BULL. AUSTRAL. MATH. SOC. 34C25, 46A50
VOL. 40 (1989) [357–361]

PERIODIC SOLUTIONS OF A SECOND ORDER NONLINEAR DIFFERENTIAL EQUATION

BAHMAN MEHRI

We consider the following non-linear nonautonomous second order differential equation

$$x''(K + h(x))x' + f(t,x) = p(t)$$

where $h(x)$ is continuous, f, p are continuous and periodic with respect to t of period w. Using the Leray-Schauder fixed point technique we prove that the above equation possesses at least one non-trivial periodic solution of period w.

It is obvious that the linear differential equation

$$x''kx' = p(t), \qquad p(t+w) \equiv p(t), \quad \int_0^w p(t)dt = 0 \tag{1}$$

possesses a w-periodic solution. It is interesting to note that the following non-linear differential equation

$$x'' + (K + h(x))x' + f(t,x) = p(t) \tag{2}$$

where h is a continuous function, f, p are continuous and periodic with respect to t of period w, also possesses a w-periodic solution. The existence of periodic solutions is proved on the basis of the Leray-Schauder fixed point technique. The conditions imposed upon the non-linear terms are not very restrictive. Therefore equation (2) with those conditions has many applications.

THEOREM 1. *Differential equation (2) admits at least one w-periodic solution if*

(i) $\int_0^w p(t)dt = 0$ *[that is, $P(t) = \int_0^t p(s)ds$ is w-periodic],*
(ii) $|H(y)| \leqslant M$ *[$H(y) = \int_0^y h(s)ds$],*
(iii) $(|f(t,x)|)/(|x|) \to 0$ *as* $|x| \to \infty$, *uniformly in* t,
(iv) $f(t,x)\operatorname{sgn} x \geqslant 0$ $(|x| \geqslant b)$.

Received 9 November, 1988
This work was completed while the author was on sabbatical leave from the Sharif University of Technology, Tehran, Iran and visiting the University of California at Davis, U.S.A.

PROOF: The proof by means of the Leray-Schauder method is simple. We consider a differential equation containing the parameter μ, $0 \leqslant \mu \leqslant 1$,

$$x'' + kx' + Cx = \mu\{p(t) - f(t,x) + Cx - x'h(x)\} \tag{3}$$

where C is an arbitrary positive constant. For $\mu = 0$ we obtain a homogeneous linear equation the only w-periodic solution of which is the trivial one; for $\mu = 1$ equation (2) is identical with the original one (1). It is a well-known fact (see [**1**, **2**, **3**]) that equation (3) adimits at least one periodic solution for each parameter value $\mu \in [0,1]$, if for $0 < \mu < 1$ all periodic solutions as well as their first derivatives are uniformly bounded. Consequently the stated theorem can be proved with the aid of an a priori estimate.

Let $x(t) \equiv x(t+w)$ be a solution of equation (3) and let $0 < \mu < 1$. We write

$$R = \max_{0 \leqslant t \leqslant w} |x(t)|, \qquad F = F(R) = \max_{|x| \leqslant R,\ 0 \leqslant t \leqslant w} |f(t,x)|.$$

The derivative $y = x'$ satisfies the equation

$$y' + ky = \mu\{e(t) - f(t,x(t)) - h(x(t))x'(t)\} - (1-\mu)Cx(t).$$

Introducing the Green's function

$$G(t;s) = \begin{cases} \dfrac{e^{k(s-t-w)}}{-1+e^{-kw}}; & 0 \leqslant t \leqslant s \leqslant w \\ \dfrac{e^{k(s-t)}}{-1+e^{-kw}}; & 0 \leqslant s \leqslant t \leqslant w \end{cases}$$

$[G(t+0,t) - G(t-0,t) = 1]$ of the boundary value problem

$$y' + ky = q(t)$$
$$y(0) = y(w)$$

where

$$q(t) = \mu\{e(t) - f(t,x(t)) - h(x(t)) \cdot x'(t)\} - (1-\mu) \cdot e \cdot x(t)$$

is periodic and $q(t+w) = q(t)$, we obtain the following representation of the solution of $y(t)$

$$y(t) = \int_0^w G(t;s)q(s)ds. \tag{4}$$

https://doi.org/10.1017/S0004972700017378 Published online by Cambridge University Press

Replacing $q(t)$ by the term $h(x(t)) \cdot x'(t)$ which occurs in the expression for $q(t)$ we obtain

$$\begin{aligned} y(t) &= \int_0^w G(t;s)h(x(s)) \cdot x'(s)ds \\ &= H(x(t))G(t;s)\,]_0^{t-0} + H(x(t))G(t;s)\,]_{t+0}^{w} - \int_0^w G_t(t,s)H(x(s))ds \\ &= H(x(t)) - \int_0^w G_t(t;s)H(x(s))ds. \end{aligned}$$

Inserting the explicit expression for $q(t)$ in equation (4) we derive estimates of the type

$$|y(t)| \leqslant \rho(m + F(R) + 2M + CR)$$

where $\rho = \max\{1, 1/k\}$, $m = \max_{0\leqslant t\leqslant w} |p(t)|$.

Now a term by term integration of differential equation (2) (for the periodic solution) yields

$$[x'(t) + Kx(t) + \mu H(x(t)) - P(t)]_0^w + \int_0^w \{(1-\mu)Cx(t) = \mu f(t, x(t))\}dt = 0,$$

or

$$\int_0^w \{(1-\mu)C \cdot x(t) + \mu f(t, x(t))\}dt = 0.$$

Since $1 - \mu > 0$ and we have

$$\{(1-\mu)\cdot c \cdot x(t) + \mu f(t, x(t))\}\operatorname{sgn} x = (1-\mu)C|x| + \mu f(t,x)\operatorname{sgn} x > 0$$

for $|x| \geqslant b$, $t \in [0, w]$, it follows that $|x(t)| \geqslant b$ for $0 \leqslant t \leqslant w$ is excluded. Therefore there exists τ, $0 < \tau < w$, such that $|x(\tau)| < b$. Applying the mean-value thorem to an arbitrary interval $[\tau, t] \subseteq [\tau, \tau + w]$, we have

$$\begin{aligned} |x(t) - x(\tau)| &= |t - \tau|\,|y(\tau + \theta(t-\tau))| \\ &\leqslant w \cdot \rho \cdot (m + F(R) + 2M + CR), \end{aligned}$$

or

$$|x(t)| < b + w \cdot \rho \cdot (m + F(R) + 2M + CR).$$

Hence

$$\max_{0\leqslant t\leqslant w} |x(t) = R < b + w \cdot \rho \cdot (m + F(R) + 2M + CR).$$

https://doi.org/10.1017/S0004972700017378 Published online by Cambridge University Press

Choosing $0 < C < 1/(w \cdot \rho)$, we obtain

$$(5) \qquad 1 < \frac{b + w \cdot \rho \cdot (m + m)}{1 - w \cdot \rho \cdot C} \frac{1}{R} + \frac{w \cdot \rho}{1 - w \cdot \rho \cdot C} \frac{F(R)}{R}.$$

An immediate consequence of assumption (iii) is

$$\frac{F(R)}{R} \to 0 \text{ as } R \to \infty.$$

Therefore we conclude from (5)

$$R = \max_{0 \leqslant t \leqslant w} |x(t)| \leqslant R_0 \text{ (independently of } \mu),$$
$$F(R) = \max_{|x| \leqslant R,\ t \in [0,w]} |f(t,x) \leqq F_0 = \max_{|x| \leqslant R_0,\ t \in [0,w]} |f(t,w)|.$$

The resulting a priori estimates

$$|x(t)| \leqslant R_0, \qquad |x'(t)| \leqslant \rho \cdot (m + F_0 + 2M + CR_0)$$

ensure the existence of a periodic solution of equation (2) as we stated in our theorem. □

Remark. In the case

$$(\text{iv}') \qquad f(t,x) \operatorname{sgn} x \leqslant 0, \qquad (|x| \geqslant b)$$

we introduce a new independent variable

$$\tau = -t$$

and obtain a differential equation of the previous type. Thus Theorem 1 remains valid if assumption (1) is replaced by (iv)′.

As an application of our theorem consider the following differential equation

$$x'' + (1 + \sin x)x' + x^{1/3} \sin^2 t = \sin t$$

which occurs in electric circuit theory. Obviously

$$p(t) = \sin t, \qquad \int_0^{2\pi} \sin t \, dt = 0, \quad h(x) = \sin x, \qquad H(x) = -\cos x + 1$$
$$|H(x)| \leqslant 2, \qquad f(t,x) = x^{1/3} \sin^2 t, \qquad x f(t,x) \geqslant 0 \quad (b = 0),$$

and

$$\lim_{|x| \to \infty} \frac{|f(t,x)|}{|x|} = 0,$$

that is, all assumptions of our theorem are satisfied. Hence there exists at least one 2π-periodic solution of equation (6).

https://doi.org/10.1017/S0004972700017378 Published online by Cambridge University Press

References

[1] L. Cesari, 'Functional analysis and periodic solution of nonlinear differential equation', *Contributions to Differential Equations* **1** (1963), 149–187.

[2] R. Reissig, 'On the existence of periodic solutions of certain non-autonomous differential equation', *Ann. Mat. Pura Appl.* **85** (1970), 235–240.

[3] G. Villari, 'Contirbuti allo studio dell, esistenza di Soluzione periodiche per i Sistemi di equazioni differenziali ardinasie', *Ann. Mat. Pura Appl.* **69** (1965), 171–190.

Department of Mathematics,
University of California Davis,
Davis CA95616,
United States of America.