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ON SOME P-ESTIMATES FOR BANACH SPACES

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Relations between l_p-type estimates of Khamsi and a uniform version of the Kadec-Klee property are studied. Khamsi's result on normal structure is strengthened.

INTRODUCTION

The notion of normal structure was introduced by Brodskii and Milman in [1]. Let us recall that a Banach space X has normal structure if every bounded convex subset A of X with $\operatorname{diam} A > 0$ contains an element x such that

$$\sup\{\|y - x\| : y \in A\} < \operatorname{diam} A.$$

This property turned out to have important applications in fixed point theory (see [5]) and many conditions were found which imply normal structure. One of them was defined in [8]. Namely for a Banach space X let us put

$$D(X) = \sup\{\limsup_{n\to\infty} \operatorname{dist}(x_{n+1}, \operatorname{co}\{x_k\}_{k=1}^{n})\},$$

where the supremum is taken over all sequences (x_n) in X with $\operatorname{diam}\{x_n\} = 1$. It was proved that if $D(X) < 1$, then X is reflexive and has normal structure [8]. Some classes of Banach spaces X for which $D(X) < 1$ were considered in [6].

Another geometric property stronger than normal structure was invented by Khamsi in [4]. The aim of this paper is to study relations between a property from [6] and that of Khamsi. As a consequence we shall see that for reflexive spaces Khamsi's property actually gives the condition $D(X) < 1$.

PRELIMINARIES

In [3] Huff introduced a generalisation of uniform convexity. He called it nearly uniform convexity.

A slight modification of a result from [3] shows that a Banach space X is nearly uniformly convex if and only if X is reflexive and for every $\epsilon > 0$ there exists $\delta > 0$

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such that whenever (x_n) is a sequence in the unit ball B_X of X with $x_n \xrightarrow{w} x$ and $\inf\{\|x_n - x\| : n \geqslant 1\} > \epsilon$, then

$$\|x\| \leqslant 1 - \delta.$$

Spaces which satisfy the last condition are called uniformly Kadec-Klee spaces. In [**2**] weakly uniformly Kadec-Klee spaces were defined. The following version of that definition agrees with the one given in [**6**].

A Banach space X is WUKK$'$ provided there exist constants $\epsilon, \delta \in (0, 1)$ such that if (x_n) is a sequence in B_X with $x_n \xrightarrow{w} x$ and $\inf\{\|x_n - x\| : x \geqslant 1\} > \epsilon$, then

$$\|x\| \leqslant 1 - \delta.$$

In this paper we shall deal mainly with Banach spaces which have finite dimensional decompositions (FDD in short). Let us recall that a sequence (X_n) of finite dimensional subspaces of a Banach space X is said to be an FDD of X if every $x \in X$ has a unique representation of the form

$$x = \sum_{k=1}^{\infty} x_k,$$

where $x_k \in X_k$, for all k.

The element $x \neq 0$ is called a *block* if the set $\operatorname{supp} x = \{k\colon x_k \neq 0\}$ is finite. The FDD (X_n) is *bimonotone* whenever

$$\max\{\|x\|, \|y\|\} \leqslant \|x + y\|$$

for any two blocks x, y with $\max\operatorname{supp} x < \min\operatorname{supp} y$.

The following definition is a modification of the condition which was considered in [**4**].

DEFINITION: A Banach space X with an FDD has property (K) if there exist constants $p \in [1, \infty)$, $\lambda \in (0, 2)$ such that

$$\|x\|^p + \|y\|^p \leqslant \lambda \|x + y\|^p$$

for any two blocks $x, y \in X$ with $\max\operatorname{supp} x + 1 < \min\operatorname{supp} y$.

Analysis of the James space (see [**4**]) shows that the property (K) is essentially weaker than that of Khamsi. However it is easy to see that all the results in [**4**] remain valid if one uses property (K).

Results

Let Y be a closed subspace of a Banach space X. We denote by $\operatorname{codim}_X Y$ the dimension of the quotient space X/Y.

Proposition 1. *If Y is a closed subspace of a Banach space X with $\operatorname{codim}_X Y < \infty$, then*

$$D(X) = D(Y).$$

Proof: Let Y be a closed subspace of X with $\operatorname{codim}_X Y < \infty$. Clearly $D(Y) \leqslant D(X)$.

In order to show the opposite inequality let us fix a positive $\epsilon < D(X)$ and choose a sequence (x_n) in X so that

$$D(X) - \epsilon \leqslant \limsup_{n\to\infty} \operatorname{dist}(x_{n+1}, \operatorname{co}\{x_k\}_{k=1}^n)$$

and $\operatorname{diam}\{x_n\} = 1$.

By our assumption there exists a finite dimensional subspace Z of X such that $X = Y \oplus Z$. Therefore for each n we have a decomposition $x_n = y_n + z_n$, where $y_n \in Y$, $z_n \in Z$. Moreover the sequence (z_n) is bounded. Passing to a subsequence, we can assume that $\|z_m - z_n\| \leqslant \epsilon$ for all m, n. Consequently

$$\|y_n - y_m\| \leqslant \|x_n - x_m\| + \|z_m - z_n\| \leqslant 1 + \epsilon$$

for all m, n. Hence $d = \operatorname{diam}\{y_n\} \leqslant 1 + \epsilon$ and clearly $d > 0$.

Let us put $y'_n = (1/d) y_n$. Then $\operatorname{diam}\{y'_n\} = 1$. Moreover for any nonnegative $\lambda_1, \ldots, \lambda_n$ with $\sum \lambda_i = 1$ we have

$$\begin{aligned}\left\|y'_{n+1} - \sum_{i=1}^n \lambda_i y'_i\right\| &\geqslant \frac{1}{d}\left(\left\|x_{n+1} - \sum_{i=1}^n \lambda_i x_i\right\| - \left\|z_{n+1} - \sum_{i=1}^n \lambda_i z_i\right\|\right) \\ &\geqslant \frac{1}{d}(\operatorname{dist}(x_{n+1}, \operatorname{co}\{x_i\}_{i=1}^n) - \epsilon).\end{aligned}$$

It follows that

$$\limsup_{n\to\infty} \operatorname{dist}\left(y'_{n+1}, \operatorname{co}\{y'_i\}_{i=1}^n\right) \geqslant \frac{1}{d}(D(X) - 2\epsilon).$$

In view of the definition of $D(Y)$ this shows that

$$D(Y) \geqslant \frac{1}{d}(D(X) - 2\epsilon) \geqslant \frac{1}{1+\epsilon}(D(X) - 2\epsilon).$$

Since $\epsilon > 0$ may be arbitrarily small, we finally obtain

$$D(Y) \geqslant D(X).$$

□

Theorem 2. *Let X be a Banach space with an FDD. If X has property* (K), *then X is* WUKK'.

Proof: Let us assume that X has property (K) and let $p \geqslant 1$, $\lambda \in (0, 2)$ be as in Definition 1.

Consider a sequence (x_n) in B_X with $x_n \xrightarrow{w} x$ and $\inf\{\|x_n - x\| : n \geqslant 1\} > \epsilon$, where $\epsilon = (\lambda/2)^{1/p}$. Given $\eta > 0$, we can find a block $u \in X$ for which $\|x - u\| < \eta$. Since the sequence $(x_n - x)$ converges weakly to zero, there exists an index n and a block v such that $\max\operatorname{supp} u + 1 < \min\operatorname{supp} v$ and $\|x_n - x - v\| < \eta$. By our assumption we have

$$\|u\|^p + \|v\|^p \leqslant \lambda \|u + v\|^p .$$

Therefore

$$(\|x\| - \eta)^p + (\|x_n - x\| - \eta)^p \leqslant \lambda(\|x_n\| + 2\eta)^p.$$

It follows that

$$(\|x\| - \eta)^p + (\epsilon - \eta)^p \leqslant \lambda(1 + 2\eta)^p.$$

Passing to the limit with $\eta \to 0$, we finally get

$$\|x\|^p \leqslant \lambda - \epsilon^p.$$

Thus it suffices to put $\delta = 1 - (\lambda/2)^{1/p}$. □

In the particular case of reflexive spaces the above results and [**6**, Theorem 4] give us the following strengthened version of Theorem 3 of [**4**].

Corollary 1. *Let Y be a closed subspace with an* FDD *of a reflexive space X. If $\operatorname{codim}_X Y < \infty$ and Y has property* (K), *then*

$$D(X) < 1.$$

Now we turn to a result which is a partial converse of Theorem 2. For stating it we first need to recall the following two notions.

Let (X_n) be an FDD of a Banach space X. A sequence (Y_n) of finite dimensional subspaces of X is called a blocking of (X_n) if there exists an increasing sequence of integers (n_k) with $n_1 = 0$ such that

$$Y_k = X_{n_k+1} \oplus \cdots \oplus X_{n_{k+1}}$$

for all k. Clearly the sequence (Y_k) is an FDD of X.

An FDD (X_n) of a space X is shrinking (see [**7**] for the definition) if and only if every bounded sequence of blocks (z_n), with $k \leqslant \min\operatorname{supp} z_k$ for all k, is weakly convergent to zero. This is the case for example if the space X is reflexive.

Proposition 3. *Let X be a Banach space with a shrinking* FDD (X_n). *If X is* WUKK$'$, *then there exists a constant $c > 0$ and a blocking (Y_k) of (X_n) such that*

$$1 + c \leqslant \|x + y\|$$

whenever $x, y \in X$ are two blocks with $\|x\| = \|y\| = 1$ and $\max\operatorname{supp} x + 1 < \min\operatorname{supp} y$, *where the supports are taken with respect to (Y_k).*

Proof: If X is WUKK$'$, then from the definition we obtain some constants $\epsilon, \delta \in (0, 1)$. If suffices to show that there exists a constant $c > 0$ which satisfies the following property (compare to [9]). For every n there is $m > n$ such that

$$1 + c \leqslant \|x + y\|$$

whenever $\|x\| = \|y\| = 1$ and $\max\operatorname{supp} x \leqslant n$, $m \leqslant \min\operatorname{supp} y$, where the supports are taken with respect to (X_n).

Let us assume the contrary. Then for every positive $c < 1/(\max\{\epsilon, 1 - \delta\}) - 1$ we can find an integer n and two sequences of blocks (x_k), (y_k) such that

$$\|x_k + y_k\| < 1 + \frac{1}{2}c,$$

$\|x_k\| = \|y_k\| = 1$, $\max\operatorname{supp} x_k \leqslant n$ and $k \leqslant \min\operatorname{supp} y_k$ for all k.

The sequence (x_k) is contained in the unit sphere of a finite dimensional subspace of X. Consequently, passing to a subsequence, we can assume that it converges to some x.

Let us consider a sequence of elements $z_k = (1 + c)^{-1}(x + y_k)$. By our assumption it converges weakly to $z = (1 + c)^{-1}x$. Moreover $\|z_k - z\| = (1 + c)^{-1} > \epsilon$ for all k and $\|z_k\| \leqslant 1$ for sufficiently large k. On the other hand $\|z\| = (1 + c)^{-1} > 1 - \delta$, which contradicts the assumption that X is WUKK$'$. ∎

Theorem 4. *Let X be a Banach space with a bimonotone shrinking FDD (X_n). If X is WUKK$'$, then there exists a blocking (Y_k) of (X_n) such that the space X with the FDD (Y_k) has property (K).*

Proof: Let us assume that X is WUKK$'$ and let (Y_k) and $c > 0$ be as in Proposition 3. We shall prove that X has a property which is even stronger than (K).

Let us fix an arbitrary $p \in [1, \infty)$ and two blocks x, y such that $\max\operatorname{supp} x + 1 < \min\operatorname{supp} y$, where the supports are taken with respect to (Y_k). We shall show that

$$\|x\|^p + \|y\|^p \leqslant (1 + a^p)\,\|x + y\|^p,$$

where $a = (1 + c)^{-1} < 1$.

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To this end let us put $t = \|x\|$, $s = \|y\|$, $x' = (1/t)x$ and $y' = (1/s)y$. In the case when $t/s \leqslant a$ we apply the assumption that (X_n) is bimonotone. Namely

$$\|x + y\|^p \geqslant s^p \geqslant (t^p + s^p)(1 + a^p)^{-1}.$$

Let us in turn assume that $a < t/s \leqslant 1$. By Proposition 3 we obtain

$$\begin{aligned}\|x + y\|^p &= s^p \left\| \frac{t}{s} x' + y' \right\|^p \\ &\geqslant s^p \left(\|x' + y'\| - 1 + \frac{t}{s} \right)^p \geqslant s^p \left(\frac{1}{a} - 1 + \frac{t}{s} \right)^p \\ &\geqslant \left(\frac{t}{a} \right)^p \geqslant (t^p + s^p)(1 + a^p)^{-1}.\end{aligned}$$

The remaining cases may be handled in a similar way. □

The assumption that an FDD is bimonotone cannot be omitted. In general even nearly uniform convexity does not imply property (K).

EXAMPLE. Let us consider a norm in the plane given by the formula

$$\|(\alpha, \beta)\|_0 = \max\left\{ \left| \frac{1}{4}\alpha + \beta \right|, \left(\alpha^2 + \left(\frac{1}{2}\beta \right)^2 \right)^{1/2} \right\},$$

where α, β are real numbers.

By X we denote the space l_2 with the following equivalent norm

$$\|x\| = \sup\left\{ \left(\|(\alpha_1, \alpha_k)\|_0^2 + \left(\frac{1}{2} \|Q_k x\|_2 \right)^2 \right)^{1/2} : k > 1 \right\},$$

where $x = (\alpha_n) \in l_2$, $Q_k x = (\alpha_n)_{n>k}$ and $\|\cdot\|_2$ is the l_2-norm.

We shall show that X is nearly uniformly convex and does not have property (K) for any FDD. For this purpose let us fix $\epsilon > 0$ and consider a sequence (x_n) in B_X weakly convergent to x with $\inf\{\|x_n - x\| : n \geqslant 1\} > \epsilon$. Since $(x_n - x)$ converges weakly to zero, for every $\eta > 0$ we can find an index n and two blocks u, v such that

$$\|x - u\| < \eta,\ \|x_n - x - v\| < \eta$$

and $\max \operatorname{supp} u < \min \operatorname{supp} v$, where the supports are taken with respect to the natural basis of X. Clearly

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$$\|u\|^2 + \left(\frac{1}{2}\|v\|\right)^2 \leqslant \|u\|^2 + \left(\frac{1}{2}\|v\|_2\right)^2 \leqslant \|u+v\|^2.$$

Therefore

$$(\|x\| - \eta)^2 + \left(\frac{1}{2}(\|x_n - x\| - \eta)\right)^2 \leqslant (\|x_n\| + 2\eta)^2$$

and consequently

$$(\|x\| - \eta)^2 + \left(\frac{1}{2}(\epsilon - \eta)\right)^2 \leqslant (1 + 2\eta)^2.$$

By passing to the limit with $\eta \to 0$, it follows that

$$\|x\| \leqslant \left(1 - \left(\frac{1}{2}\epsilon\right)^2\right)^{1/2}.$$

This shows that the space X is uniformly Kadec-Klee. Since X is reflexive, it is also nearly uniformly convex.

Let us now assume that the space X with some FDD has property (K) with constants $p \geqslant 1$ and $\lambda \in (0, 2)$. We consider the elements

$$x = \left(-\frac{4}{5}, 0, \dots\right), \quad y_n = \left(0, \dots, 0, \frac{6}{5}, 0, \dots\right),$$

where $6/5$ is the $(n+1)$th coordinate. Straightforward computations show that $\|x\| = 4/5$, $\|y_n\| = 6/5$ and $\|x + y_n\| = 1$ for all n. Moreover the sequence (y_n) converges weakly to zero. Therefore an argument similar to that in the proof of Theorem 2 gives

$$\left(\frac{4}{5}\right)^p + \left(\frac{6}{5}\right)^p \leqslant \lambda,$$

which contradicts the assumption that $\lambda < 2$.

Let us remark that, using an argument from [**6**], one can show that the space X has so called property (β) (see [**10**]) which is even stronger than nearly uniform convexity.

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