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On the Injectivity of C^1 Maps of the Real Plane

Milton Cobo, Carlos Gutierrez and Jaume Llibre

Abstract. Let $X\colon \mathbb{R}^2 \to \mathbb{R}^2$ be a C^1 map. Denote by $\operatorname{Spec}(X)$ the set of (complex) eigenvalues of DX_p when p varies in $\mathbb{R}^2$. If there exists $\epsilon > 0$ such that $\operatorname{Spec}(X) \cap (-\epsilon, \epsilon) = \varnothing$, then X is injective. Some applications of this result to the real Keller Jacobian conjecture are discussed.

1 Introduction

Let $X\colon \mathbb{R}^2 \to \mathbb{R}^2$ be a map of class C^1. We shall denote by $\operatorname{Spec}(X)$ the set of (complex) eigenvalues of the derivative DX_p when p varies in $\mathbb{R}^2$. We will refer to $X\colon \mathbb{R}^2 \to \mathbb{R}^2$ as a *Keller map* if X is a polynomial map and the Jacobian determinant of X is identically equal to one in $\mathbb{R}^2$. It is important to observe that if $X\colon \mathbb{R}^2 \to \mathbb{R}^2$ is a Keller map, then $\operatorname{Spec}(X) \subset \mathbb{S}^1 \cup (\mathbb{R} \setminus \{0\})$. The bidimensional Real Keller Conjecture claims that if $X\colon \mathbb{R}^2 \to \mathbb{R}^2$ is a Keller map, then X is injective. For more details about Keller maps and the Jacobian conjecture see the recent book of van den Essen [5].

Our main result is the following:

Theorem A *Let $X\colon \mathbb{R}^2 \to \mathbb{R}^2$ be a C^1 map. Suppose that, for some $\epsilon > 0$, $\operatorname{Spec}(X) \cap (-\epsilon, \epsilon) = \varnothing$. Then X is injective.*

Relevant to this theorem, we may say:

(1) It is optimal because if the assumptions are relaxed to $0 \notin \operatorname{Spec}(X)$, the conclusion—even for a polynomial map X—is not true anymore, as shown by Pinchuck's counterexample [15] (See also [5], page 241).

(2) It confirms in a stronger way, the following Chamberland's conjecture [3] in dimension 2: *Let $Y\colon \mathbb{R}^n \to \mathbb{R}^n$ be a C^1 map. Suppose that there exists an $\epsilon > 0$ such that, for all $\lambda \in \operatorname{Spec}(Y)$, $|\lambda| > \epsilon$. Then Y is injective.*

(3) It does not imply the bidimensional real Keller Conjecture because, given n an even natural, the polynomial Keller map

$$X(x, y) = (-y, x + y^n)$$

satisfies $\operatorname{Spec}(X) = \mathbb{S}^1 \cup (\mathbb{R} \setminus \{0\})$ (that is, $\operatorname{Spec}(X)$ is the biggest possible for Keller maps). This example will be studied in Section 8.

(4) Campbell [2] classified the two-dimensional C^1 maps whose eigenvalues are both 1. All such maps have an explicit inverse. The class of functions considered in Theorem A is much broader, but no explicit inverse is given.

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Theorem A is proved in Section 3. A key point in its proof is notion of *half-Reeb components* which will be introduced in Section 2.

The next result is for C^1-maps having at least one component polynomial of the form $p(x)q(y)$.

Theorem B *Let $g\colon \mathbb{R}^2 \to \mathbb{R}$ be a C^1-map, let $p, q\colon \mathbb{R} \to \mathbb{R}$ be polynomial maps and let*

$$X(x,y) = \big(f(x,y), g(x,y)\big) = \big(p(x)q(y), g(x,y)\big).$$

Then if $0 \notin \operatorname{Spec}(X)$, X is injective.

Theorem B is proved in Section 4.

For polynomial maps we have the following results.

Theorem C *Let $X\colon \mathbb{R}^2 \to \mathbb{R}^2$ be a polynomial map. Suppose that, for some $\epsilon > 0$, either $\operatorname{Spec}(X) \cap (-\epsilon, 0] = \varnothing$ or $\operatorname{Spec}(X) \cap [0, \epsilon) = \varnothing$. Then X is injective.*

This is a sharper version of Theorem A in the case of polynomial maps.

Theorem D *Let $X = (f, g)\colon \mathbb{R}^2 \to \mathbb{R}^2$ be a polynomial map such that $\operatorname{Spec}(X) \cap \{0\} = \varnothing$ and denote*

$$\Gamma = \{(x,y) \in \mathbb{R}^2 : \operatorname{Trace}(\mathrm{DX})(x,y) = 0\}.$$

Then the following statements hold.

(a) *If $f|_\Gamma$ or $g|_\Gamma$ is a proper map, then X is injective.*
(b) *X is injective if and only if $(f^2 + g^2)|_\Gamma$ is a proper map.*

Theorems C and D are proved in Section 7. Their proof uses the notion of pair of aligned half-Reeb components which will be introduced in Section 5.

In Section 8 we will give some examples and applications of the results above.

2 Half-Reeb Components and Injectivity

Let $f\colon \mathbb{R}^2 \to \mathbb{R}$ be a C^1 submersion. For $q \in \mathbb{R}^2$ we denote by $X_f(q) = \big(-f_y(q), f_x(q)\big)$ the planar Hamiltonian vector field with Hamiltonian f. As usual $\nabla f(p) = \big(f_x(p), f_y(p)\big)$ denotes the gradient of f. Let $g(x,y) = xy$ and consider the set

$$B = \{(x,y) \in [0,2] \times [0,2] : x + y \le 2\} \setminus \{(0,0)\}.$$

Definition 1 We will say that $\mathcal{A} \subset \mathbb{R}^2$ is a *half-Reeb component* for X_f (or simply a hRc for X_f) if there is a homeomorphism $h\colon B \to \mathcal{A}$ which is a topological equivalence between $X_f|_{\mathcal{A}}$ and $X_g|_B$ and such that

(1) The segment $\{(x,y) \in B : x + y = 2\}$ is sent by h onto a transversal section for the flow of X_f in the complement of $h(1,1)$; this section is called the *compact edge* of $\mathcal{A}$.

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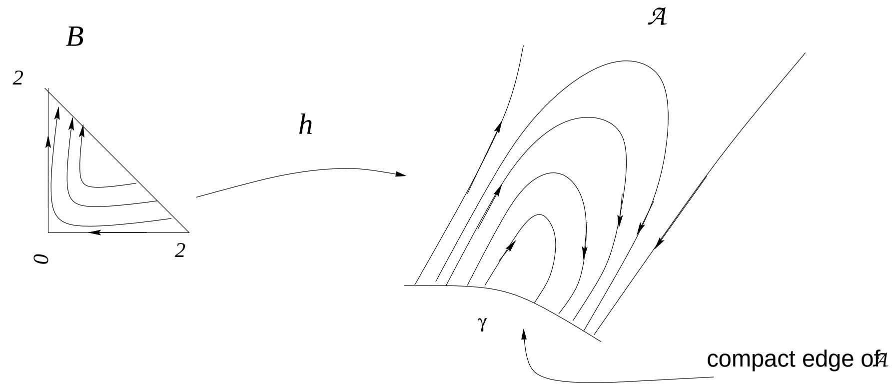

Figure 1: A half-Reeb component.

(2) Both segments $\{(x, y) \in B : x = 0\}$ and $\{(x, y) \in B : y = 0\}$ are sent by h onto full half-trajectories of X_f. These two semi-trajectories of X_f are called the *non-compact edges* of $\mathcal{A}$.

The connection between half-Reeb components and injectivity is given by the following result.

Proposition 1 *Suppose that* $X = (f, g)\colon \mathbb{R}^2 \to \mathbb{R}^2$ *is a* C^1 *map such that* $0 \notin \operatorname{Spec}(X)$. *If* X *is not injective, then both* X_f *and* X_g *have* hRc*'s.*

Proof Suppose by contradiction that X_f has no half-Reeb components. By assumption, the Hamiltonian vector field X_f, induced by f, has no singularities. Hence, by Kaplan's classification of planar foliations [11], we obtain that X_f is topologically equivalent to the horizontal foliation of $\mathbb{R}^2$. This and the fact that f is a submersion imply that each nonempty level curve of f must have exactly one connected component. As g restricted to each level curve of f is strictly monotone, we arrive at the contradiction that X is injective. This finishes the proof of the proposition. ∎

For each $\theta \in \mathbb{R}$ let R_θ denote the linear rotation

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

We will use in the sequel the following proposition.

Proposition 2 *Let* $X = (f, g)\colon \mathbb{R}^2 \to \mathbb{R}^2$ *be a non-injective* C^1*-map such that* $0 \notin \operatorname{Spec}(X)$. *Let* $\mathcal{A}$ *be a* hRc *of* X_f *and let* $(f_\theta, g_\theta) = R_\theta \circ X \circ R_{-\theta}$, $\theta \in \mathbb{R}$. *Then there is an* $\epsilon > 0$ *such that for all* $\theta \in (-\epsilon, 0) \cup (0, \epsilon)$, X_{f_θ} *has a* hRc *whose projection on the x-axis is an interval of infinite length.*

The proof of this proposition is contained in [10, Lemma 2.5].

3 Proof of Theorem A

Suppose by contradiction that $X = (f, g)$ is not injective. Hereafter we will use the fact that non-injectivity and the assumptions of Theorem A are open in the Whitney C^1 topology; in particular we shall assume, from now on, that X is smooth.

By Proposition 1, X_f has a half-Reeb component $\mathcal{A}$. Let $\Pi\colon \mathbb{R}^2 \to \mathbb{R}$ be the projection on the first coordinate. By composing with a rotation if necessary, in the way stated in Proposition 2, we may assume that $\Pi(\mathcal{A})$ is an interval of infinite length. To simplify matters, let us suppose that $[b, \infty) \subset \Pi(\mathcal{A})$.

By Thom's Transversality Theorem for jets [7], we can assume the following:

(a1) the set

$$T = \{(x, y) \in \mathbb{R}^2 : f_y(x, y) = 0\}$$

is made up of regular curves;

(a2) There is a discrete subset Δ of T such that if $p \in T \setminus \Delta$ (resp. $p \in \Delta$), X_f has quadratic contact (resp. cubic contact) with the vertical foliation of $\mathbb{R}^2$.

Then, if $a > b$ is large enough,

(b) for any $x \geq a$, the vertical line $\Pi^{-1}(x)$ intersects exactly one trajectory $\alpha_x \subset \mathcal{A}$ of $X_f|_{\mathcal{A}}$ such that $\Pi(\alpha_x) \cap (x, \infty) = \varnothing$; in other words, x is the maximum for the restriction $\Pi|_{\alpha_x}$.

It follows that

(c) if $x \geq a$ and $p \in \alpha_x \cap \Pi^{-1}(x)$ then $p \in T \cap \mathcal{A} \setminus \Delta$.

Let T_x be the set of $p \in \mathcal{A}$ such that $p \in \alpha_x \cap \Pi^{-1}(x)$, $x \geq a$. Notice that, for every $x \geq a$, $\alpha_x \cap \Pi^{-1}(x)$ is a finite set; nevertheless, by (b), (c) and by using Thom's Transversality Theorem for jets, we may get the following stronger statement:

(d) There is a sequence $F = \{a_1, a_2, \dots, a_i, \dots\}$ in $[a, \infty)$, which may be either empty or finite or else countable, such that if $x \in F$ (resp. $x \in [a, \infty) \setminus F$), then $\Pi^{-1}(x) \cap T_x$ is a two-point-set (resp. a one-point-set).

If $x \in [a, \infty) \setminus F$, define $\eta(x) = \big(x, \eta_2(x)\big) = \Pi^{-1}(x) \cap T_m$. Observe that $\eta\colon [a, \infty) \setminus F \to T_m$ is a smooth embedding. As $f|_{\mathcal{A}}$ is bounded,

(e) $F \circ \eta$ extends continuously to a strictly increasing bounded map defined in $[a, \infty)$ such that, for all $x \in [a, \infty) \setminus F$, $f_x\big(\eta(x)\big)$ has constant sign.

Therefore, there exists a real constant K such that

$$\begin{aligned} K &= \int_{a_1}^{\infty} \frac{d}{dx} f\big(\eta(x)\big)\, dx = \sum_{i=1}^{\infty} \int_{a_i}^{a_{i+1}} \frac{d}{dx} f\big(\eta(x)\big)\, dx \\ &= \sum_{i=1}^{\infty} \int_{a_i}^{a_{i+1}} f_x\big(\eta(x)\big) \end{aligned}$$

This and (e) imply that, for some sequence $x_n \to \infty$, $\lim_{n\to\infty} f_x\big(\eta(x_n)\big) = 0$. This is a contradiction with the assumption $\operatorname{Spec}(X) \cap (-\epsilon, \epsilon) = \varnothing$. In short we have proved Theorem A. ■

4 Proof of Theorem B

By the assumptions we have that

(a) $f_x(x, y) = p'(x)q(y)$ and $f_y(x, y) = p(x)q'(y)$.

As $0 \notin \operatorname{Spec}(X)$, we obtain that

(b) for all $y \in \mathbb{R}$, either $q(y) \neq 0$ or $q'(y) \neq 0$.

Now we assume, by contradiction, that X is not injective; then, by Proposition 1, there is a half-Reeb component $\mathcal{A}$ for X_f. As $0 \notin \operatorname{Spec}(X)$, f is a submersion and so the non-compact edges of $\mathcal{A}$ must accumulate at infinity. This implies that the projection of $\mathcal{A}$ on at least one of the coordinate axis has infinite length; let us consider only the case in which this happens for the x-axis. Let $T = \{w \in \mathbb{R}^2 : f_y(w) = 0\}$. Similarly to the proof of Theorem A, we obtain that the projection of $T \cap \mathcal{A}$ on the x-axis is an interval of infinite length, say $[a, \infty)$. Let $G = \{y_1, y_2, \ldots, y_m\}$ and $F = \{x_1, x_2, \ldots, x_n\}$ be all the real roots of the polynomials $q'(y)$ and $p(x)$ respectively. Under these conditions, using (a) and (b), we obtain,

(c) if $(x, y) \in T \cap \mathcal{A}$ and $x \notin F$, then $y \in G$ and $q(y) \neq 0$.

Since f restricted to $\mathcal{A}$ is bounded, there is a constant $M > 0$ such that, for all $(x, y) \in \mathcal{A}$, $|f(x, y) = p(x)q(y)| \leq M$. Therefore, by (c), if $(x, y) \in T \cap \mathcal{A}$, and $x \notin F$, we have that

$$|p(x)| \leq \frac{M}{\min_{y_i \in G} |q(y_i)|},$$

which implies that $p(x)$ is identically constant. Hence, for all $(x, y) \in \mathbb{R} \times G$, $f_x(x, y) = f_y(x, y) = 0$, and so $0 \in \operatorname{Spec}(X)$. This contradiction proves that X is injective.

5 Aligned and Adjacent Half-Reeb Components

Definition 2 Let $\gamma = (\gamma_1, \gamma_2) \colon [0, 3] \to \mathbb{R}^2$ be a compact edge of a half-Reeb component $\mathcal{A}$ for X_f and let $\gamma(t_0)$ be the unique point where the curve γ is tangent to the flow X_f. Consider the vector $\dot{\gamma}(t_0)^{\perp} = \big(-\dot{\gamma}_2(t_0)\dot{\gamma}_1(t_0)\big)$ and the straight line $r(s) = s\dot{\gamma}(t_0)^{\perp} + \gamma(t_0)$, $s \in \mathbb{R}$, passing through $\gamma(t_0)$ with direction $\dot{\gamma}(t_0)^{\perp}$. We will say that $\mathcal{A}$ is *on the left* (resp. *on the right*) of γ, if there is an interval $[0, \delta)$ (resp. $(-\delta, 0]$) such that $r\big([0, \delta)\big) \subset \mathcal{A}$ (resp. $r\big((-\delta, 0]\big) \subset \mathcal{A}\big)$).

In order to prove Theorems C and D, we need to introduce some definitions and state some results. First at all we introduce the notion of aligned half-Reeb components.

Definition 3 Let $f \colon \mathbb{R}^2 \to \mathbb{R}$ be a C^1 submersion and let $\mathcal{A}$ and $\mathcal{B}$ be disjoint half-Reeb components for X_f. We say that $\mathcal{A}$ and $\mathcal{B}$ are *aligned* and denote $\{\mathcal{A}, \mathcal{B}\}$, if there is a smooth embedded curve $\gamma \colon [1, 2] \to \mathbb{R}^2$ such that the following properties are satisfied:

(a1) For some $1 < s_0 \leq r_0 < 2$, $\gamma|_{[s_0, r_0]}$ is transversal to X_f;

(a2) $\gamma([1, s_0])$ and $\gamma([r_0, 2])$ are the compact edges for $\mathcal{A}$ and $\mathcal{B}$, respectively, and $\gamma\big((s_0, r_0)\big)$ is disjoint of $\mathcal{A}$ and $\mathcal{B}$;

(a3) $\mathcal{A}$ and $\mathcal{B}$ are both either on the left or on the right of the curve γ.

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The curve γ will be said to be *an aligning path for the pair* $\{\mathcal{A}, \mathcal{B}\}$ with connecting interval $[s_0, r_0]$. See Figure 2. Let α_p^+ (resp. α_p^-) denote the positive (resp. negative) half-trajectory of X_f starting at $p \in \mathbb{R}^2$. If $\alpha^+_{\gamma(s_0)}$ and $\alpha^+_{\gamma(r_0)}$ (resp. $\alpha^-_{\gamma(s_0)}$ and $\alpha^-_{\gamma(r_0)}$) are non-compact edges of $\mathcal{A}$ and $\mathcal{B}$, respectively, then γ, as right above, will be said to be a *positive* (resp. negative) aligning path for the pair $\mathcal{A}$ and $\mathcal{B}$.

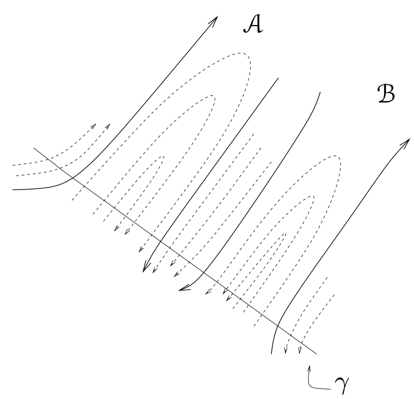

Figure 2: A pair of aligned half Reeb components

We shall need the following result of Gutierrez (see [10, Theorem D]):

Proposition 3 *Suppose that* $X\colon \mathbb{R}^2 \to \mathbb{R}^2$ *is a* C^1 *map such that* $0 \notin \operatorname{Spec}(X)$ *and, for all* $\theta \in \mathbb{R}$, X_{f_θ} *has no pair of aligned* hRc*'s, where* $(f_\theta, g_\theta) = R_\theta \circ X \circ R_{-\theta}$. *Then,* X *is injective.*

To introduce the notion of *adjacent* half-Reeb components we will consider the following compactification of the plane.

Let $\mathbb{S}^2 = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + (z-1)^2 = 1\}$, $\mathbb{S}^2_- = \{(x, y, z) \in \mathbb{S}^2 : 0 \le z < 1\}$, and $\mathbb{S}^1 = \{(x, y, 1) \in \mathbb{S}^2\}$. Let $\varphi\colon \mathbb{S}^2 \setminus \mathbb{S}^1 \to \mathbb{R}^2$ be the 2-to-1 map given by

$$\varphi(x, y, z) = \left(\frac{x}{1-z}, \frac{y}{1-z}\right). \tag{1}$$

We shall denote by $\mathrm{Cl}(\mathbb{R}^2)$ the compact disc made up of the union of $\mathbb{R}^2$ and $\mathbb{S}^1$ by identifying $(x, y, z) \in \mathbb{S}^2_-$ with $\varphi(x, y, z)$ and by borrowing from $\overline{\mathbb{S}^2_-} = \mathbb{S}^2_- \cup \mathbb{S}^1$ its topology.

If $f\colon \mathbb{R}^2 \to \mathbb{R}$ is a submersion then given a half trajectory α of X_f, it follows from the arguments of the Poincaré-Bendixson Theorem and from the fact that X_f has no singularities, that the limit set

$$\mathcal{L}(\alpha) := \bar{\alpha} \setminus \alpha \tag{2}$$

of α—as a subset of $\mathrm{Cl}(\mathbb{R}^2)$—, is either $\mathbb{S}^1$ or a nonempty closed sub-interval of it. Also if $\mathcal{A}$ is a half-Reeb component of X_f, the difference $\mathcal{L}(\mathcal{A}) := \bar{\mathcal{A}} \setminus \mathcal{A}$ (in $\mathrm{Cl}(\mathbb{R}^2)$) is a compact connected subset of $\mathrm{Cl}(\mathbb{R}^2)$ which may or may not be properly contained in $\mathbb{S}^1$.

Given $p \in \mathbb{R}^2$ we will denote by α_p the trajectory of X_f passing through p. Also, α_p^+ and α_p^- will denote the positive and negative half-trajectories of X_f, respectively, starting at p. If $\gamma\colon [a,b] \to \mathbb{R}^2$ is an embedded smooth curve transversal to X_f and and $a \le t_1 \le t_2 \le b$, we will denote

$$\mathcal{R}^+\big(\gamma([t_1,t_2])\big) = \bigcup_{t_1 \le s \le t_2} \alpha^+_{\gamma(s)}, \quad \mathcal{R}^-\big(\gamma([t_1,t_2])\big) = \bigcup_{t_1 \le s \le t_2} \alpha^-_{\gamma(s)}.$$

Definition 4 Let $\mathcal{A}$ and $\mathcal{B}$ be a pair of aligned hRc's for X_f and let $\gamma\colon [1,2] \to \mathbb{R}^2$ be an aligning path for the pair $\{\mathcal{A},\mathcal{B}\}$ with connecting interval $[s_0, r_0]$. We say that $\mathcal{A}$ and $\mathcal{B}$are *adjacent* half-Reeb components for X_f if the set

$$\mathcal{L}\Big(\mathcal{R}^+\big(\gamma([s_0,r_0])\big)\Big) = \overline{\bigcup_{s_0 \le s \le r_0} \alpha^+_{\gamma(s)}} \setminus \bigcup_{s_0 \le s \le r_0} \alpha^+_{\gamma(s)}$$

is contained in the unit circle $\mathbb{S}^1$ at infinity (resp. $\mathcal{L}\Big(\mathcal{R}^-\big(\gamma([s_0,r_0])\big)\Big) \subset \mathbb{S}^1$). Set $\Omega = \mathcal{A} \cup \mathcal{R}^+(\gamma[s_0,r_0]) \cup \mathcal{B}$ (resp. $\Omega = \mathcal{A} \cup \mathcal{R}^-(\gamma[s_0,r_0]) \cup \mathcal{B}$). Notice that $\mathcal{A}$ and $\mathcal{B}$ are adjacent if and only if the subset $\Omega \cup \mathcal{L}(\Omega)$ of $\mathrm{Cl}(\mathbb{R}^2)$ is homeomorphic to a bidimensional compact disc.

6 Polynomial Maps

Hereafter we will consider only polynomial maps of the plane. If $f\colon \mathbb{R}^2 \to \mathbb{R}$ is a polynomial submersion, then it may be seen that

(a) $\mathcal{L}\big(\mathcal{R}^+(\gamma[s_0,r_0])\big)$ is just one point of $\mathbb{S}^1$; in fact, otherwise, $\mathcal{L}\big(\mathcal{R}^+(\gamma[s_0,r_0])\big)$ would contain an open subinterval $I \subset \mathbb{S}^1$ which in turn would imply the contradiction that f is bounded along every ray approaching I (see Figure 3).

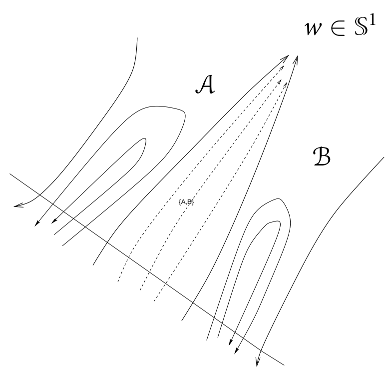

Figure 3: A pair of adjacent half Reeb components

(b) There exists a finite set $S_f = \{z_1, \dots, z_k\}$ of points in the unit circle $\mathbb{S}^1$ at infinity such that if α is a half-trajectory of X_f then $\mathcal{L}(\alpha) = z_i$ for some $i \in \{1, \dots, k\}$.

(c) If $\mathcal{A}$ is a hRc of X_f then $\mathcal{L}(\mathcal{A}) \cap \mathbb{S}^1 \subset S_f$. Each point $z_i \in S_f$ will be called a *limit point* for X_f.

Furthermore, if $\varphi\colon \mathbb{S}^2 \setminus \mathbb{S}^1 \to \mathbb{R}^2$ is the map defined by equation (1), by studying the Poincaré compactification $\tilde{X}_f$—via φ—of the polynomial vector field X_f, we may conclude, using Dumortier's work [4], that

(d) X_f has finitely many hRc's.

Within the proof of next result we will use the following notation. If $\gamma \subset \mathrm{Cl}(\mathbb{R}^2)$ is an arc and $p, q \in \gamma$ we denote by $[p, q]_\gamma$ (resp. $(p, q)_\gamma$) the closed (resp. open) sub-interval of γ with endpoints p and q.

Proposition 4 *Let $X = (f, g)\colon \mathbb{R}^2 \to \mathbb{R}^2$ be a non-injective polynomial map such that $0 \notin \mathrm{Spec}(X)$. Then there exists $\theta \in \mathbb{R}$ such that X_{f_θ} has a pair $\{\mathcal{A}, \mathcal{B}\}$ of adjacent half-Reeb components such that $e^{\pm\frac{\pi}{2}i} \notin \mathcal{L}(\mathcal{A}) \cup \mathcal{L}(\mathcal{B})$, where $(f_\theta, g_\theta) = R_\theta \circ X \circ R_{-\theta}$.*

Proof By Proposition 3 there exists $\mu \in \mathbb{R}$ such that X_{f_μ} has a pair of aligned hRc's. Proceed assuming that $f = f_\mu$ and consider the case in which the limit set of one of the hRc's of X_f contains $e^{\pm\frac{\pi}{2}i}$.

By [10, Lemma 2.5], we may find $\epsilon > 0$ small, such that if $\theta \in (-\epsilon, +\epsilon)$, then X_{f_θ} has a pair $\{\mathcal{A}_\theta, \mathcal{B}_\theta\}$ of aligned hRc's. We claim that

(a) if $\epsilon > 0$ is small enough, for all $\theta \in (-\epsilon, 0) \cup (0, \epsilon)$ the limit set of both $\mathcal{A}_\theta$ and $\mathcal{B}_\theta$ is disjoint of $e^{\pm\frac{\pi}{2}i}$.

In fact, let f_n and g_k be the highest degree homogeneous part of f and g respectively and let L_θ denote the straight line passing through the origin with slope $\tan(\pi/2 + \theta)$. The assumptions imply that $f_n(L_0) \equiv g_k(L_0) \equiv 0$. Also, it may be seen that

(b) if $\epsilon > 0$ is small and $\theta \in (-\epsilon, 0) \cup (0, \epsilon)$, then both, $f_n(L_\theta \setminus \{0\})$ and $g_k(L_\theta \setminus \{0\})$ are disjoint of $\{0\}$.

To fix ideas, suppose that $n \geq k$. In this way, if $\theta \neq 0$ is small and $(f_\theta)_n$ and $(g_\theta)_n$ denote the highest degree homogeneous part of f_θ and g_θ, respectively, then

$$(f_\theta)_n = (\cos\theta) f_n \circ R_{-\theta} - [k/n](\sin\theta) g_k \circ R_{-\theta}$$
$$(g_\theta)_n = (\sin\theta) f_n \circ R_{-\theta} + [k/n](\cos\theta) g_k \circ R_{-\theta}$$

where $[k/n]$ denotes the integer part of k/n. If we assumed that for some $\theta \in (-\epsilon, 0) \cup (0, \epsilon)$ $\mathcal{A}_\theta$ accumulated at infinity at a direction corresponding to L_0, then we would conclude that $(f_\theta)_n(L_0) \equiv 0$. As g_θ, restricted to $\mathcal{A}$, is bounded, $(g_\theta)_n(L_0) \equiv 0$. Therefore, we would obtain

$$f_n(L_{-\theta}) = f_n \circ R_{-\theta}(L_0)) = 0 = g_n \circ R_{-\theta}(L_0) = g_n(L_{-\theta}).$$

This contradiction with (b) proves (a).

Let $\theta \in (\mu - \epsilon, \mu + \epsilon)$ be such that X_{f_θ} has a pair $\mathcal{A}_1$ and $\mathcal{A}_2$ of aligned hRc's with $\mathrm{e}^{\pm\frac{\pi}{2}i} \notin \mathcal{L}(\mathcal{A}_1) \cup \mathcal{L}(\mathcal{A}_2)$. Let $\gamma\colon [1, 2] \to \mathbb{R}^2$ be a smooth aligning path for $\mathcal{A}_1$ and $\mathcal{A}_2$, with connecting interval $[s_0.r_0]$, which we shall assume to be positive.

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We will prove that if $\mathcal{A}_1$ and $\mathcal{A}_2$ are not already adjacent, then we can construct a new pair $\{\mathcal{B}_1, \mathcal{B}_2\}$ of aligned half-Reeb components for X_{f_θ} such that $\mathcal{B}_1$ is either equal to $\mathcal{A}_1$ or $\mathcal{A}_2$ and the interior of $\mathcal{B}_2$ is contained in $\mathcal{R}^+(\gamma[s_0, r_0])$, *i.e.*, $\mathcal{B}_2$ is different from $\mathcal{A}_1$ and $\mathcal{A}_2$. As f_θ is polynomial, there are only a finitely many hRc's for X_{f_θ} and so, proceeding inductively, we will, eventually, arrive to a pair of adjacent hRc's for X_{f_θ}.

Let $p \in \mathbb{R}^2 \cap \mathcal{L}\big(\mathcal{R}^+(\gamma[s_0, r_0])\big)$ and let $\eta\colon [-1, 1] \to \mathbb{R}^2$ be a regular curve satisfying

$$\eta(0) = p \quad \text{and} \quad \dot\eta(0) = \nabla f(p).$$

We may assume that $f|_\eta$ is strictly monotone and that $\mathcal{R}^+(\gamma[s_0, r_0])$ intersects η but $\alpha^+_{s_0}$ and $\alpha^+_{r_0}$ do not intersect η. Let $s_0 < t_1 < t_2 < t_0$ be numbers such that $\alpha^+_{\gamma(t_1)}$ and $\alpha^+_{\gamma(t_2)}$ intersect the curve η at points p_1 and p_2 respectively. As f restricted to $[p_1, p_2]_\eta$ and $[\gamma(t_1), \gamma(t_2)]_\gamma$ is strictly monotonous, there is a *flow box* between the trajectories $\alpha^+_{t_1}$ and $\alpha^+_{t_2}$ and the arcs $[p_1, p_2]_\eta$ and $[\gamma(t_1), \gamma(t_2)]_\gamma$. In particular, $[p_1, p_2]_\eta \subset \mathcal{R}^+(\gamma[s_0, r_0])$ and all the semi-trajectories $\alpha^+_{\gamma(s)}$ with $s \in [t_1, t_2]$ intersect (only) once the curve η inside the arc $[p_1, p_2]_\eta$. Therefore $p \notin [p_1, p_2]_\eta$. Consider only the case in which $p_1 \in [p_2, p]_\eta$ (resp. $p_2 \in [p_1, p]_\eta$), $p_1 = \eta(d_1)$ and $p_2 = \eta(d_2)$ with $-1 < d_2 < d_1 < 0$. See Figure 4.

It is easy to see that we can construct a smooth curve η_0 linking the points $\gamma(t_1)$ and p_1 (resp. $\gamma(t_2)$ and p_2) and having only one quadratic tangency with the foliation of X_{f_θ}. Moreover, η_0 can be chosen in such a way that the curve $\zeta = [\gamma(0), \gamma(t_1)]_\gamma \cup \eta_0 \cup [p_1, p]_\eta$ (resp. $\zeta = [\gamma(t_2), \gamma(3)]_\gamma \cup \eta_0 \cup [p_2, p]_\eta$) is smooth. Observe that ζ has exactly two (quadratic) tangency points with the foliation of X_{f_θ}, one inside the arc $[\gamma(0), \gamma(t_1)]_\gamma$ (resp. $[\gamma(t_2), \gamma(3)]_\gamma$) corresponding to $\mathcal{A}_1$ (resp. $\mathcal{A}_2$) and the other inside the curve η_0. Let t_0 be the supremum of the number $s_0 < t < t_1$ such that $\alpha^+_{\gamma(t)}$ intersect the curve η. By the assumptions $s_0 < t_0 < t_1$ and $\alpha^+_{\gamma(t_0)} \cap \eta = \varnothing$.

Let

$$d_0 = \sup\{d_1 < d < 0 : \eta(d) \in \mathcal{R}^+(\gamma[s_0, r_0])\}.$$

Clearly the trajectory $\alpha_{\eta(d_0)}$ does not intersect the arc $[\gamma(s_0), \gamma(r_0)]_\gamma$.

It is easy to see there is a new hRc $\mathcal{B}_2$ of X_{f_θ} whose compact edge is the arc $[\gamma(t_0), \eta(d_0)]_\zeta$ and whose non-compact edges are $\alpha^+_{\gamma(t_0)}$ and $\alpha^-_{\eta(d_0)}$. Clearly the pair $\{\mathcal{A}_1, \mathcal{B}_2\}$ (resp. $\{\mathcal{A}_2, \mathcal{B}_2\}$) is aligned in the sense defined before. This concludes the proof of the proposition. ■

We shall need the following well known result essentially due to Hadamard (see for more details [5]).

Proposition 5 *Let $X = (f, g)\colon \mathbb{R}^2 \to \mathbb{R}^2$ be a polynomial map such that* $\mathrm{Spec}(X) \cap \{0\} = \varnothing$. *Then X is a diffeomorphism if and only if $f^2 + g^2\colon \mathbb{R}^2 \to \mathbb{R}$ is a proper map.*

We will see in Section 8, that Theorem D give us a better criterium, than that of Proposition 5 to find out whether a polynomial map such that $\mathrm{Spec}(X) \cap \{0\} = \varnothing$ is injective.

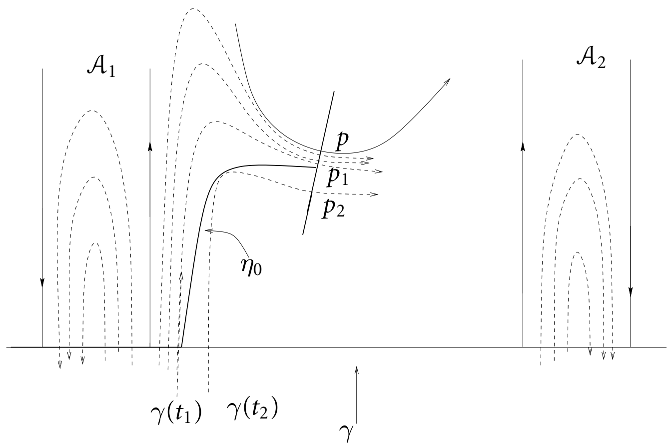

Figure 4: The curve η_0.

7 Proof of Theorems C and D

Let us set up the preliminaries for the proof of Theorems C and D. First of all, if X is a non-injective polynomial map then, by Proposition 4, there exists $\theta \in \mathbb{R}$ such that if $X_\theta = (f_\theta, g_\theta) = R_\theta \circ X \circ R_{-\theta}$, the flow X_{f_θ} has a pair $\{\mathcal{A}_1, \mathcal{A}_2\}$ of adjacent hRc's such that $e^{\pm\frac{\pi}{2}i} \notin \mathcal{L}(\mathcal{A}_1) \cup \mathcal{L}(\mathcal{A}_2)$. Notice that $\operatorname{Spec}(X_\theta) = \operatorname{Spec}(X)$. As f_θ is polynomial and $\mathcal{A}_1$ and $\mathcal{A}_2$ are adjacent, $\mathcal{L}(\mathcal{A}_1) \cap \mathcal{L}(\mathcal{A}_2) = \{w\} \subset \mathbb{S}^1$, with $w \neq e^{\pm\frac{\pi}{2}i}$. Therefore, Proposition 2 implies $\Pi(\mathcal{A}_1) \cap \Pi(\mathcal{A}_2)$ contains an interval of infinite length, where $\Pi\colon \mathbb{R}^2 \to \mathbb{R}$ is the projection on the first coordinate. To fix ideas, let us suppose that $\Pi(\mathcal{A}_1) \cap \Pi(\mathcal{A}_2)$ contains the interval $[b, \infty)$, see figure 5.

As f_θ is a polynomial map,

(a) the algebraic curve

$$T_\theta = \{p \in \mathbb{R}^2 : (f_\theta)_y(p) = 0\}$$

is made up of finitely many regular curves and finitely many singular points.

Similarly to the proof of Theorem A, if $a > b$ is large enough, we have that

(b) For any $x \geq a$, the vertical line $\Pi^{-1}(x)$ intersects exactly one trajectory $\alpha_x^i \subset \mathcal{A}_i$ of $X_{f_\theta}|_{\mathcal{A}_i}$, $i = 1, 2$, such that $\Pi(\alpha_x^i) \cap (x, \infty) = \varnothing$, $i = 1, 2$; in other words, x is the maximum for the restriction $\Pi|_{\alpha_x^i}$, $i = 1, 2$.

It follows that

(c) If $x \geq a$ and $p_i \in \alpha_x^i \cap \Pi^{-1}(x)$, for $i = 1, 2$, then there is a "parabolic" tangency between X_{f_θ} and $\Pi^{-1}(x)$ at p_i. In particular, $p_i \in T$, *i.e.*, $\frac{\partial f_\theta}{\partial y}(p_i) = 0$.

Let T_x^i, $i = 1, 2$ be the set of $p \in \mathcal{A}_i$ such that $p \in \alpha_x^i \cap \Pi^{-1}(x)$, $x \geq a$. Notice that, for every $x \geq a$, T_x^i is a finite set. By using (a) we may define analytic functions $\eta_i\colon [a, \infty) \to \mathcal{A}_i$, $i = 1, 2$ in such a way that $\eta_i(x) \in T_x^i$ and there are no other points of T_x^i inside the arc of $\Pi^{-1}(x)$ connecting $\eta_1(x)$ and $\eta_2(x)$. In this way, as the flow X_f is continuous (see Figure 5)

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(d) The collinear vectors $\nabla f_\theta\big(\eta_1(x)\big)$ and $\nabla f_\theta\big(\eta_2(x)\big)$ have opposed orientations; *i.e.*, $\frac{\partial f_\theta}{\partial x}\big(\eta_1(x)\big)\cdot\frac{\partial f_\theta}{\partial x}\big(\eta_2(x)\big)<0$.

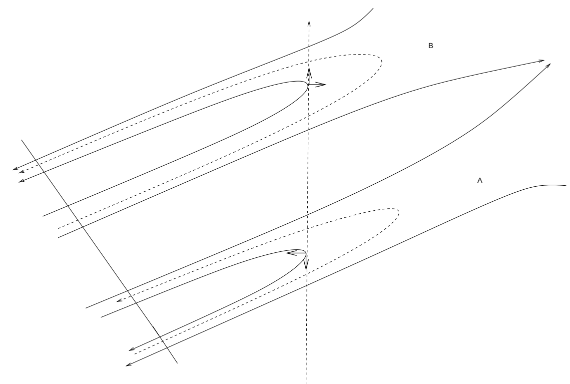

Figure 5: $[b,\infty)\subset\Pi(\mathcal{A}_1)\cap\Pi(\mathcal{A}_2)$.

7.1 Proof of Theorem C

If X is a non-injective polynomial map, proceeding as in the proof of Theorem A, we obtain that

$$\infty>\int_{[a,\infty)}\frac{d}{ds}f_\theta\big(\eta_i(s)\big)\,ds=\int_{[a,\infty)}(f_\theta)_x\big(\eta_i(s)\big)\,ds,\quad i=1,2,$$

which implies that $\lim_{n\to\infty}(f_\theta)_x\big(\eta_1(x_n)\big)=0$ and $\lim_{n\to\infty}(f_\theta)_x\big(\eta_2(y_n)\big)=0$, for some sequences $x_n,y_n\to\infty$. This fact together with (d) imply that there are positive and negative values of $\mathrm{Spec}(X_\theta)$ $(=\mathrm{Spec}(X))$ arbitrarily close to zero. Therefore if—for some $\epsilon>0$—either $(-\epsilon,0]\cap\mathrm{Spec}(X)=\varnothing$ or $\mathrm{Spec}(X)\cap[0,\epsilon)=\varnothing$, X is injective.

7.2 Proof of Theorem D

If X is a non-injective polynomial map such that $\mathrm{Spec}(X)\cap\{0\}=\varnothing$ then the map $X_\theta=R_\theta\circ X\circ R_{-\theta}$ is also a non-injective polynomial map such that $\mathrm{Spec}(X)\cap\{0\}=\varnothing$ and the condition (d) implies that the signs of $\mathrm{Trace}\Big(\mathrm{D}X_\theta\big(\eta_1(x)\big)\Big)$ and $\mathrm{Trace}\Big(\mathrm{D}X_\theta\big(\eta_2(x)\big)\Big)$ are opposed. Henceforth, for each $x\geq a$ there is a point $\big(x,q(x)\big)$ in the arc joining $\eta_1(x)$ and $\eta_2(x)$ which belongs to the set

$$\Gamma_\theta=\{p\in\mathbb{R}^2:\mathrm{Trace}(\mathrm{D}X_\theta)(p)=0\}.$$

Let $\gamma\colon [1,2] \to \mathbb{R}^2$ be an aligning path for $\mathcal{A}_1$ and $\mathcal{A}_2$ as in Definition 3. To fix ideas let us suppose that X_{f_θ} is oriented in such a way that $\alpha^+_{\gamma(s_0)}$ and $\alpha^+_{\gamma(r_0)}$ belong to $\mathcal{A}_1$ and $\mathcal{A}_2$ respectively. In this way, the curve $\Sigma_\theta = \big\{ \big(x, q(x)\big) : x \geq a\big\} \subset \Gamma_\theta$ accumulates at infinity inside the region $\Omega = \mathcal{A}_1 \cup \mathcal{R}^+(\gamma[s_0, r_0]) \cup \mathcal{A}_2$. We observe that all of $f_\theta|_\Omega, g_\theta|_\Omega$ and $(f_\theta^2 + g_\theta^2)|_\Omega$ are bounded maps. Therefore, $X_\theta(\Sigma_\theta)$ is a bounded subset of $\mathbb{R}^2$.

On the other hand, the relation $\mathrm{DX}_\theta(q) = R_\theta \cdot \mathrm{DX}\big(R_{-\theta}(q)\big) \cdot R_{-\theta}$, implies that

$$\Gamma_\theta = R_\theta(\Gamma) \quad \text{where} \quad \Gamma = \{p \in \mathbb{R}^2 : \mathrm{Trace}(\mathrm{DX})(p) = 0\}. \tag{3}$$

Let us prove item (a) of Theorem D. Suppose that X is a non-injective polynomial map such that $\mathrm{Spec}(X) \cap \{0\} = \varnothing$ and let $\Sigma = R_{-\theta}(\Sigma_\theta) \subset \Gamma$. Since Σ_θ is non-compact, Σ is non-compact. On the other hand, $X(\Sigma) = R_{-\theta}\big(X_\theta(\Sigma_\theta)\big)$ is a bounded subset of $\mathbb{R}^2$ which implies that all $f|_\Gamma$, $g|_\Gamma$ and $(f^2 + g^2)|_\Gamma$ are not proper maps (because $\Sigma \subset \Gamma$). Therefore if one of $f|_\Gamma$ or $g|_\Gamma$ is a proper map, X is injective.

Let us prove item (b) of Theorem D. By Proposition 5 if X is an injective polynomial map such that $\mathrm{Spec}(X) \cap \{0\} = \varnothing$, then $(f^2 + g^2)|_\Gamma$ is a proper map. On the other hand, if X is a non-injective polynomial map such that $\mathrm{Spec}(X) \cap \{0\} = \varnothing$, proceeding as in item (a) we conclude that $(f^2 + g^2)|_\Gamma$ is a not proper map.

8 Examples and Applications. Keller Maps

Example 1 Let $X = (f_x, f_y)\colon \mathbb{R}^2 \to \mathbb{R}^2$ be the gradient of a polynomial map $f\colon \mathbb{R}^2 \to \mathbb{R}$. If X is an orientation preserving and locally diffeomorphic map, then Theorem C implies that X is injective.

Indeed, as the matrix $\mathrm{DX}(p)$ is non-singular and symmetric for all $p \in \mathbb{R}^2$, its eigenvalues are real and non-null. As X preserves orientation, $\mathrm{Spec}(X) \subset (-\infty, 0)$ or $\mathrm{Spec}(X) \subset (0, +\infty)$. Therefore, by Theorem C, X is injective.

Example 2 Let $n > 1$ be a natural number and let

$$f = x - 2y + y^n,$$
$$g = x - y + y^n.$$

Then $X = (f, g)\colon \mathbb{R}^2 \to \mathbb{R}^2$ is an injective Keller map. Moreover, if n is even, $\mathrm{Spec}(X) = \mathbb{S}^1 \cup (\mathbb{R} \setminus \{0\})$. That is, $\mathrm{Spec}(X)$ is the biggest possible for Keller maps

Notice that $\mathrm{Trace}(\mathrm{DX})(x, y) = ny^{n-1}$. This means that

$$\Gamma = \{(x, y) : \mathrm{Trace}(\mathrm{DX})(x, y) = 0\} = \{(x, y) : y = 0\}.$$

We check that both $f|_\Gamma$ and $g|_\Gamma$ are proper maps, then by Theorem D, X is injective. The characteristic polynomial of $\mathrm{DX}(x, y)$ is $\lambda^2 - ny^{n-1}\lambda + 1$. Therefore $\mathrm{Spec}(X)$ is given by all the numbers $1/2 \cdot (ny^{n-1} \pm \sqrt{n^2 y^{2n-2} - 4})$, $y \in \mathbb{R}$. It is easy to see that $\mathbb{R} \setminus \{0\} \subset \mathrm{Spec}(X)$ if n is even and then $\mathrm{Spec}(X) = \mathbb{S}^1 \cup (\mathbb{R} \setminus \{0\})$ (because $\mathrm{Spec}(X)$

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is a connected set of $\mathbb{R}^2$). By taking an odd natural $n \geq 3$, in this example, we get that $\operatorname{Spec}(X)$ is made up of the union of the sets $\mathbb{S}^1 \cap \{(x, y) : x > 0\}$ and $(0, \infty)$.

The following example (see [3]) shows the existence of analytic non-injective Keller maps.

Example 3 Let

$$f = \sqrt{2}e^{x/2} \cos\left(\frac{y}{e^x}\right)$$
$$g = \sqrt{2}e^{x/2} \sin\left(\frac{y}{e^x}\right). \tag{4}$$

Then $X = (f, g) \colon \mathbb{R}^2 \to \mathbb{R}^2$ is a non-injective analytic Keller map.

It is easy to check that X is a Keller map. It is also non-injective, for instance, observe that for all $k \in \mathbb{Z}$ and all $y \in \mathbb{R}$, $X(0, y + 2k\pi) = \sqrt{2}\big(\cos(y), \sin(y)\big)$.

Let us proceed to describe the adjacent half-Reeb components of X_f and X_g. In fact, both X_f and X_g have infinitely many adjacent half-Reeb components (see Definition 4). In fact, observe that f vanishes along all curves of the form

$$C_k(t) := \{(x, y) : x = t, y = (\pi/2 + \pi k) \cdot \mathrm{e}^t\}, \quad t \in \mathbb{R},\ k \in \mathbb{Z}.$$

Also $f(0, y) = \sqrt{2}\cos(y)$ and then $\frac{\partial}{\partial y} f(0, y)$ vanishes only once in the segment

$$S_k := \{(x, y) : x = 0, \pi/2 + \pi k \leq y \leq \pi/2 + \pi(k + 1)\}$$

that connects $C_k(0)$ and $C_{k+1}(0)$. We observe that f is bounded in the semi-plane $\{(x, y) : x \leq 0\}$ and unbounded in $\{(x, y) : x > 0\}$. In this way, for all $k \in \mathbb{Z}$, X_f has a half-Reeb component $\mathcal{A}_k$ bounded its non-compact edges $\{C_k(t), t \leq 0\}$ and $\{C_{k+1}(t), t \leq 0\}$ and its compact edge S_k. All consecutive pairs $\{\mathcal{A}_k, \mathcal{A}_{k+1}\}$ are adjacent. Similarly, X_g has a Half-Reeb component $\mathcal{B}_k$ between consecutive curves of the form

$$D_k(t) := \{(x, y) : x = t, y = (\pi + \pi k) \cdot \mathrm{e}^t\}, \quad t \in \mathbb{R},\ k \in \mathbb{Z}$$

and the segment

$$T_k := \{(x, y) : x = 0, \pi + \pi k \leq y \leq \pi/2 + \pi(k + 1)\},$$

and all consecutive pairs $\{\mathcal{B}_k, \mathcal{B}_{k+1}\}$ are adjacent. Observe that the curve $D_k(t), t \leq 0$ is the only semi-trajectory of X_g which is completely contained in the half-Reeb component $\mathcal{A}_k$ of X_f.

Example 4 If $X = (f, g) \colon \mathbb{R}^2 \to \mathbb{R}^2$ is the Pinchuck non-injective polynomial map (see [15]), then, by Theorem C, $\operatorname{Spec}(X)$ meets the real line at arbitrarily small positive and negative numbers.

Let us see this by directly describing the foliation induced by X_f, where

$$f = (xy - 1)\big(x(xy - 1) + 1\big) + \big(x(xy - 1) + 1\big)^2\big((xy - 1)^2 + y\big)$$

After an extensive use of symbolic computation, one may see that:

(4,1) for every $c < 0$, $\{f = c\}$ is connected;

(4,2) for every $c \geq 0$, $\{f = c\}$ has 3 connected components;

(4,3) the set $\{f \leq 0\}$ is usually known as a Reeb component of X_f and the set $\{f < 0\}$ is homeomorphic to a disc;

(4,4) for every $c \in \mathbb{R}$, a connected component of $\{f = c\}$ can approach to infinity, only at one of the following directions: the positive x-axis, the negative x-axis and the negative $y-$axis.

(4,5) $\{f = 0\}$ consists of 3 connected components A_1, A_2, A_3 such that

- A_1 is contained in $\{(x, y) : xy > 0\} \cup \{(0, 0\}$ and approaches infinity in the directions of the positive $x-$axis and the negative $x-$axis.
- A_2 is contained in $\{(x, y) : x < 0, y < 0\}$ and approaches infinity in the directions of the negative $x-$axis and the negative $y-$axis.
- A_3 is contained in $\{(x, y) : y > 0\}$ meeting the $x-$axis exactly at $(1, 0)$; it approaches infinity in the directions of the positive $x-$axis and the negative $y-$axis.

(4,5) the Reeb component $\{f \leq 0\}$ contains 2 half-Reeb components, one of which approaches the positive $x-$axis and the other the negative $x-$axis. By observing, as in the proof of Theorem C, the gradient vector field (f_x, f_y) along this half-Reeb components, it can be seen that there are sequences $\{a_n\}$ and $\{b_n\}$ such that, for all n, $f_x(a_n) > 0$, $f_y(a_n) = 0 = f_y(b_n)$, $f_x(b_n) < 0$ and moreover $f_x(a_n) \to 0$ and $f_x(b_n) \to 0$. In other words, $\operatorname{Spec}(X)$ meets the real line at arbitrarily small positive and negative numbers.

Example 5 Let $X = (f, g)\colon \mathbb{R}^2 \to \mathbb{R}^2$ be an orientation preserving and locally diffeomorphic polynomial map. If X is injective then it follows, from the main result of either of the papers [1] and [12], that X is a diffeomorphism. Therefore, X_f and X_g have no Reeb components.

In fact, if X_f has a hRc, say $\mathcal{A}$, then, we may see that $X(\mathcal{A})$ is bounded. This is not possible because $\mathcal{A}$ is unbounded and X is a diffeomorphism.

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Departamento de Matemáticas
IBILCE-UNESP
São José do Rio Preto (SP)
Brazil
e-mail: milton@mat.ibilce.unesp.br

ICMC-USP
São Carlos & IMPA
Rio de Janeiro
Brazil
e-mail: gutp@icmc.sc.usp.br

Departament de Matemàtiques
Universitat Autònoma de Barcelona
08193 Bellaterra
Barcelona
Spain
e-mail: jllibre@mat.uab.es