Canad. Math. Bull. Vol. **28** (1), 1985

ON QUASISIMILARITY FOR TOEPLITZ OPERATORS

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ABSTRACT. In this article we give a sufficient condition for quasisimilar analytic Toeplitz operators to be unitarily equivalent. We also use a result of Deddens and Wong to give a sufficient condition for an operator intertwining two analytic Toeplitz operators to intertwine their inner parts too. Analytic Toeplitz operators with univalent symbols satisfying a suitable normalization that are quasisimilar are shown to have equal symbols.

1. **Introduction**. Let H^2 denote the Hilbert space of functions f analytic in the open unit disk $\mathbb{D}$ which satisfy $\sup_{0\leq r<1} \int |f(re^{i\theta})|^2\, d\theta < \infty$. Let H^∞ be the space of bounded analytic functions on $\mathbb{D}$, and for φ in H^∞ let T_φ denote the operator on H^2 defined by $T_\varphi f = \varphi f$. The operator T_φ is said to be an *analytic Toeplitz operator*. In this article we consider the following questions. If an operator intertwines two analytic Toeplitz operators does it necessarily intertwine their inner parts too? Do quasisimilar analytic Toeplitz operators have equal essential spectra? Does quasisimilarity imply unitary equivalence? Although the study of Toeplitz operators has been extensive, little seems to be known about their quasisimilarity. Our purpose is to answer parts of the above questions. In particular, we give a sufficient condition for an operator intertwining two analytic Toeplitz operators to intertwine their inner parts too. We also give a sufficient condition for quasisimilarity to imply unitary equivalence.

If $\mathcal{H}$ is a separable Hilbert space, let $\mathcal{B}(\mathcal{H})$ denote the Banach algebra of all bounded linear operators on $\mathcal{H}$. If $\mathcal{H}_1$ and $\mathcal{H}_2$ are Hilbert spaces and $X:\mathcal{H}_1 \to \mathcal{H}_2$ is a bounded operator having trivial kernel and dense range, then X is said to be quasi-invertible. If A_1, A_2 are operators on $\mathcal{H}_1$, $\mathcal{H}_2$, then A_1 is *quasisimilar* to $A_2 (A_1 \sim A_2)$ if there are quasi-invertible operators $X:\mathcal{H}_1 \to \mathcal{H}_2$ and $Y:\mathcal{H}_2 \to \mathcal{H}_1$ satisfying $XA_1 = A_2X$ and $A_1Y = YA_2$. If A_1 and A_2 are unitarily equivalent we write $A_1 \cong A_2$.

Raphael [8] has shown that quasisimilar cyclic subnormal operators have equal essential spectra. Williams [9] has shown that quasisimilar quasinormal operators have equal essential spectra. However the equality of essential spectra under quasisimilarity for general subnormal operators is still open. We study the intertwining operators and suggest the pertinent questions to be considered.

A function g in H^∞ is said to be *inner* if $\lim_{r\to 1} |g(re^{i\theta})| = 1$ for almost every θ. A

Received by the editors January 20, 1984 and, in revised form, May 18, 1984.

AMS Subject Classification: Primary, 47B20; Secondary, 47B35.

Key words and phrases: Toeplitz operator, quasisimilarity, inner-outer factorization.

The author had a postdoctoral fellowship from the University of Calgary during the preparation of this paper.

function F in H^∞ is *outer* if T_F has dense range. Every φ in H^∞ has a factorization $\varphi = gF$ where g is inner and F is outer [7].

An isometry T on H^2 is a *pure isometry* if $\bigcap_{n=0}^{\infty} T^n H^2 = \{0\}$. For $g \in H^\infty$, T_g is a pure isometry if and only if g is a nonconstant inner function. For an inner function g in H^∞ we obtain the decomposition $H^2 = \sum_{k=0}^{\infty} \oplus \ g^k(H^2 \theta gH^2)$. If $\{u_k\}_{k=1}^{n}$ (n possibly ∞) is a basis for $H^2 \theta gH^2$, then $\{u_k g^m\}_{m=0}^{\infty}{}_{k=1}^{n}$ is a basis for H^2, and with respect to this basis the matrix for T_g has a block form with an (n by n) identity matrix on its subdiagonal. Also any X in the commutant $\{T_g\}'$ of T_g has a lower triangular block form which is constant along its diagonals. We denote this X by the purely formal sum $\sum_{n=0}^{\infty} \hat{X}_n T_g^n$ with X_n on the nth subdiagonal. Throughout the rest of this paper we assume φ_1 and φ_2 are in H^∞ and have the inner-outer factorization $\varphi_i = g_i F_i$ ($i = 1, 2$) unless otherwise stated.

2. **Quasisimilarity**. The following simple lemma is essential to our purposes.

LEMMA 2.1 *For $\varphi \in H^\infty$, let $\varphi = gF$ be its inner-outer factorization. Then T_φ is Fredholm if and only if T_g is Fredholm and T_F is invertible.*

PROOF. If T_g is Fredholm and T_F is invertible then it readily follows that $T_\varphi = T_g T_F$ is Fredholm. On the other hand if T_φ is Fredholm, then $\ker T_\varphi^* = \ker T_g^*$ is finite dimensional and since T_g is an isometry it follows that T_g is Fredholm. Because $T_F = T_g^* T_\varphi$, it follows that T_F is Fredholm. Hence T_F has closed range and therefore it is invertible. Q.E.D.

We would like to point out that if $\varphi = gF$ is the inner-outer factorization of φ then T_φ is Fredholm if and only if g is a finite Blaschke product and F is invertible in H^∞. The index of T_g is the negative of the number of zeros of g counting multiplicity.

Note that if $T_{\varphi_1} \sim T_{\varphi_2}$ then $T_{g_1} \cong T_{g_2}$. Indeed $T_{\varphi_1} \sim T_{\varphi_2}$ implies that $H^2 \theta g_1 H^2$ and $H^2 \theta g_2 H^2$ have the same dimension and we know that any two pure isometries of the same multiplicity are unitarily equivalent. Therefore the inner parts are strongly related to each other. In the next lemma we investigate this property further.

LEMMA 2.2. *If $X : H^2 \to H^2$ is an operator such that $XT_{\varphi_1} = T_{\varphi_2}X$ and $XT_{g_1} = T_{g_2}X$ then $XT_{F_1} = T_{F_2}X$.*

PROOF. Since $XT_{\varphi_1} = T_{\varphi_2}X$, we have $XT_{g_1}T_{F_1} = T_{g_2}T_{F_2}X$. Using the relation $XT_{g_1} = T_{g_2}X$ we get $T_{g_2}XT_{F_1} = T_{g_2}T_{F_2}X$. Because T_{g_2} is an isometry, it follows that $XT_{F_1} = T_{F_2}X$. Q.E.D.

REMARKS. Let $T_{\varphi_1} \sim T_{\varphi_2}$. If we can show that $T_{g_1} \sim T_{g_2}$ in such a way that the same quasi-invertible operators intertwining T_{φ_1} and T_{φ_2} also intertwine T_{g_1} and T_{g_2} then Lemma 2.2 shows that $T_{F_1} \sim T_{F_2}$. Using a result of Clary [1] we conclude that $\sigma(T_{F_1}) = \sigma(T_{F_2})$. Therefore T_{F_1} is invertible if and only if T_{F_2} is invertible. In other words, T_{φ_1} is Fredholm if and only if T_{φ_2} is Fredholm.

The following lemma is a slight extension of Lemma 2 of [6] and will be used in the sequel.

LEMMA 2.3. *Let N, N' be nilpotent operators on K, K' respectively and let $X_0 = \lambda I_K + N$, $X_0' = \lambda I_{K'} + N'$ where λ is a nonzero complex number. If B, A_0, A_1, $A_2, \ldots \in \mathcal{B}(K, K')$ satisfy*

(a) $\|A_k\| \leq M, \quad k = 0, 1, 2, \ldots,$ and

(b) $A_kX_0 = X_0'A_{k-1} + B, \quad k = 1, 2, 3, \ldots,$

then $A_0 = A_1 = A_2 = \ldots$.

PROOF. Write $K = \sum_{i=1}^{n} \oplus K_i$ ($K' = \sum_{i=1}^{m} \oplus K_i'$) such that $X_0(X_0')$ has a lower triangular operator-valued matrix with diagonal entries $\lambda I_i (\lambda I_i')$ and repeat the proof of Lemma 2 of [6]. Q.E.D.

The next lemma says that the intertwining operator should be lower triangular.

LEMMA 2.4. *Assume $XT_{\varphi_1} = T_{\varphi_2}X$ and for $i = 1, 2$ write $H_i^2 = \sum_{n=0}^{\infty} g_i^n(H^2 \theta g_i H^2)$ then $X: H_1^2 \to H_2^2$ is lower triangular.*

PROOF. It suffices to show that $X^*: H_2^2 \to H_1^2$ is upper triangular. Equivalently we will show that X^* maps the subspaces $M_{2n} = \sum_{k=0}^{n} \oplus g_2^k(H^2 \theta g_2 H^2)$ into the subspaces $M_{1n} = \sum_{k=0}^{n} \oplus g_1^k(H^2 \theta g_1 H^2)$.

Now $XT_{\varphi_1} = T_{\varphi_2}X$ implies $XT_{\varphi_1}^{n+1} = T_{\varphi_2}^{n+1}X$ and hence $X^*T_{\varphi_2}^{*n+1} = T_{\varphi_1}^{*n+1}X^*$. Thus X^* maps the kernel of $T_{\varphi_2}^{*n+1}$ into the kernel of $T_{\varphi_1}^{*n+1}$. But

$$\ker T_{\varphi_i}^{*n+1} = \ker T_{g_i}^{*n+1} = H^2 \theta g_i^{n+1} H^2 = M_{in}\ (i = 1, 2)$$

Hence X is lower triangular. Q.E.D.

Note that a necessary and sufficient condition for the relation $XT_{g_1} = T_{g_2}X$ to hold is that $X: H_1^2 \to H_2^2$ be constant along its diagonals. Indeed writing T_{g_i} ($i = 1, 2$) in its block form and carrying out the necessary computations it follows that $X_{k+i,k} = X_{i0}$ $i, k = 0, 1, 2, \ldots$.

LEMMA 2.5. *Assume $XT_{\varphi_1} = T_{\varphi_2}X$ where $X: H_1^2 \to H_2^2$ is a bounded operator. If $T_{F_1} = \sum_{n=0}^{\infty} \hat{T}_n T_{g_1}^n$ and $T_{F_2} = \sum_{n=0}^{\infty} \hat{T}_n' T_{g_2}^n$ where $T_0 = \lambda I + N$, $T_0' = \lambda I' + N'$ with N and N' nilpotent, then $XT_{g_1} = T_{g_2}X$.*

PROOF. By Lemma 2.4 X is lower triangular. We will show that X is constant along its diagonals by inductively proving that $X_{k,0} = X_{k+1,1} = X_{k+2,2} = \ldots$ for $k = 0, 1, 2, \ldots$. We also remark that $\|X_{k+i,i}\| \leq \|X\|$ for all $k, i = 0, 1, 2, \ldots$.

If $1 \leq j < i$ then the (i, j) entry of $(XT_{g_1})T_{F_1} = T_{F_2}(T_{g_2}X)$ is

$$X_{i,j+1}T_0 + X_{i,j+2}T_1 + \ldots + X_{i,i}T_{i-j-1} = T_{i-j-1}'X_{j,j} + T_{i-j-2}'X_{j+1,j} + \ldots + T_0'X_{i-1,j}. \tag{2.6}$$

If $i = j + 1$ then $X_{j+1,j+1}T_0 = T_0'X_{j,j}$ and by Lemma 2.3 we obtain $X_{0,0} = X_{1,1} = X_{2,2} = \ldots$. To apply induction let us now assume that $X_{p,0} = X_{p+1,1} = X_{p+2,2} = \ldots$ for all $p \leq k$. Setting $i = j + k + 2$ in (2.6) we get

$$X_{k+1+j+1,j+1}T_0 = T_0'X_{k+1+j,j} + [T_1'X_{k,0} + \ldots + T_{k+1}'X_{0,0} - X_{k,0}T_1 - \ldots - X_{0,0}T_{k+1}].$$

An application of Lemma 2.3 gives us $X_{k+1,0} = X_{k+2,1} = \dots$ and hence by induction $X_{k,0} = X_{k+1,1} = \dots$ for all $k = 0, 1, 2, \dots$. Therefore $XT_{g_1} = T_{g_2}X$. Q.E.D.

The following theorem is the main result of this section. The idea of the proof is due to Deddens and Wong [6].

THEOREM 2.7. *Let φ_1, φ_2 be in H^∞ with inner-outer factorization $\varphi_i = g_iF_i$ $(i = 1, 2)$. Suppose there is a λ in $\mathbb{C}$ such that g_i factors as $g_{i1}g_{i2} \dots g_{in}$ and such that $F_i - \lambda$ is divisible by each g_{ij}, that is $F_i - \lambda = g_{ij}h_{ij}$ for $i = 1, 2$ and $j = 1, 2, \dots, n$. If $X: H_1^2 \to H_2^2$ is a bounded operator such that $XT_{\varphi_1} = T_{\varphi_2}X$ then $XT_{g_1} = T_{g_2}X$.*

PROOF. Write $T_{F_i} = \sum_{n=0}^{\infty} \hat{T}_{n,i}T_{gi}^n$ $(i = 1, 2)$. We will show that $T_{0,i} = \lambda I_i + N_i$ where N_i is nilpotent $(i = 1, 2)$. Then Lemma 2.5 implies that $XT_{g_1} = T_{g_2}X$.

For convenience set $T_{F_i} = T_i$, $T_{g_{ij}} = T_{ij}$ and $H_i^2 = \mathcal{H}_i$. Then $T_{g_i} = T_{i1}T_{i2} \dots T_{in}$. Since $T_{0,i}^*$ is the restriction of $T_{F_i}^*$ to

$$\mathcal{H}_i \theta T_{g_i}\mathcal{H}_i = (\mathcal{H}_i\theta T_{i1}\mathcal{H}_i) \oplus T_{i1}(\mathcal{H}_{i1}\theta T_{i2}\mathcal{H}_i) \oplus \dots \oplus T_{i1} \dots T_{in-1}(\mathcal{H}_i\theta T_{in}\mathcal{H}_i),$$

it follows that $T_{0,i}^*$ is upper triangular and hence that $T_{0,i}$ is lower triangular. Let $(T_{0,i})_{jj}$ be the compression of $T_{0,i}$ to $T_{i1} \dots T_{ij-1}(\mathcal{H}_i\theta T_{ij}\mathcal{H}_i)$. If $f, g \in \mathcal{H}_i\theta T_{ij}\mathcal{H}_i$ then since $T_i = \lambda + T_{ij}T_{h_{ij}}$ we obtain

$$T_i(T_{i1}T_{i2} \dots T_{ij-1}f) = (T_{i1}T_{i2} \dots T_{ij-1})T_if = \lambda T_{i1}T_{i2} \dots T_{ij-1}f + T_{i1}T_{i2} \dots T_{ij-1}T_{ij}T_{h_{ij}}f.$$

But $(T_{ij}T_{h_{ij}}T_{i1} \dots T_{ij-1}f, T_{i1}T_{i2} \dots T_{ij-1}g) = 0$, hence $(T_{0,i})_{jj} = \lambda I_j$. This shows that $T_{0,i} - \lambda I_i$ is nilpotent. Q.E.D.

COROLLARY 2.8. *Suppose φ_1, φ_2 are in H^∞ with inner-outer factorization $\varphi_i = g_iF_i$ $(i = 1, 2)$. If $g_i(z) = (a - z)^n(1 - \bar{a}z)^{-n}$ $(i = 1, 2; n \geq 0, |a| < 1)$, $F_1(a) = F_2(a)$ and $X: H^2 \to H^2$ is a bounded operator such that $XT_{\varphi_1} = T_{\varphi_2}X$ then $XT_{g_1} = T_{g_2}X$.*

PROOF. Factor $g_i = g_{i1}g_{i2} \dots g_{in}$ where $g_{ij}(z) = (a - z)(1 - \bar{a}z)^{-1}$, $i = 1, 2$ and $j = 1, 2, \dots, n$. Then $F_i - \lambda$ is divisible by each g_{ij}, where $\lambda = F_i(a)$. Applying Theorem 2.7 we obtain $XT_{g_1} = T_{g_2}X$. Q.E.D.

COROLLARY 2.9. *Suppose $\varphi_i = z^nF_i$ $(i = 1, 2)$, $F_1(0) = F_2(0)$ and $XT_{\varphi_1} = T_{\varphi_2}X$. Then $XT_{z^n} = T_{z^n}X$.*

Even though the following result looks like the uniqueness statement in the Riemann mapping theorem, we would like to point out that from the relation $T_{\varphi_1} \sim T_{\varphi_2}$ it only follows that the two sets $\varphi_1(\mathbb{D})$ and $\varphi_2(\mathbb{D})$ have the same closures and this does not convey any information about the equality of the two sets themselves. Also $G = \varphi_1(\mathbb{D})$ being simply connected might not have the property that $(\bar{G})^\circ = G$. Examples are easy to construct.

PROPOSITION 2.10. *Let $\varphi_1 \in H^\infty$ be univalent, $T_{\varphi_1} \sim T_{\varphi_2}$ and assume the normalization $\varphi_1(0) = \varphi_2(0)$, $\varphi_1'(0) = \varphi_2'(0) > 0$ holds. Then $\varphi_1 = \varphi_2$.*

PROOF. Since dim (ker $T^*_{\varphi_2-\lambda}$) = dim (ker $T^*_{\varphi_1-\lambda}$) = 1 or 0 for every $\lambda \in \mathbb{C}$ we conclude that the number of zeros of $\varphi_2 - \lambda$ in $\mathbb{D}$ is at most 1. Hence φ_2 is univalent. Also $XT_{\varphi_1} = T_{\varphi_2}X$ implies $XT_{\varphi_1-\varphi_1(0)} = T_{\varphi_2-\varphi_2(0)}X$. Let $\varphi_i - \varphi_i(0) = zF_i$ be the inner-outer factorizations. Since $F_1(0) = \varphi_1'(0) = \varphi_2'(0) = F_2(0)$, by Corollary 2.9 we obtain $XT_z = T_zX$. Hence $X = T_h$, h outer in H^∞. Since $XT_{\varphi_1} = T_{\varphi_2}X$ we conclude that $\varphi_1 h = \varphi_2 h$, so $\varphi_1 = \varphi_2$.

3. **Unitary Equivalence**. In this section we consider the following problem. If two analytic Toeplitz operators are quasisimilar, must they be unitarily equivalent? The answer to this question is still unknown. However, in certain special cases an improvement is possible. For example, Conway [3] has shown that if S is the unilateral shift of multiplicity one, $S \sim T_\varphi$ then $S \cong T_\varphi$ ($\varphi \in H^\infty$). We use results of Conway ([2], [4, p. 220] and Clary [1]) to obtain an extension of a result of Cowen [5]. Note that if u is an inner function, we say that the *order of* u is n if u is a finite Blaschke product of order n; otherwise we say the order of u is infinity. For any operator $A \in \mathcal{B}(\mathcal{H})$, let $P^\infty(A)$ denote the weak * closed subalgebra of $\mathcal{B}(\mathcal{H})$ generated by A. That is, $P^\infty(A)$ is the weak * closure in $\mathcal{B}(\mathcal{H})$ of the polynomials in A.

THEOREM 3.1. *Suppose φ and ψ are in H^∞ and there are inner functions u and v such that $P^\infty(T_\varphi) = P^\infty(T_u)$ and $P^\infty(T_\psi) = P^\infty(T_v)$. Then the following are equivalent:*

(*a*) $T_\varphi \cong T_\psi$,

(*b*) $T_\varphi \sim T_\psi$,

(*c*) *there are functions g in H^∞ and w_1, w_2 inner such that $\varphi = g \circ w_1$, $\psi = g \circ w_2$ and order w_1 = order w_2.*

PROOF. (*c*) implies (*a*) follows from Theorem 1 of [5], and (*a*) implies (*b*) is clearly true, so we only need to prove that (*b*) implies (*c*).

By hypothesis, there are inner functions u and v so that $P^\infty(T_\varphi) = P^\infty(T_u)$ and $P^\infty(T_\psi) = P^\infty(T_v)$. We will show that u and v have the same order. Let n = order u and m = order v. It is easy to see that $P^\infty(T_\varphi) = P^\infty(T_u) = \{T_{g\circ u} : g \in H^\infty\}$ and $P^\infty(T_\psi) = P^\infty(T_v) = \{T_{g\circ v} : g \in H^\infty\}$. Let X and Y be quasi-invertible operators such that $XT_\varphi = T_\psi X$ and $YT_\psi = T_\varphi Y$. By a result of Conway [2] (see also [4, p. 220]) there is an isometric isomorphism $F : P^\infty(T_\varphi) \to P^\infty(T_\psi)$ having the following properties

(1) $F(T_\varphi) = T_\psi$,

(2) $XA = F(A)X$ and $YF(A) = AY$ for every A in $P^\infty(T_\varphi)$, and

(3) F is a weak * homeomorphism.

Now F induces an algebra isomorphism $\Phi : H^\infty \circ u \to H^\infty \circ v$ given by $\Phi(g \circ u) = h \circ v$ where $T_{h\circ v} = F(T_{g\circ u})$. Let $w = \Phi(u) = q \circ v$ and $w_0 = \Phi^{-1}(v) = q_0 \circ u$. Since Φ is weak * continuous we have $v = \Phi(w_0) = \Phi(q_0 \circ u) = q_0 \circ w = q_0 \circ q \circ v$. Because v is inner, we have $q_0(q(z)) = z$, z in $\mathbb{D}$. Also $w = q \circ v = q \circ \Phi(w_0) = q \circ \Phi(q_0 \circ u) = q \circ q_0 \circ w$. Moreover since $T_u \sim T_w$, we have by a result of Clary [1], $\sigma(T_u) = \sigma(T_w)$ Hence $\overline{w(\mathbb{D})} = \overline{u(\mathbb{D})} = \overline{\mathbb{D}}$. Therefore $q(q_0(z)) = z$, z in $\mathbb{D}$. But

$\overline{q(\mathbb{D})} = \overline{q(v(\mathbb{D}))} = \overline{w(\mathbb{D})} = \overline{\mathbb{D}}$ and $\overline{q_0(\mathbb{D})} = \overline{\mathbb{D}}$. Hence q is a Möbius transformation of $\mathbb{D}$ onto $\mathbb{D}$ and w is an inner function of order m. But $T_u \sim T_v$ implies dim (ker T_u^*) = dim (ker T_w^*). Thus order u = order w, so $n = m$.

Since $T_\varphi \in P^\infty(T_\varphi)$, there is $g \in H^\infty$ such that $\varphi = g \circ u$. Since Φ is weak * continuous we have $\Phi(\varphi) = g \circ \Phi(u) = g \circ w$, but we have $F(T_\varphi) = T_\psi$ so $\Phi(\varphi) = \psi$. Therefore, we have $\varphi = g \circ u$ and $\psi = g \circ w$ where $g \in H^\infty$ and order u = order $w = n$ so the conclusion follows with $w_1 = u$ and $w_2 = w$.

Q.E.D.

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