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A NOTE ON STRONG RIESZ SUMMABILITY

BY

B. THORPE

ABSTRACT. This note proves that if $1 \le p < \infty$ and $1 - 1/p < k < 2 - 1/p$ then the space of sequences strongly Riesz summable $[R, \lambda, k]_p$ to 0 has AK. Using general results of Jakimovski and Russell it is then possible to deduce a best possible limitation condition and a convergence factor result for $[R, \lambda, k]_p$.

1. In a recent paper Jakimovski and Tzimbalario obtained a mean value theorem for absolute Riesz summability (Theorem 8 of [6]) that played an analogous role to the Riesz mean value theorem in ordinary Riesz summability. That is to say, just as Peyerimhoff was able to use the Riesz mean value theorem to deduce that the space $(R, \lambda, k)_0$ of sequences that are summable to $0(R, \lambda, k)$, has AK in the case $0 < k \le 1$ (see Satz 8.2 of [9]), Jakimovski and Tzimbalario were able to use their absolute Riesz mean value theorem to prove the corresponding result for $|R, \lambda, k|_0$, the space of sequences absolutely Riesz summable to 0 (see the case $p = 0$ of Theorem 5 in [6]). Kratz and Shawyer have recently proved a strong Riesz mean value theorem (see [7], [10]) and used it to obtain summability factor results for strong Riesz summability $[R, \lambda, k]_p$. However it does not appear to be possible to investigate the AK property of $[R, \lambda, k]_p$ from their inequality. It is the purpose of this note to give a direct proof of the AK property of strong Riesz summability. The main result extends Theorem 5 of [8] from strong Cesàro summability to strong Riesz summability and as applications a limitation condition and a convergence factor result are given. Using the ideas of this paper it is possible to obtain an alternative strong Riesz mean value theorem to that given in [7] and this will be investigated elsewhere.

2. In this section we give the notation and some basic properties. Let λ denote an unbounded, monotonic, strictly increasing positive sequence $\{\lambda_n\}_{n \ge 0}$. Suppose $1 \le p < \infty$ and $k > 1 - 1/p$. We say $\sum_{n=0}^{\infty} a_n = l\ [R, \lambda, k]_p$ if and only if

$$\int_{\lambda_0}^{X} |R^{(k-1)}(w) - l|^p \, dw = o(X)$$

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as $X \to \infty$, where

$$R^{(k-1)}(w) = \frac{A^{(k-1)}(w)}{w^{k-1}} = \sum_{\lambda_n < w} \left(1 - \frac{\lambda_n}{w}\right)^{k-1} a_n$$

(see [3], [11]). If $\mathbf{s} = \{s_n\}_{n \geq 0}$ where $s_n = \sum_{\nu=0}^{n} a_\nu$ let

$$o[R, \lambda, k]_p = \left\{\mathbf{s} : \sum_{n=0}^{\infty} a_n = 0[R, \lambda, k]_p\right\}$$

and for $\mathbf{s} \in o[R, \lambda, k]_p$ define

$$\|\mathbf{s}\| = \sup_{X > \lambda_0} \left(\frac{1}{X} \int_{\lambda_0}^{X} |R^{(k-1)}(w)|^p \, dw\right)^{1/p}. \tag{1}$$

It is not hard to prove, using Fatou's lemma, that with this definition of a norm, $o[R, \lambda, k]_p$ is a Banach space with continuous coordinate mappings *i.e.* $\mathbf{s} \mapsto s_n$ is continuous for each $n \geq 0$. (c.f. the proof for ordinary Riesz summability in P. 46 of [9]).

Also using integration by parts and the fact that $k > 1 - (1/p)$, an equivalent norm to (1) is

$$\|\mathbf{s}\|_1 = \sup_{X > \lambda_0} \left(X^{(1-k)p-1} \int_{\lambda_0}^{X} |A^{(k-1)}(w)|^p \, dw\right)^{1/p}. \tag{2}$$

We say $o[R, \lambda, k]_p$ has AK if $\{\boldsymbol{\delta}^m\}_{m \geq 0}$ is a Schauder basis, where $\boldsymbol{\delta}^m$ denotes the sequence with 1 in the mth coordinate and 0's elsewhere i.e. for every $\mathbf{s} \in o[R, \lambda, k]_p$

$$\left\|\mathbf{s} - \sum_{m=0}^{n} s_m \boldsymbol{\delta}^m\right\| \to 0 \tag{3}$$

as $n \to \infty$. This concept was introduced into summability by Zeller in [12] for more general sequence spaces.

As in page 47 of [9] we see, if $A(t) = \sum_{\lambda_m < t} a_m$, that

$$\begin{aligned}
\left\|\mathbf{s} - \sum_{m=0}^{n-1} s_m \boldsymbol{\delta}^m\right\|_1 &= \sup_{X > \lambda_n} \left(X^{(1-k)p-1} \int_{\lambda_n}^{X} \left|\frac{d}{dw}\left(\int_{\lambda_n}^{w} (w-t)^{k-1} A(t)\, dt\right)\right|^p dw\right)^{1/p} \\
&= \sup_{X > \lambda_n} \left(X^{(1-k)p-1} \int_{\lambda_n}^{X} \left|A^{(k-1)}(w)\right.\right. \\
&\qquad \left.\left. - \frac{d}{dw}\left(\int_0^{\lambda_n} (w-t)^{k-1} A(t)\, dt\right)\right|^p dw\right)^{1/p}. \\
&= \sup_{x > \lambda_n} \left(X^{(1-k)p-1} \int_{\lambda_n}^{X} \left|A^{(k-1)}(w)\right.\right. \\
&\qquad \left.\left. -(k-1) \int_0^{\lambda_n} (w-t)^{k-2} A(t)\, dt\right|^p dw\right)^{1/p}.
\end{aligned}$$

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Now $s \in o[R, \lambda, k]_p$ implies that $\int_{\lambda_0}^X |A^{(k-1)}(w)|^p\, dw = o(X^{1-(1-k)p})$ and so

$$\lim_{n\to\infty} \sup_{X>\lambda_n} \left(X^{(1-k)p-1} \int_{\lambda_n}^X |A^{(k-1)}(w)|^p\, dw \right)^{1/p} = 0 \tag{4}$$

If $k=1$, this proves that $o[R, \lambda, k]_p$ has AK for $p \geq 1$. If $k \neq 1$, an application of Minkowski's inequality in the case $p>1$ and trivially if $p=1$ shows, using (4), that (3) holds if and only if

$$\lim_{n\to\infty} \sup_{X>\lambda_n} X^{1-k-(1/p)} \left(\int_{\lambda_n}^X \left| \int_0^{\lambda_n} (w-t)^{k-2} A(t)\, dt \right|^p dw \right)^{1/p} = 0. \tag{5}$$

3. We now state and prove the main result.

THEOREM 1. *Let* $1 \leq p < \infty$. *If* $1-1/p < k < 2-1/p$ *then* $o[R, \lambda, k]_p$ *has AK.*

Proof. The case $k=1$ was pointed out in Section 2 above and so in what follows we assume $k \neq 1$. If $1<k<2$ then we replace the inner integral in (5) by

$$\int_0^{\lambda_n} (w-t)^{k-2} A(t)\, dt = \frac{(w-\lambda_n)^{k-1}}{\Gamma(k)\Gamma(2-k)} \int_0^{\lambda_n} (w-u)^{-1} (\lambda_n - u)^{1-k} A^{(k-1)}(u)\, du. \tag{6}$$

This is an identity due to M. Riesz and a proof is given on page 89 of [2]. If $0<k<1$ we can obtain the same identity as (6) by using Lemma 6 of [4] with $\kappa = \mu = k$ i.e.

$$\int_0^{\lambda_n} (w-t)^{k-2} A(t)\, dt = \int_0^{\lambda_n} (w-t)^{k-2} \left(\frac{1}{\Gamma(k)\Gamma(1-k)} \int_0^t (t-u)^{-k} A^{(k-1)}(u)\, du \right) dt$$
$$= \frac{1}{\Gamma(k)\Gamma(1-k)} \int_0^{\lambda_n} A^{(k-1)}(u) \left(\int_u^{\lambda_n} (w-t)^{k-2} (t-u)^{-k}\, dt \right) du$$

(6) now follows by either using the same techniques used by Bosanquet in [2] or, as Professor D. Borwein showed me, by writing the inner integrand as

$$\left(\frac{w-u}{w-t} - 1 \right)^{-k} (w-t)^{-2}$$

and evaluating the integral directly. Thus, if $1-(1/p) < k < 2-(1/p)$, $k \neq 1$, then a necessary and sufficient condition for $o[R, \lambda, k]_p$ to have AK is that

$$\lim_{n\to\infty} \sup_{X>\lambda_n} X^{1-k-(1/p)} \left(\int_{\lambda_n}^X (w-\lambda_n)^{(k-1)p} \times \left| \int_0^{\lambda_n} (\lambda_n - u)^{1-k} (w-u)^{-1} A^{(k-1)}(u)\, du \right|^p dw \right)^{1/p} = 0. \tag{7}$$

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We first prove that

$$\lim_{n\to\infty}\sup_{X>2\lambda_n} X^{1-k-(1/p)}\Big(\int_{2\lambda_n}^{X}(w-\lambda_n)^{(k-1)p} \times\Big|\int_0^{\lambda_n}(\lambda_n-u)^{1-k}(w-u)^{-1}A^{(k-1)}(u)\,du\Big|^p\,dw\Big)^{1/p}=0 \tag{8}$$

To do this, notice that $(w-u)^{-1}\le(w-\lambda_n)^{-1}$ and so

$$\sup_{X>2\lambda_n} X^{1-k-(1/p)}\Big(\int_{2\lambda_n}^{X}(w-\lambda_n)^{(k-1)p}\Big|\int_0^{\lambda_n}(\lambda_n-u)^{1-k}(w-u)^{-1}A^{(k-1)}(u)\,du\Big|^p\,dw\Big)^{1/p}$$

$$\le\sup_{X>2\lambda_n} X^{1-k-(1/p)}\Big(\int_{2\lambda_n}^{X}(w-\lambda_n)^{(k-2)p}\,dw\Big)^{1/p}\Big(\int_0^{\lambda_n}(\lambda_n-u)^{1-k}\,|A^{(k-1)}(u)|\,du\Big)$$

$$\le\frac{M}{\lambda_n}\int_0^{\lambda_n}(\lambda_n-u)^{1-k}\,|A^{(k-1)}(u)|\,du \tag{9}$$

where M denotes a constant (independent of n) that may be different at each appearance. If $p>1$, apply Hölder's inequality to (9) (where $1/q+1/p=1$) to get

$$\frac{1}{\lambda_n}\int_0^{\lambda_n}(\lambda_n-u)^{1-k}\,|A^{(k-1)}(u)|\,du\le\frac{1}{\lambda_n}\Big(\int_0^{\lambda_n}(\lambda_n-u)^{(1-k)q}\,du\Big)^{1/q}\times\Big(\int_0^{\lambda_n}|A^{(k-1)}(u)|^p\,du\Big)^{1/p}$$

$$\le\frac{M}{\lambda_n}\lambda_n^{1-k+(1/q)}\Big(\int_0^{\lambda_n}|A^{(k-1)}(u)|^p\,du\Big)^{1/p} \tag{10}$$

since $1-1/p<k<2-1/p$. Also $\mathbf{s}\in o[R,\lambda,k]_p$ so that

$$\Big(\int_0^{\lambda_n}|A^{(k-1)}(u)|^p\,du\Big)^{1/p}=o(\lambda_n^{k-1+(1/p)}) \tag{11}$$

and putting this in (10) gives (8) in the case $p>1$. If $p=1$, then since $0<k<1$ a trivial estimate in (9) gives (8) (using (11).)

To complete the proof it is sufficient to show that, with the same integrand as in (7),

$$\lim_{n\to\infty}\sup_{X>\lambda_n}\Big(\int_{\lambda_n}^{\min(X,2\lambda_n)}\Big|\int_0^{\lambda_n}(\,.\,.)\,du\Big|^p\,dw\Big)^{1/p}=0. \tag{12}$$

By an application of Minkowski's inequality if $p>1$ and the triangle inequality if $p=1$, it is sufficient for (12) that

$$\lim_{n\to\infty}\sup_{X>\lambda_n}\Big(\int_{\lambda_n}^{\min(X,2\lambda_n)}\Big|\int_0^{2\lambda_n-w}(\,.\,.)\,du\Big|^p\,dw\Big)^{1/p}=0 \tag{13}$$

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and

(14) $$\lim_{n\to\infty}\sup_{X>\lambda_n}\left(\int_{\lambda_n}^{\min(X,\,2\lambda_n)}\left|\int_{2\lambda_n-w}^{\lambda_n}(\,.\,.\,)\,du\right|^p dw\right)^{1/p}=0.$$

To prove (13), since $u\le\lambda_n\le w$ we have

(15) $$(\lambda_n-u)^{1-k}(w-u)^{-1}\le(\lambda_n-u)^{-k}$$

and putting this in the inner integral in (13) gives

(16) $$\left|\int_0^{2\lambda_n-w}(\lambda_n-u)^{1-k}(w-u)^{-1}A^{(k-1)}(u)\,du\right|\le\int_0^{2\lambda_n-w}(\lambda_n-u)^{-k}\,|A^{(k-1)}(u)|\,du.$$

Now, if $p>1$, choose $\eta>0$ so that $(\eta-k)<-(1/q)$ i.e. $0<\eta<k-(1-(1/p))$ and apply Hölder's inequality to (16) to get

$$\int_0^{2\lambda_n-w}(\lambda_n-u)^{-k}\,|A^{(k-1)}(u)|\,du\le\left(\int_0^{2\lambda_n-w}(\lambda_n-u)^{-\eta p}\,|A^{(k-1)}(u)|^p\,du\right)^{1/p}$$
$$\times\left(\int_0^{2\lambda_n-w}(\lambda_n-u)^{(\eta-k)q}\,du\right)^{1/q}\le M(w-\lambda_n)^{\eta-k+(1/q)}$$
$$\times\left(\int_0^{2\lambda_n-w}(\lambda_n-u)^{-\eta p}\,|A^{(k-1)}(u)|^p\,du\right)^{1/p}.$$

Hence the LHS of (13) becomes

$$\le M\lim_{n\to\infty}\sup_{X>\lambda_n}X^{1-k-(1/p)}\left(\int_{\lambda_n}^{\min(X,\,2\lambda_n)}(w-\lambda_n)^{\eta p-1}\right.$$
$$\left.\times\left\{\int_0^{2\lambda_n-w}(\lambda_n-u)^{-\eta p}\,|A^{(k-1)}(u)|^p\,du\right\}dw\right)^{1/p}$$
$$=M\lim_{n\to\infty}\sup_{X>\lambda_n}X^{1-k-(1/p)}\left(\int_0^{\lambda_n}(\lambda_n-u)^{-\eta p}\,|A^{(k-1)}(u)|^p\right.$$
$$\left.\times\left\{\int_{\lambda_n}^{\min(X,\,2\lambda_n-u)}(w-\lambda_n)^{\eta p-1}\,dw\right\}du\right)^{1/p}$$
$$\le M\lim_{n\to\infty}\sup_{X>\lambda_n}X^{1-k-(1/p)}\left(\int_0^{\lambda_n}(\lambda_n-u)^{-\eta p}\,|A^{(k-1)}(u)|^p\{(\lambda_n-u)^{\eta p}\}\,du\right)^{1/p}$$
$$\le M\lim_{n\to\infty}\lambda_n^{1-k-(1/p)}\left(\int_0^{\lambda_n}|A^{(k-1)}(u)|^p\,du\right)^{1/p}$$

and so (13) holds by using (11). If $p=1$ then we take $1/q$ to mean 0 and use the inequality

$$\int_0^{2\lambda_n-w}(\lambda_n-u)^{-k}\,|A^{(k-1)}(u)|\,du\le(w-\lambda_n)^{\eta-k}\int_0^{2\lambda_n-w}(\lambda_n-u)^{-\eta}\,|A^{(k-1)}(u)|\,du$$

instead of Hölder's inequality in the above.

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To prove (14), since $\lambda_n \le w \le \min(X, 2\lambda_n)$ and $2\lambda_n - w \le u \le \lambda_n$ we have

$$(w-u)^{-1} \le (w-\lambda_n)^{-1}$$

and putting this in the inner integral in (14) gives

$$\left|\int_{2\lambda_n-w}^{\lambda_n} (\lambda_n-u)^{1-k}(w-u)^{-1}A^{(k-1)}(u)\,du\right| \le M(w-\lambda_n)^{-1} \times \int_{2\lambda_n-w}^{\lambda_n} (\lambda_n-u)^{1-k}\,|A^{(k-1)}(u)|\,du. \tag{17}$$

If $p>1$, choose $\eta>0$ so that $(1-k-\eta)>-(1/q)$ i.e. $0<\eta<(2-(1/p))-k$ and apply Hölder's inequality to (17) to get

$$\int_{2\lambda_n-w}^{\lambda_n} (\lambda_n-u)^{1-k}\,|A^{(k-1)}(u)|\,du \le \left(\int_{2\lambda_n-w}^{\lambda_n} (\lambda_n-u)^{\eta p}\,|A^{(k-1)}(u)|^p\,du\right)^{1/p} \times \left(\int_{2\lambda_n-w}^{\lambda_n} (\lambda_n-u)^{(1-k-\eta)q}\,du\right)^{1/q}$$

$$\le M(w-\lambda_n)^{1-k-\eta+(1/q)}\left(\int_{2\lambda_n-w}^{\lambda_n} (\lambda_n-u)^{\eta p}\,|A^{(k-1)}(u)|^p\,du\right)^{1/p}$$

Hence the LHS of (14) becomes

$$\le M \limsup_{n\to\infty}\sup_{X>\lambda_n} X^{1-k-(1/p)}\left(\int_{\lambda_n}^{\min(X,2\lambda_n)} (w-\lambda_n)^{-\eta p-1} \times \left\{\int_{2\lambda_n-w}^{\lambda_n} (\lambda_n-u)^{\eta p}\,|A^{(k-1)}(u)|^p\,du\right\}dw\right)^{1/p}$$

$$= M \limsup_{n\to\infty}\sup_{X>\lambda_n} X^{1-k-(1/p)}\left(\int_{\max(0,2\lambda_n-X)}^{\lambda_n} (\lambda_n-u)^{\eta p}\,|A^{(k-1)}(u)|^p \times \left\{\int_{2\lambda_n-u}^{\min(X,2\lambda_n)} (w-\lambda_n)^{-\eta p-1}\,dw\right\}du\right)^{1/p}$$

$$\le M \limsup_{n\to\infty}\sup_{X>\lambda_n} X^{1-k-(1/p)}\left(\int_{\max(0,2\lambda_n-X)}^{\lambda_n} (\lambda_n-u)^{\eta p}\,|A^{(k-1)}(u)|^p\{(\lambda_n-u)^{-\eta p}\}\,du\right)^{1/p}$$

$$\le M \lim_{n\to\infty} \lambda_n^{1-k-(1/p)}\left(\int_0^{\lambda_n} |A^{(k-1)}(u)|^p\,du\right)^{1/p}$$

and so (14) holds by using (1)). If $p=1$ then simple modifications to the above (similar to those given in the proof of (13)) give the result in this case also. Thus the theorem is completely proved.

In contrast to Theorem 1 we have the following

PROPOSITION. (i) *If* $k>1$ *then* $\exists\lambda$ *such that* $o[R, \lambda, k]_1$ *does not have* AK.

(ii) *If* $1<p<\infty$ *and* $k \ge 2-(1/p)$ *then* $\exists\lambda$ *such that* $o[R, \lambda, k]_p$ *does not have* AK.

Proof. If $\lambda_n = (n+1)$ then strong Riesz summability is equivalent to strong Cesàro summability and so (i) and (ii) follow from Theorems 9 and 10 of [8].

I can add in this connection that Professor B. Kuttner has recently shown me a proof that, under the 'high indices' condition on λ i.e. for $n \geq 1$, $\lambda_{n+1} \geq c\lambda_n$ where c is a fixed constant strictly greater than 1, then $o[R, \lambda, k]_p = c_0$ if $k > 1-(1/p)$, where c_0 is the space of convergent to zero sequences. Thus $o[R, \lambda, k]_p$ has AK in this case for all $k > 1-(1/p)$.

4. As a first application we have the following limitation condition.

THEOREM 2. *Let* $1 \leq p < \infty$ *and* $1-(1/p) < k < 2-(1/p)$. *If* $s_n \to l\,[R, \lambda, k]_p$ *then*

$$s_n - l = o(\Lambda_n^{k-1+(1/p)}) \tag{18}$$

where

$$\Lambda_n = \frac{\lambda_{n+1}}{\lambda_{n+1}-\lambda_n}$$

and this result is best possible in the sense that given any unbounded sequence $\{\theta_n\}$, $\exists \mathbf{s} \in [R, \lambda, k]_p$ *such that* $\theta_n(s_n - l) \neq o(\Lambda_n^{k-1+(1/p)})$.

Proof. If $s_n \to l\ [R, \lambda, k]_p$ then $\mathbf{s} - l\boldsymbol{\delta} \in o[R, \lambda, k]_p$ where $\boldsymbol{\delta}$ is the constant sequence of all 1's. Thus by Theorem 1 above and Corollary 1(b) to Theorem 3 of [5] the best possible limitation condition (in the sense described above) is $s_n - l = o(\|\boldsymbol{\delta}^n\|^{-1})$. Using the equivalnt norm given by (2),

$$\|\boldsymbol{\delta}^n\|_1 = \max\Bigg(\sup_{\lambda_n \leq X \leq \lambda_{n+1}} X^{1-k-(1/p)} \left(\int_{\lambda_n}^{X} (w-\lambda_n)^{(k-1)p}\, dw \right)^{1/p},$$

$$\sup_{X \geq \lambda_{n+1}} X^{1-k-(1/p)}$$

$$\times \left\{ \int_{\lambda_n}^{\lambda_{n+1}} (w-\lambda_n)^{(k-1)p}\, dw + \int_{\lambda_{n+1}}^{X} |(w-\lambda_n)^{k-1} - (w-\lambda_{n+1})^{k-1}|^p\, dw \right\}^{1/p} \Bigg)$$

Now

$$\sup_{\lambda_n \leq X \leq \lambda_{n+1}} X^{1-k-(1/p)} \left(\int_{\lambda_n}^{X} (w-\lambda_n)^{(k-1)p}\, dw \right)^{1/p} = M \sup_{\lambda_n \leq X \leq \lambda_{n+1}} \left(1 - \frac{\lambda_n}{X}\right)^{k-1+(1/p)}$$
$$= M\left(\frac{\lambda_{n+1}-\lambda_n}{\lambda_{n+1}}\right)^{k-1+(1/p)}.$$

and similarly

$$\sup_{X \geq \lambda_{n+1}} X^{1-k-(1/p)} \left(\int_{\lambda_n}^{\lambda_{n+1}} (w-\lambda_n)^{(k-1)p}\, dw \right)^{1/p} = M\left(\frac{\lambda_{n+1}-\lambda_n}{\lambda_{n+1}}\right)^{k-1+(1/p)}.$$

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Thus if $k=1$, $\|\boldsymbol{\delta}^n\|_1 = M\Lambda_n^{1/p}$, which gives (18). If $k \neq 1$ then

$$\begin{aligned}
&\sup_{X\geq\lambda_{n+1}} X^{1-k-(1/p)}\left(\int_{\lambda_{n+1}}^{X} |(w-\lambda_n)^{k-1}-(w-\lambda_{n+1})^{k-1}|^p\,dw\right)^{1/p}\\
&= M \sup_{X\geq\lambda_{n+1}} X^{1-k-(1/p)}\left(\int_{\lambda_{n+1}}^{X}\left|\int_{\lambda_n}^{\lambda_{n+1}}(w-u)^{k-2}\,du\right|^p dw\right)^{1/p}\\
&\leq M \sup_{X\geq\lambda_{n+1}} X^{1-k-(1/p)}\int_{\lambda_n}^{\lambda_{n+1}}\left(\int_{\lambda_{n+1}}^{X}(w-u)^{(k-2)p}\,dw\right)^{1/p} du
\end{aligned}$$

by Minkowski's inequality (if $p>1$),

$$\begin{aligned}
&\leq M \sup_{X\geq\lambda_{n+1}} X^{1-k-(1/p)}\int_{\lambda_n}^{\lambda_{n+1}}(\lambda_{n+1}-u)^{k-2+(1/p)}\,du\\
&\leq M\,\Lambda_n^{-(k-1+(1/p))}.
\end{aligned}$$

Hence, another use of Minkowski's inequality (if $p>1$) *shows that* $\|\boldsymbol{\delta}^n\|_1$ lies between two positive constant multiples of $\Lambda_n^{-(k-1+(1/p))}$ and so (18) follows.

As a final application we give the following convergence factor result.

THEOREM 3. *Let* $1\leq p<\infty$ *and* $1-(1/p)<k<2-(1/p)$. *Then* $\sum_{n=0}^{\infty} s_n\varepsilon_n$ *is convergent for all* $\mathbf{s}\in o[R, \lambda, k]_p$ *if and only if*

$$\varepsilon_n = \int_{\lambda_n}^{\infty}\left(1-\frac{\lambda_n}{t}\right)^{k-1}\alpha(t)\,dt - \int_{\lambda_{n+1}}^{\infty}\left(1-\frac{\lambda_{n+1}}{t}\right)^{k-1}\alpha(t)\,dt \tag{19}$$

where α *satisfies* $\sum_{n=0}^{\infty} M_n(\alpha, p)<\infty$ *and*

$$M_n(\alpha, p) = \begin{cases} \operatorname*{ess\,sup}_{2^n<t<2^{n+1}} |t\alpha(t)| & \text{if } p=1\\ \left(2^{-n}\int_{2^n}^{2^{n+1}} |t\alpha(t)|^q\,dt\right)^{1/q} & \text{if } p>1\end{cases} \tag{20}$$

Proof. If $\mathbf{s}\in o[R, \lambda, k]_p$ then by Theorem 1 $\mathbf{s}=\sum_{n=0}^{\infty} s_n\boldsymbol{\delta}^n$ and so for every continuous linear functional $f\in o[R, \lambda, k]_p^*$, $f(\mathbf{s})=\sum_{n=0}^{\infty} s_n f(\boldsymbol{\delta}^n)$.

Conversely, if $\sum_{n=0}^{\infty} s_n\varepsilon_n$ is convergent for every $\mathbf{s}\in o[R, \lambda, k]_p$ then $\mathbf{s}\mapsto \sum_{n=0}^{\infty} s_n\varepsilon_n$ defines a continuous linear functional on $o[R, \lambda, k]_p$. Moreover, from the definition of $o[R, \lambda, k]_p$ there is an isometry from $o[R, \lambda, k]_p$ on to a closed subspace of W_p, where W_p is defined as in [1]. Using the representation of W_p^* obtained in [1] and the Hahn-Banach theorem, we see that $f\in o[R, \lambda, k]_p^*$ if and only if

$$f(\mathbf{s}) = \int_0^{\infty} R^{(k-1)}(t)\alpha(t)\,dt$$

where α satisfies $\sum_{n=0}^{\infty} M_n(\alpha, p)<\infty$ for $M_n(\alpha, p)$ defined as in (20).

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Thus

$$f(\boldsymbol{\delta}^n)=\int_{\lambda_n}^{\lambda_{n+1}}\left(1-\frac{\lambda_n}{t}\right)^{k-1}\alpha(t)\,dt+\int_{\lambda_{n+1}}^{\infty}\left\{\left(1-\frac{\lambda_n}{t}\right)^{k-1}-\left(1-\frac{\lambda_{n+1}}{t}\right)^{k-1}\right\}\alpha(t)\,dt$$

and (9) will follow if we show

$$\int_{\lambda_n}^{\infty}\left(1-\frac{\lambda_n}{t}\right)^{k-1}\alpha(t)\,dt$$

exists. Now convergence at infinity of this integral follows since the convergence of $\sum_{n=0}^{\infty} M_n(\alpha, p)$ implies that $\int_0^\infty |\alpha(t)|\,dt<\infty$ and convergence at λ_n (in the case $k<1$) follows by an application of Hölder's inequality if $p>1$ and an easy estimate if $p=1$ (using (20)). Hence the result.

COROLLARY. *Let* $1\le p<\infty$ *and* $1-(1/p)<k<2-(1/p)$. *Then* (19) *is a necessary and sufficient condition for* $\sum_{n=0}^{\infty} s_n\varepsilon_n$ *to be convergent for all sequences* $\mathbf{s}$ *summable* $[R, \lambda, k]_p$.

Proof. If $s_n\to l'[R, \lambda, k]_p$ then $(s-l'\boldsymbol{\delta})\in o[R, \lambda, k]_p$ and so the result will follow immediately from Theorem 3 provided we show

$$\lim_{n\to\infty}\int_{\lambda_n}^{\infty}\left(1-\frac{\lambda_n}{t}\right)^{k-1}\alpha(t)\,dt=0 \tag{21}$$

(since then $\sum_{n=0}^{\infty}\varepsilon_n$ will converge). If $k=1$ then (21) is clear. If $k>1$ then Lebesgue's dominated convergence theorem proves (21). If $1-(1/p)<k<1$, $p>1$, $\lambda_n\in[2^m, 2^{m+1})$ then for $l>m+1$

$$\begin{aligned}\left|\int_{2^l}^{2^{l+1}}\left(1-\frac{\lambda_n}{t}\right)^{k-1}\alpha(t)\,dt\right| &\le\left(\int_{2^l}^{2^{l+1}}\left(1-\frac{\lambda_n}{t}\right)^{(k-1)p}dt\right)^{1/p}\left(\int_{2^l}^{2^{l+1}}|\alpha(t)|^q\,dt\right)^{1/q}\\ &\le M2^{l/p}\left(\int_{2^l}^{2^{l+1}}|\alpha(t)|^q\,dt\right)^{1/q}\end{aligned}$$

and so

$$\left|\int_{2^{m+2}}^{\infty}\left(1-\frac{\lambda_n}{t}\right)^{k-1}\alpha(t)\,dt\right|\le M\sum_{l=m+2}^{\infty}2^{l/p}\left(\int_{2^l}^{2^{l+1}}|\alpha(t)|^q\,dt\right)^{1/q}\to 0$$

as $n\to\infty$. Similarly

$$\left|\int_{\lambda_n}^{2^{m+2}}\left(1-\frac{\lambda_n}{t}\right)^{k-1}\alpha(t)\,dt\right|\le M2^{m/p}\left(\int_{2^m}^{2^{m+2}}|\alpha(t)|^q\,dt\right)^{1/q}\to 0$$

as $n\to\infty$. If $p=1$ then an easier estimate gives the same result, and so (21) is proved and hence the result.

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Department of Pure Mathematics,
The University,
Birmingham.
B15 2TT.
England.