of equations.

THEOREM 3.1. *If* (1.1) *is oscillatory as* $t \to +\infty$, *then the following statements are equivalent:*

(a) $\lambda(t)$ *is a constant.*

(b) *For any solution* $u(t)$, $|u'(t)|$ *has the same value at the zeros of* $u(t)$.

(c) *For any non-trivial solution* $u(t)$, *there exists a non-trivial solution* $u^*(t)$ *such that* $u^{*\prime}(t)$ *is zero whenever* $u(t)$ *is zero.*[1]

Proof. The assumption that $\lambda(t)$ is constant is equivalent, under the oscillatory hypothesis, to having the zeros of any non-trivial solution equally spaced since two different non-trivial solutions cannot have their zeros spaced by a different amount owing to the separation theorem.

To prove (3.1), we need some identities which are valid in general. If the subscripts denote the appropriate partial derivatives, then from (2.2) and (2.3) we have

$$F_1(t_1, t_2) = -G(t_1, t_2), \qquad F_2(t_1, t_2) = G(t_2, t_1), \tag{3.2}$$

$$G_1(t_1, t_2) = R(t_1)F(t_1, t_2). \tag{3.3}$$

By showing the right-hand side is independent of t_3 we obtain

$$F(t_1, t_2) = F(t_3, t_2)G(t_3, t_1) - F(t_3, t_1)G(t_3, t_2) \tag{3.4}$$

and from this we obtain by differentiation

$$G(t_1, t_2) = G(t_1, t_3)G(t_3, t_2) - F(t_3, t_2)G_2(t_3, t_1) \tag{3.5}$$

and as a special case

$$G(t_1, t_2)G(t_2, t_1) - F(t_1, t_2)G_2(t_1, t_2) = 1. \tag{3.6}$$

Now, if $\lambda(t)$ is defined, we show from (2.10), (3.2), and (3.6) that

$$1 + \lambda'(t) = G^2(t, t + \lambda(t)) \tag{3.7}$$

[1]The referee has observed that if $-\infty < t < \infty$ is replaced by, say, $1 \leqslant t < \infty$ then (c) does not imply (a) or (b). An example is given by $u'' + au/t^2 = 0$ where $r^2 = a - 1/4 > 0$, for which $F(t_1,t_2) = (t_1 t_2)^{1/2} r^{-1} \sin[r \ln(t_2/t_1)]$ and $\lambda(t) = [\exp(\pi/r) - 1]\, t$.

where

$$G(t, t + \lambda(t))G(t + \lambda(t), t) = 1. \tag{3.8}$$

Suppose $\lambda(t)$ is a constant, then $G(t, t + \lambda) = G(t + \lambda, t) = -1$ and setting $t_1 = t_0$, $t_2 = t$, $t_3 = t + \lambda$ in (3.4) we have $F(t_0, t) = -F(t_0, t + \lambda)$ and therefore every solution $u(t)$ has the property that $u(t) = -u(t + \lambda)$ and both (b) and (c) follow from this relationship.

Let u_1, u_2 be solutions with initial conditions (2.5), then $F(t_0, t) = u_1(t)$, $G(t, t_0) = u_1'(t)$. Now $t_0 + \lambda(t_0)$ is a zero of $u_1(t)$ and consequently if (b) holds then $|G(t_0 + \lambda(t_0), t_0)| = 1$ and by (3.8) and (3.7), $\lambda'(t_0) = 0$; but since t_0 was arbitrary, $\lambda(t)$ is a constant.

If (c) holds, then with the same u_1, u_2 we have $G(t_0, t) = u_2(t)$ and $G_2(t_0, t_0 + \lambda(t_0)) = u_2'(t_0 + \lambda(t_0)) = 0$. But since t_0 was arbitrary $G_2(t, t + \lambda(t)) = 0$ for all t. Also $G_1(t, t + \lambda(t)) \equiv 0$ by (3.3) and (2.10). Consequently, $G(t, t + \lambda(t))$ is a constant and since $\lambda(t) > 0$ for all t, $\lambda(t)$ is a constant by (3.7) which completes the proof.

It is clear from the proof that (3.1) characterizes equations (1.1) whose indicatrix has the origin as centre and closes on itself after one revolution of the radius vector. However, a simple way to compute examples is by association of (1.1) with the following non-linear equation:

THEOREM 3.9. *If $g(t)$ is any solution of*

$$g'' - (g')^2 + e^{4g} = R(t) \tag{a}$$

then

$$\begin{cases} u_1(t) = e^{-g(t)} \sin\left[\int_{t_0}^{t} e^{2g(\tau)}\, d\tau\right] \\ u_2(t) = e^{-g(t)} \cos\left[\int_{t_0}^{t} e^{2g(\tau)}\, d\tau\right] \end{cases} \tag{b}$$

are independent solutions to (1.1) *with* $W(u_1, u_2) = u_1'u_2 - u_1u_2' = 1$; *also, if u_1, u_2 are solutions to* (1.1) *with Wronskian* $W(u_1, u_2) = 1$, *then*

$$g = \ln [u_1^2 + u_2^2]^{-1/2} \tag{c}$$

is a solution to (3.9a).

A proof is obtained by direct substitution and will therefore be omitted. The function $F(t_1, t_2)$ is given by

$$F(t_1, t_2) = e^{-g(t_1)}\, e^{-g(t_2)} \sin\left[\int_{t_1}^{t_2} e^{2g(\tau)}\, d\tau\right] \tag{3.10}$$

where $g(t)$ is any solution to (3.9a).

Let λ be any positive constant and let $g(t)$ be any periodic function of class C^2 with period λ and such that $\int_0^\lambda e^{2g(\tau)} d\tau = \pi$, then $R(t)$ given by (3.9a) yields an example of those equations characterized by (3.1). Every example may be obtained in this manner.

Those equations characterized by (3.1) for which $R(t)$ is non-negative will be called Minkowskian. To justify this name note that, by (2.15), the curvature (therefore the euclidean curvature) is non-negative and the indicatrix is a closed convex curve with centre z. If we normalize the equation setting $R^*(t) = (\lambda/\pi)^2 R(\lambda t/\pi)$, then the indicatrix will enclose area π and F^* and G^* are precisely the Minkowskian sine and cosine functions in the Minkowski geometry with the indicatrix as unit circle (see **6**).

If (1.1) is oscillatory as $t \to +\infty$, we define the amplitude function $A(t)$. Set $A(t_0) = \max F(t_0, t)$ for $t_0 \leqslant t \leqslant t_0 + \lambda(t_0)$, then $A(t_0)$ is twice the area of the maximal triangle with base $zP(t_0)$ and the third vertex on the arc of the indicatrix from t_0 to $t_0 + \lambda(t_0)$. If in addition (1.1) is oscillatory as $t \to -\infty$ and the solution $u(t)$ has the property that $|u'(t)|A(t)$ has the same value at every zero of $u(t)$, then $u(t)$ is said to have constant amplitude. It is clear that every solution of an equation characterized by (3.1) has constant amplitude, but this condition by itself is somewhat more general. Consider, for example, an indicatrix which is an admissible simple closed curve whose convex closure has the origin as centre but the indicatrix does not have this property; yet every solution has constant amplitude but $\lambda(t)$ is not a constant.

THEOREM 3.11. *If* (1.1) *is oscillatory as* $t \to \pm\infty$ *and* $R(t)$ *is non-negative, then* (1.1) *is Minkowskian if and only if every solution has constant amplitude.*

Proof. It is only necessary to prove that the latter property implies the equation is Minkowskian under the conditions stated. Let $u(t_0) = 1$, $u'(t_0) = 0$; then since $R(t)$ is non-negative the amplitude of the solution is attained whenever $u'(t) = 0$. Consequently, $|u(t)| = |G(t_0, t)| \leqslant 1$ for all t. Since t_0 was arbitrary $|G(x, y)| \leqslant 1$ for all x and y and $\lambda(t)$ is a constant by (3.8) and (3.7) which completes the proof.

A number of results of Minkowski geometry (**6**) may now be interpreted in terms of (1.1). For a Minkowski equation $\alpha(t) = A(t + m(t))$ so that these periodic functions have the same range of values.

THEOREM 3.12. *If* (1.1) *is a Minkowski equation then*

$$\frac{\lambda}{\pi} \leqslant \max \alpha = \max A < \frac{\lambda}{2}$$

$$\frac{\lambda}{4} < \min \alpha = \min A < \frac{\lambda}{3}$$

where the left-hand equality for max A *occurs if and only if* $R(t)$ *is a positive constant. Also*

$$8 < \lambda \int_0^\lambda R(t)\, dt \leqslant \pi^2$$

where the right-hand equality holds if and only if $R(t)$ *is a positive constant.*

Proof. We first notice that from the triangle area interpretation of $\alpha(t)$, $\lambda/4 < \alpha(t) < \lambda/2$, since λ is the area enclosed by the indicatrix. Now an affine regular hexagon may be inscribed in the indicatrix (see **6**) and by the properties of the Minkowski metric and (2.14) we have $\int_0^{2\lambda}\alpha^{-1}(t)dt > 6$ and consequently $\min\alpha < \lambda/3$. The product $\lambda\int_0^{\lambda}R(t)dt$ may be interpreted as the product of the area of the indicatrix and the area of its polar reciprocal with respect to its centre z. However, this product is bounded above and below by π^2 and 8 respectively (see **5** and **7**). The number 8 is obtained only for a parallelogram, a situation ruled out in our case by conditions of differentiability. However, π^2 is obtained if and only if the indicatrix is an ellipse; but the indicatrix is an ellipse with centre at the origin z if and only if $R(t)$ is a positive constant. Finally, to show that $\max\alpha \geqslant \lambda/\pi$ we note that the above product of areas is invariant under normalization; and it follows from the inequality involving π^2 and the interpretation (2.12) that $\max\alpha^* \geqslant 1$ or $\max\alpha \geqslant \lambda/\pi$.

The following property of the solutions is somewhat stronger:

THEOREM 3.13. *If* (1.1) *is oscillatory as* $t \to +\infty$, *then the following two conditions are equivalent.*

(a) *The equation is Minkowskian and the indicatrix is strictly convex.*

(b) *For any non-trivial solution* $u(t)$, *there exists a non-trivial solution* $u^*(t)$ *such that* $u(t)$ *is zero whenever* $u^{*\prime}(t)$ *is zero.*

Proof. If (a) holds, then $F(t_0, t)$ may be paired with $F(t_0 + m(t_0), t)$ to prove (b). Consider the solutions u_1, u_2 with initial conditions (2.5). If (b) holds, then $u_1(t)$ is zero whenever $u_2'(t)$ is zero. But the zeros of u_1 and u_2 separate each other and by Rolle's theorem $u_2'(t)$ vanishes whenever $u_1(t)$ is zero for $t \geqslant t_0$. However, if t_1 is any zero of $u_1(t)$ then u^*_2 with $u^*_2(t_1) = 1$, $u^*_2{}'(t_1) = 0$ is dependent on $u_2(t)$ and consequently by (3.1c), $\lambda(t)$ is a constant.

Suppose the convex closure of the indicatrix contains a line segment on its boundary, then there exists $t_1 < t_2$ such that $t_1 + m(t_1) = t_2 + m(t_2)$, and by the separation theorem no non-trivial solution can have more than one zero in the interval $t_1 \leqslant t \leqslant t_2 + m(t_2)$. However, the derivative of a non-trivial solution, with $t_1 + m(t_1)$ as a zero, vanishes at t_1 and t_2 and we obtain a contradiction by (b). Thus the indicatrix is strictly convex and this completes the proof.

The assumption that $\alpha(t)$ is constant is even more restrictive.

THEOREM 3.14. *With reference to* (1.1) *the following conditions are equivalent:*

(a) *The equation is oscillatory as* $t \to +\infty$ *and* $A(t)$ *is a constant.*

(b) *Every point has a focal point which follows it and* $\alpha(t)$ *is a constant.*

(c) *The equation is oscillatory as* $t \to +\infty$ *and for any solution* $u(t)$ *there exists a solution* $u^*(t)$ *such that each of the pairs* $u(t)$, $u^{*\prime}(t)$, *and* $u'(t)$, $u^*(t)$ *has exactly the same set of zeros.*

Proof. We need some additional identities which are valid when each point has a focal point which follows it. From (2.7), (3.3), (3.6), (2.8), and (3.2) we calculate:

$$1 + m'(t) = R(t)\alpha^2(t), \tag{3.15}$$

$$\alpha'(t) = G(t + m(t), t)R(t)\alpha^2(t), \tag{3.16}$$

$$\frac{d}{dt}[G(t + m(t), t)] = \alpha^{-1}(t)\,[1 - \alpha^4(t)\,R(t)\,R(t + m(t))]. \tag{3.17}$$

We first show (a) implies (b). For a given t_0, let M_0 be a number such that $A = F(t_0, t_0 + M_0)$, $0 < M_0 < \lambda(t_0)$. By definition of $A(t)$ and (3.2), $G(t_0 + M_0, t_0) = 0$. We will assume for the moment that $R(t_0 + M_0) \neq 0$, then by (3.3) and the implicit function theorem there exists, in a neighbourhood of t_0, $M(t)$ with a continuous derivative such that $M(t_0) = M_0$, $0 < M(t) < \lambda(t)$ and $G(t + M(t), t) \equiv 0$. Set $A^*(t) = F(t, t + M(t))$, then by (3.2), $A^{*\prime}(t) = -G(t, t + M(t))$. Since $A^*(t) \leqslant A$ and $A^*(t_0) = A$, we have $G(t_0, t_0 + M_0) = 0$. Now $m(t_0)$ and M_0 are both less than $\lambda(t_0)$ and by (2.7) and the fact that the solution $G(t_0, t)$ cannot have two zeros in the interval from t_0 to $t_0 + \lambda(t_0)$ we conclude that $m(t_0) = M_0$ and $G(t_0 + m(t_0), t_0) = 0$. Returning to the assumption that $R(t_0 + M_0) \neq 0$, consider the set of curves (G-curves) in the xy plane given by $G(x, y) = 0$. By (3.3) and (3.6), each curve may be expressed as a single-valued function $y(x)$ of class C^1 where the range of x is a finite or infinite open interval and if (x, y) is a point on such a curve then $dy/dx = F^2(x, y)R(x)$. Moreover, the number of such curves is at most countable since $G(t_0, t)$ has at most a countable number of zeros. A standard argument in real-variable theory shows that the set of all t_0 such that $y = t_0$ is a horizontal tangent to a G-curve is a set of measure zero and it follows by continuity that $G(t + m(t), t) = 0$ for all t. Therefore, by (3.16), $\alpha(t)$ is a constant.

Suppose (b) holds, by (3.16) and (3.17) the function $G^*(t) = G(t + m(t), t)$ is either zero or its derivative is positive. However, $R(t)$ cannot vanish identically in any interval $(t_0, +\infty)$ since t_0 must have a focal point which follows it. Consequently, whenever $G^*(t_0) = 0$ then $G^*(t) = 0$ for all $t \geqslant t_0$. If a minimum zero of G^* exists, then $R(t)$ must vanish (by continuity) at this point and we obtain a contradiction by (3.17). Therefore $G(t + m(t), t)$ is identical to zero and $R(t) > 0$ for all t. The two solutions $F(t_0, t), G(t_0 + m(t_0), t)$ must be dependent and the focal point $t_0 + m(t_0) + m(t_0 + m(t_0))$ of $t_0 + m(t_0)$ is, by (2.7), a conjugate point of t_0 and therefore (1.1) is oscillatory as $t \to +\infty$. Let $u(t)$, $u^*(t)$ be non-trivial solutions such that $u(t_0) = 0$, $u^{*\prime}(t_0) = 0$. Now it follows that $t_0 + m(t_0)$ is a zero of both $u'(t)$ and $u^*(t)$; we repeat the argument and conclude by induction that both pairs $u(t)$, $u^{*\prime}(t)$ and $u'(t)$, $u^*(t)$ have the same set of zeros for $t \geqslant t_0$. If $t_1 < t_0$ is a zero of any one of these four functions we may repeat the argument above and conclude that the two solutions involved are constant multiples of $u(t)$ and $u^*(t)$. Therefore (b) implies (c).

Suppose (c) holds, then by (3.13) the equation is Minkowskian with the additional property that the tangents to the indicatrix at t_0 and $t_0 + m(t_0)$ are parallel to the radius vectors at $t_0 + m(t_0)$ and t_0 respectively. Therefore, by triangle area interpretation $\alpha(t) \equiv A(t)$. But since $A(t) = F(t, t + m(t)) = \max F(t, y)$ for $t \leqslant y \leqslant t + \lambda(t)$ it follows from (3.2) and (3.16) that $\alpha(t) = A(t)$ is a constant and this completes the proof.

An equation (1.1) characterized by (3.14) will be called a Radon equation, since the indicatrix is a Radon or self-conjugate curve. The general construction of such curves is given in **(6)**. Smooth ones require additional constraints. In the cartesian plane let C be a smooth convex arc with continuous positive curvature from (1, 0) to (0, 1), inclusive, and remaining within the unit square in the first quadrant. In addition, we require C to be tangent to $x = 1$ at (1, 0) and tangent to $y = 1$ at (0, 1) and the curvature at (0, 1) is required to be the reciprocal of the curvature at (1, 0). Now rotate the polar reciprocal of C with respect to the origin through 90° and complete the closed convex curve by reflection through the origin. It is evident from the construction that the results of this section must necessarily be theorems in the large.

By (3.15), (3.12), and the proof of (3.14) we have

THEOREM 3.18. *If* (1.1) *is a Radon equation, then*

(a) $$t + m(t) = m(0) + \alpha^2 \int_0^t R(\tau)\, d\tau$$

(b) $$R(t)\, R(t + m(t)) = \alpha^{-4}$$

(c) $$\frac{\lambda}{\pi} \leqslant A = \alpha < \frac{\lambda}{3}$$

(d) $$\pi^2 \geqslant \lambda \int_0^\lambda R(\tau)\, d\tau > 9$$

where the left-hand equality is obtained in (c) *and* (d) *if and only if* $R(t)$ *is a positive constant.*

Here, the lower bound in (d) is obtained from (a) and (c) and with the interpretation as a product of areas it corresponds among the Radon curves to a regular affine hexagon as indicatrix.

Finally, the fact that the regular n-gons with $n \equiv 2 \bmod 4$ are Radon curves while those with $n \equiv 0 \bmod 4$ are not, suggests the following result:

COROLLARY 3.19. *If* (1.1) *is a Radon equation and* $R(t)$ *is not a positive constant, then* λ *is an odd multiple of the smallest positive period of* $R(t)$.

Proof. For a Radon equation we have $m(t) + m(t + m(t)) = \lambda$ for all t or

$$2m(t) + \int_t^{t+m(t)} m'(\tau)\, d\tau = \lambda.$$

The function $Q(x) = 2x + \int_t^{t+x} m'(\tau) d\tau$ is, for a given t, a strictly increasing function of x since $dQ/dx = 1 + \alpha^2 R(t + x) > 0$. Therefore, if $\lambda = 2nT$ where T is the smallest positive period of $R(t)$, then since $m(t)$ has period T it follows that $m(t) = nT$ for all t and by (3.18a), $R(t)$ is a positive constant which completes the proof.

References

1. H. Busemann, *The foundation of Minkowski geometry*, Comm. Math. Helv., *24* (1950), 156–187.
2. ——— *The isoperimetric problem in the Minkowski plane*, Amer. J. Math., *69* (1947), 863–871.
3. P. Finsler, *Ueber eine Verallgemeinerung des Satzes von Meusnier*, Vierteljschr. Naturforsch. Ges., Zürich, *85* (1940), 155–164.
4. L. W. Green, *Proof of Blaschke's sphere conjecture*, Bull. Amer. Math. Soc., *67* (1961), 156–158.
5. K. Mahler, *Ein Uebertragungsprinzip für konvexe Körper*, Časopis Pěst. Mat. Fys., *68* (1939), 93–102.
6. C. M. Petty, *On the geometry of the Minkowski plane*, Rivista di Matematica della Universita di Parma, *6* (1955), 269–292.
7. L. A. Santaló, *Un invariante afin para los cuerpos convexos del espacio de n-dimensiones*, Portugaliae Math., *8* (1949), 155–161.

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