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# SPACE-LIKE SUBMANIFOLDS WITH PARALLEL MEAN CURVATURE IN DE SITTER SPACES

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**Abstract**

This paper investigates complete space-like submanifold with parallel mean curvature vector in the de Sitter space. Some pinching theorems on square of the norm of the second fundamental form are given.

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## 1. Introduction

A de Sitter space $S_p^{n+p}(1)$ is an $(n+p)$-dimensional connected complete pseudo-Riemannian manifold of index $p$ with constant curvature 1. Goddard [3] conjectured that complete space-like hypersurface in $S_1^{n+1}(1)$ with constant mean curvature $H$ must be totally umbilical. In 1987, Akutagawa [1] and Ramanathan [6] proved independently the conjecture is true if $H^2 \leq 1$ when $n = 2$ and $n^2H^2 < 4(n-1)$ when $n \geq 3$. This statement has been generalized by Cheng [2] to complete space-like submanifolds in $S_p^{n+p}(1)$ with parallel mean curvature vector. In [5], we proved that complete space-like hypersurface $M$ in $S_1^{n+1}(1)$ with constant mean curvature is totally umbilical if $S \leq 2\sqrt{n-1}$, where $S$ is the square of the second fundamental form. Moreover, $S = 2\sqrt{n-1}$ only if $n = 2$ and $M$ is flat.

In the present paper we shall prove the following

THEOREM 1. *Let $M$ be a complete space-like $n$-dimensional submanifold in the de Sitter space $S_p^{n+p}(1)$ with parallel mean curvature vector $\eta$. Denote by $S$ the square of norm of second fundamental form.*

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(i) *If* $S \leq \left((2n\sqrt{n-1}/(n+2\sqrt{n-1})\right)(1+\|\eta\|^2)$, *then* $M$ *is totally umbilical and lies in a totally geodesic submanifold* $S_1^{n+1}(1)$ *of* $S_p^{n+p}(1)$. *Moreover,* $M$ *is isometric to a sphere* $S^n\left(\sqrt{n/(n-S)}\right)$ *of radius* $\sqrt{n/(n-S)}$ *or a plane* $\mathbb{R}^2$ *in case* $S = n = 2$.

(ii) *If* $S \leq \left((2n\sqrt{n-1}/(n-2)\right)(1-\|\eta\|^2)$ $(n > 2)$, *then* $M$ *lies in a totally geodesic submanifold* $S_1^{n+1}(1)$ *of* $S_p^{n+p}(1)$.

## 2. Preliminaries

Let $M$ be an $n$-dimensional space-like submanifold of $S_p^{n+p}(1)$. Locally we choose a pseudo-Riemannian orthonormal frame $\{e_1, \dots, e_{n+p}\}$ in $S_p^{n+p}(1)$ such that, restricted to $M$, $e_1, \dots, e_n$ is tangent to $M$. Throughout this paper the following convention on the ranges of indices is used unless otherwise stated

$$1 \leq A, B, C, D, \dots \leq n+p, \quad 1 \leq i, j, k, l, \dots \leq n, \quad n+1 \leq \alpha, \beta, \dots \leq n+p.$$

Let $\{\omega_1, \dots, \omega_{n+p}\}$ be the dual coframe of $\{e_A\}$. The pseudo-Riemannian metric on $S_p^{n+p}(1)$ is

$$ds^2 = \sum_A \varepsilon_A\, \omega_A^2 \tag{2.1}$$

where $\varepsilon_1 = \dots = \varepsilon_n = 1$, $\varepsilon_{n+1} = \dots = \varepsilon_{n+p} = -1$. The structure equations are

$$d\omega_A = -\sum_B \omega_{AB} \wedge \omega_B, \qquad \varepsilon_A\, \omega_{AB} + \varepsilon_B\, \omega_{BA} = 0, \tag{2.2}$$

$$d\omega_{AB} = -\sum_C \omega_{AC} \wedge \omega_{CB} + \varepsilon_B\, \omega_A \wedge \omega_B. \tag{2.3}$$

Restricted to $M$ we have

$$ds^2 = \sum_i (\omega_i)^2 \tag{2.4}$$

$$\omega_{\alpha i} = \sum_j h_{ij}^\alpha \omega_j \tag{2.5}$$

$$R_{ijkl} = \delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk} - \sum_\alpha (h_{ik}^\alpha h_{jl}^\alpha - h_{il}^\alpha h_{jk}^\alpha) \tag{2.6}$$

$$R_{\alpha\beta jk} = -\sum_i (h_{ij}^\alpha h_{ik}^\beta - h_{ik}^\alpha h_{ij}^\beta) \tag{2.7}$$

where $R_{ijkl}$ are the components of the curvature tensor of $M$, $R_{\alpha\beta jk}$ the components of the curvature tensor of the normal bundle $T^\perp M$, and $h_{ij}^\alpha$ the components of the second fundamental form $\sigma = \sum_{\alpha,i,j} h_{ij}^\alpha \omega_i \otimes \omega_j \otimes e_\alpha$. We define $h_{ijk}^\alpha$ by

$$\sum_k h_{ijk}^\alpha \omega_k = d\, h_{ij}^\alpha - \sum_k h_{kj}^\alpha \omega_{ki} - \sum_k h_{ik}^\alpha \omega_{kj} + \sum_\beta h_{ij}^\beta \omega_{\alpha\beta}. \tag{2.8}$$

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Then we have the Codazzi equation

$$h^{\alpha}_{ijk} = h^{\alpha}_{ikj}. \tag{2.9}$$

The square $S$ of the norm of $\sigma$ is

$$S = \|\sigma\|^2 = \sum_{\alpha,i,j} (h^{\alpha}_{ij})^2 \tag{2.10}$$

and the mean curvature vector $\eta$ of $M$ is given by

$$\eta = \frac{1}{n}\operatorname{tr}\sigma = \frac{1}{n}\sum_{\alpha,i} h^{\alpha}_{ii} e_{\alpha}. \tag{2.11}$$

We need the following lemmas.

LEMMA 1 ([7]). *Let $A$, $B$ be symmetric $n \times n$ matrices satisfying $AB = BA$ and $\operatorname{tr} A = \operatorname{tr} B = 0$. Then*

$$\left|\operatorname{tr} A^2 B\right| \le \frac{n-2}{\sqrt{n(n-1)}}\left(\operatorname{tr} A^2\right)\left(\operatorname{tr} B^2\right)^{1/2}.$$

LEMMA 2 ([4, 9]). *Let $M$ be a complete Riemannian manifold whose Ricci curvature is bounded from below. If $F$ is a $C^2$-function bounded from above on $M$, then for any $\varepsilon > 0$, there is a point $x \in M$ such that*

$$\sup F - \varepsilon < F(x), \quad \|\nabla F\|(x) < \varepsilon, \quad \Delta F(x) < \varepsilon.$$

## 3. Proof of Theorem 1

Set $\|\eta\| = \sqrt{|\langle \eta, \eta\rangle|}$. Since $\|\eta\|^2 \le S/n$ and the equality holds only on set of umbilical points, the condition $S \le \left((2n\sqrt{n-1})/(n+2\sqrt{n-1})\right)(1+\|\eta\|^2)$ implies that $S \le 2\sqrt{n-1}$ and the equality holds only on the set of umbilical points. Therefore the theorem follows from [5] if $p = 1$ and [2] if $\eta = 0$.

We now suppose that $p \ge 2$ and $\eta \ne 0$. Thus we can choose the frame $e_1, \dots, e_{n+p}$ as in Section 2 with $e_{n+1} = \eta/\|\eta\|$. Then

$$\|\eta\| = \frac{1}{n}\sum_i h^{n+1}_{ii}, \quad \sum_i h^{\alpha}_{ii} = 0, \quad (\alpha > n+1). \tag{3.1}$$

Since $\eta$ is parallel in $T^{\perp}M$, we know $\|\eta\|$ is constant and $\omega_{\alpha\, n+1} = 0$. Consequently, $R_{\alpha\, n+1\, jk} = 0$. From (2.7) we have

$$\sum_i h^{\alpha}_{ij} h^{n+1}_{ik} = \sum_i h^{\alpha}_{ik} h^{n+1}_{ij}. \tag{3.2}$$

Denote by $H_\alpha$ the matrix $(h^\alpha_{ij})$ for $\alpha = n+1, \dots, n+p$. (3.2) means

$$H_\alpha H_{n+1} = H_{n+1} H_\alpha. \tag{3.3}$$

We define $h^\alpha_{ijkl}$ by

$$\sum_l h^\alpha_{ijkl}\omega_l = d\, h^\alpha_{ijk} - \sum_l (h^\alpha_{ljk}\omega_{li} + h^\alpha_{ilk}\omega_{lj} + h^\alpha_{ijl}\omega_{lk}) + \sum_\beta h^\beta_{ijk}\omega_{\alpha\beta}. \tag{3.4}$$

The Laplacian of $h^\alpha_{ij}$ is defined by

$$\Delta h^\alpha_{ij} = \sum_k h^\alpha_{ijkk}. \tag{3.5}$$

From (2.9), (3.1), and (3.4) we obtain

$$\Delta h^\alpha_{ij} = \sum_{k,l}(h^\alpha_{kl}R_{lijk} + h^\alpha_{il}R_{lkjk}) - \sum_{\beta,k} h^\beta_{ki}R_{\alpha\beta jk}. \tag{3.6}$$

Then by (2.6), (2.7), and (3.6) we have

$$\sum_{i,j} h^{n+1}_{ij}\Delta h^{n+1}_{ij} = n \operatorname{tr} H^2_{n+1} - n^2\|\eta\|^2 - n\|\eta\| \operatorname{tr} H^3_{n+1} + \sum_\beta [\operatorname{tr}(H_{n+1}H_\beta)]^2 \tag{3.7}$$

$$\sum_{i,j} h^\alpha_{ij}\Delta h^\alpha_{ij} = n \operatorname{tr} H^2_\alpha - n\|\eta\| \operatorname{tr}(H^2_\alpha H_{n+1}) - \sum_\beta \operatorname{tr}(H_\alpha H_\beta - H_\beta H_\alpha)^2 + \sum_\beta [\operatorname{tr}(H_\alpha H_\beta)]^2, \quad (\alpha > n+1). \tag{3.8}$$

Set $B = H_{n+1} - \|\eta\| I$. By means of (3.1) and (3.3) we get

$$\operatorname{tr} B = 0, \quad \operatorname{tr} H_\alpha = 0, \quad H_\alpha B = B H_\alpha, \quad (\alpha > n+1). \tag{3.9}$$

By virtue of Lemma 1,

$$|\operatorname{tr}(H^2_\alpha B)| \le \frac{n-2}{\sqrt{n(n-1)}} \operatorname{tr} H^2_\alpha \sqrt{\operatorname{tr} B^2}, \quad (\alpha > n+1). \tag{3.10}$$

Since

$$\begin{aligned} \operatorname{tr}(H^2_\alpha B) &= \operatorname{tr}(H^2_\alpha H_{n+1}) - \|\eta\| \operatorname{tr} H^2_\alpha, \quad (\alpha > n+1) \\ \operatorname{tr} B^2 &= \operatorname{tr} H^2_{n+1} - n\|\eta\|^2, \end{aligned} \tag{3.11}$$

from (3.10) we get

$$\operatorname{tr}(H^2_\alpha H_{n+1}) \le \left[\|\eta\| + \frac{n-2}{\sqrt{n(n-1)}}\sqrt{\operatorname{tr} H^2_{n+1} - n\|\eta\|^2}\right] \operatorname{tr} H^2_\alpha. \tag{3.12}$$

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Taking $A = B$ in Lemma 1 we obtain

$$|\operatorname{tr} B^3| \le \frac{n-2}{\sqrt{n(n-1)}}\left(\sqrt{\operatorname{tr} B^2}\right)^3 \tag{3.13}$$

and therefore,

$$\operatorname{tr} H_{n+1}^3 \le 3\|\eta\| \operatorname{tr} H_{n+1}^2 - 2n\|\eta\|^3 + \frac{n-2}{\sqrt{n(n-1)}}\left(\operatorname{tr} H_{n+1}^2 - n\|\eta\|^2\right)^{\frac{3}{2}}. \tag{3.14}$$

Let $T = \operatorname{tr} H_{n+1}^2$ and $U = \sum_{\alpha>n+1} \operatorname{tr} H_\alpha^2$. Then $S = T + U$ and

$$\begin{aligned}\frac{1}{2}\Delta T &= \sum_{ijk}(h_{ijk}^{n+1})^2 + \sum_{ij} h_{ij}^{n+1}\Delta h_{ij}^{n+1} \\ &\ge (T - n\|\eta\|^2)\left[n + T - 2n\|\eta\|^2 - \frac{n-2}{\sqrt{n(n-1)}}n\|\eta\|\sqrt{T - n\|\eta\|^2}\right]\end{aligned} \tag{3.15}$$

$$\begin{aligned}\frac{1}{2}\Delta U &\ge \sum_{\alpha>n+1}\sum_{ij} h_{ij}^\alpha \Delta h_{ij}^\alpha \\ &\ge nU\left[1 - \|\eta\|^2 - \frac{n-2}{\sqrt{n(n-1)}}\|\eta\|\sqrt{T - n\|\eta\|^2}\right]\end{aligned} \tag{3.16}$$

where we have used (3.7), (3.8), (3.12), and (3.14).

Since

$$\begin{aligned}&-2(n-2)\sqrt{n}\|\eta\|\sqrt{T - n\|\eta\|^2} \\ &\quad = \left[(\sqrt{n-1}+1)\sqrt{T - n\|\eta\|^2} - (\sqrt{n-1}-1)\sqrt{n}\|\eta\|\right]^2 \\ &\quad\quad + 4n\sqrt{n-1}\|\eta\|^2 - (n + 2\sqrt{n-1})T\end{aligned}$$

we have

$$\frac{1}{2}\Delta T \ge n(T - n\|\eta\|^2)\left(1 - \frac{1}{2\sqrt{n-1}}T\right), \tag{3.17}$$

$$\frac{1}{2}\Delta U \ge nU\left(1 + \|\eta\|^2 - \frac{n + 2\sqrt{n-1}}{2n\sqrt{n-1}}T\right). \tag{3.18}$$

By means of (2.6), we have

$$R_{ij} = (n-1)\delta_{ij} + \sum_{k,\alpha} h_{ik}^\alpha h_{jk}^\alpha - \sum_\alpha h_{ij}^\alpha \sum_k h_{kk}^\alpha$$

where $R_{ij}$ are the components of the Ricci tensor of $M$. Thus

$$R_{ii} \ge n - 1 + (h_{ii}^{n+1})^2 \ge (n-1) - \frac{n^2}{4}\|\eta\|^2. \tag{3.19}$$

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Taking $F = -(U+1)^{-1/2}$ in Lemma 2, we know for any $\varepsilon > 0$ there is $x \in M$ such that

$$\sup F - \varepsilon < F(x), \quad \|\nabla F\|(x) < \varepsilon, \quad \Delta F(x) < \varepsilon. \tag{3.20}$$

Since $\Delta F = -\frac{1}{2}F^3\Delta U + 3F^{-1}\|\nabla F\|^2$, we have

$$\frac{1}{2}F^4(x)\Delta U(x) = 3\|\nabla F\|^2(x) - F(x)\Delta F(x) < 3\varepsilon^2 - \varepsilon F(x). \tag{3.21}$$

Thus, for any convergent sequence $\{\varepsilon_m\}$ with $\varepsilon_m > 0$ and $\lim_{m\to\infty}\varepsilon_m = 0$, there is a point sequence $\{x_m\}$ such that $\{F(x_m)\}$ satisfies (3.20) and $\lim_{m\to\infty} F(x_m) = F_0 = \sup F$, and therefore $\lim_{m\to\infty} U(x_m) = U_0 = \sup U$.

On the other hand, from (3.21) we have

$$\frac{1}{2}F^4(x_m)\Delta U(x_m) < 3\varepsilon_m^2 - \varepsilon_m F(x_m)$$

and the right hand side converges to 0 because $-1 \le F \le 0$. Accordingly for any $\varepsilon \in (0,2)$, there is $m_\varepsilon$ such that for $m > m_\varepsilon$,

$$F^4(x_m)\Delta U(x_m) \lessgtr \varepsilon. \tag{3.22}$$

(3.18) and (3.22) yield

$$\varepsilon[U(x_m)+1]^2 > 2nU(x_m)\left[1 + \|\eta\|^2 - \frac{n+2\sqrt{n-1}}{2n\sqrt{n-1}}T(x_m)\right]. \tag{3.23}$$

Under the hypothesis of (i) in Theorem 1, we have $(1+\|\eta\|^2) \ge \big((n+2\sqrt{n-1})/(2n\sqrt{n-1})\big)(T+U)$. Hence, from (3.23) we get

$$\varepsilon[U(x_m)+1]^2 > \left(\frac{n}{\sqrt{n-1}} + 2\right)[U(x_m)]^2,$$

which implies $\{U(x_m)\}$ is bounded and $U_0 = 0$. Thus $U = 0$. Using the method of Yau [8] we know $M$ lies in a totally geodesic submanifold $S_1^{n+1}(1)$ of $S_p^{n+p}(1)$. Since $U = 0$, we know $S = T$. The inequality (3.17) becomes

$$\frac{1}{2}\Delta S \ge n(S - n\|\eta\|^2)\left(1 - \frac{1}{2\sqrt{n-1}}S\right). \tag{3.24}$$

Since $S \ge n\|\eta\|^2$ and $S \le \big((2n\sqrt{n-1})/(n+2\sqrt{n-1})\big)(1+\|\eta\|^2) \le 2\sqrt{n-1}$, (3.24) shows $\Delta S \ge 0$. Taking $F = -(S - n\|\eta\|^2 + 1)^{-1/2}$, in the same way as above we can prove $S - n\|\eta\|^2 = 0$. (Noting that $S = 2\sqrt{n-1}$ implies $S = n\|\eta\|^2$.)

So $M$ is totally umbilical. From (2.6) we know $M$ has constant sectional curvature $K = 1 - S/n$. If $n \geq 3$, then $S < n$ by $S \geq 2\sqrt{n-1}$, and $K > 0$, $M$ is isometric to the sphere $S^n(r)$ of radius $r = \sqrt{n/(n-S)}$. If $n = 2$, then either $M$ is flat (when $S = 2$) or is isometric to $S^2(\sqrt{2/(2-S)})$ (when $S < 2$).

Under the hypothesis of (ii) in Theorem 1 we have

$$(T+U) \leq \frac{2n\sqrt{n-1}}{n-2}(1-\|\eta\|^2).$$

Noting $2\sqrt{n}\|\eta\|\sqrt{T-n\|\eta\|^2} \leq T$, from (3.16) we have

$$\frac{1}{2}\Delta U \geq nU\left[1 - \|\eta\|^2 - \frac{n-2}{2n\sqrt{n-1}}T\right] \geq \frac{n-2}{2\sqrt{n-1}}U^2 \geq 0.$$

Applying Lemma 2 to $U$ we can get $U = 0$ and therefore $H_\alpha = 0$ for all $\alpha > n+1$. Using the method of Yau [8] we know $M$ lies in a totally geodesic submanifold $S_1^{n+1}(1)$ of $S_p^{n+p}(1)$. We then complete the proof of Theorem 1. □

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