BULL. AUSTRAL. MATH. SOC. MOS 80A20
VOL. 6 (1972), 83-94.

Laminar forced convection at low Péclet number

A.S. Jones

This work is concerned with the forced convection of heat in a circular tube. The fluid flow is assumed to be laminar Poiseuille flow, and the physical parameters; viscosity, density, conductivity; are assumed to be independent of temperature changes. Viscous dissipation terms are also ignored, and there are no heat sources in the fluid. The problem is treated for the case of a step change in the wall temperature, and the eigenvalues have been obtained as an expansion in powers of the Péclet number for the smaller values, and in an asymptotic form for the larger values. The temperature distribution in the fluid in the neighbourhood of the temperature jump has been calculated for two values of the Péclet number, as have the Nusselt numbers.

1. Introduction

This problem has received a great deal of attention in the case where the Péclet number is large, but not when it is small. Singh [5] calculated the negative eigenvalues for $Pé = 1$, while Abramowitz, Cahill and Wade [1] calculated the eigenvalues when $Pé = \frac{1}{2}$ together with the coefficients in the expansion of the solution. These coefficients, however, were calculated on the dubious assumption that there is no preheating of the fluid. Millsaps and Pohlhausen [3] calculated the eigenvalues and thermal properties for $Pé = 1$, but they too assumed no preheating of the fluid.

In the two-dimensional case, Agrawal [2] included the effects of

Received 13 September 1971.

preheating and calculated eigenvalues, Nusselt numbers and temperature profiles for $P\acute{e} = 1$. One criticism is that the method adopted for matching the temperature distribution upstream and downstream is unnecessarily complicated when compared with the Laplace transform method outlined below.

2. Governing equations and their solution

For the case of Poiseuille flow in a circular tube of radius a , the axi-symmetric conduction convection equation is

(2.1) $$\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{\partial^2 T}{\partial x^2} = \frac{2u_m \rho^* c_v}{\kappa}\left(1 - \frac{r^2}{a^2}\right)\frac{\partial T}{\partial x} ,$$

where

T is the fluid temperature,

u_m is the mean fluid velocity,

ρ^* is the fluid density,

c_v is the specific heat of the fluid, and

κ is the thermal conductivity of the fluid.

The variables r, x are the usual radial and axial variables in cylindrical polar coordinates, and the angular variable disappears because of the symmetry of the problem.

The boundary conditions imposed on equation 2.1 are

(2.2)
$$\begin{aligned} T &= T_0 \ ; \quad r = a \ , \quad x < 0 \\ T &= T_1 \ ; \quad r = a \ , \quad x > 0 \\ T &\to T_0 \ ; \quad x \to -\infty \ , \quad r < a \\ T &\to T_1 \ ; \quad x \to +\infty \ , \quad r < a \ . \end{aligned}$$

The equation and boundary conditions are made non-dimensional by putting

$$\rho = r/a \ , \quad \xi = x/a \ , \quad \theta = (T - T_0)/(T_1 - T_0) \ ,$$

giving

(2.3) $$\frac{\partial^2 \theta}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial \theta}{\partial \rho} + \frac{\partial^2 \theta}{\partial \xi^2} = 2P\acute{e}(1-\rho^2)\frac{\partial \theta}{\partial \xi}$$

with the boundary conditions

(2.4)
$$\begin{aligned} \theta &= 0\ ; & \rho &= 1\ , & \xi &< 0 \\ \theta &= 1\ ; & \rho &= 1\ , & \xi &> 0 \\ \theta &\to 0\ ; & \xi &\to -\infty\ , & \rho &< 1 \\ \theta &\to 1\ ; & \xi &\to +\infty\ , & \rho &< 1\ , \end{aligned}$$

where $P\acute{e}$ is the non-dimensional Péclet number(*) $u_m \rho^* c_v a/\kappa$.

The formal solution of equation (2.3) is obtained by using the double-sided Laplace transform.

Writing

$$\overline{\theta}(p,\ \rho) = \int_{-\infty}^{\infty} e^{-p\xi}\theta(\xi,\ \rho)d\xi\ ,$$

we obtain

(2.5)
$$\frac{\partial^2\overline{\theta}}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial\overline{\theta}}{\partial\rho} + \left(p^2 - 2pP\acute{e}(1-\rho^2)\right)\overline{\theta} = 0\ ,$$

with boundary conditions

(2.6)
$$\overline{\theta}(p,\ 1) = 1/p\ .$$

Equation (2.5) is a second order linear ordinary differential equation for $\overline{\theta}$ with variable ρ and parameters p, $P\acute{e}$. The equation has a regular singular point at $\rho = 0$ with indices $0,\ 0$. Consequently there is one independent solution as a power series in ρ , all other solutions having a logarithmic singularity at $\rho = 0$. Since the temperature will remain finite at $\rho = 0$, we can determine a unique solution $f(\rho;\ p,\ P\acute{e})$ of equation (2.5) from the condition $f(0;\ p,\ P\acute{e}) = 1$. The required solution for $\overline{\theta}$, satisfying the boundary conditions is then

(2.7)
$$\overline{\theta} = \frac{1}{p}\frac{f(\rho;p,P\acute{e})}{f(1;p,P\acute{e})}\ .$$

The function θ is recovered by using the inverse Laplace transform

$$\theta(\xi,\ \rho) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{\xi p}\overline{\theta}(p,\ \rho)dp\ ,$$

where

(*) Some authors define the Péclet number to be twice this value.

https://doi.org/10.1017/S0004972700044294 Published online by Cambridge University Press

$$0 < c < \text{the first positive pole of } \bar{\theta} .$$

This gives

$$(2.8) \qquad \theta(\xi, \rho) = 1 + \sum_{n=1}^{\infty} \frac{e^{\beta_n \xi}}{\beta_n} \cdot \frac{f(\rho; \beta_n, P\acute{e})}{\frac{\partial f}{\partial p}(1; \beta_n, P\acute{e})} , \quad \xi > 0$$

$$= - \sum_{n=1}^{\infty} \frac{e^{\alpha_n \xi}}{\alpha_n} \cdot \frac{f(\rho; \alpha_n, P\acute{e})}{\frac{\partial f}{\partial p}(1; \alpha_n, P\acute{e})} , \quad \xi < 0 ,$$

where $0 < \alpha_1 < \alpha_2 \ldots$ are the positive zeros of $f(1; p, P\acute{e})$ and $0 > \beta_1 > \beta_2 \ldots$ are the negative zeros.

3. Expansion of the solution function $f(\rho; p, P\acute{e})$

Assuming that the Péclet number is small, we solve equation (2.5) by expanding f in powers of $P\acute{e}$.

Putting $f(\rho; p, P\acute{e}) = \sum_{n=0}^{\infty} (P\acute{e})^n f_n(\rho, p)$, we obtain

$$(3.1) \qquad \frac{d^2 f_0}{d\rho^2} + \frac{1}{\rho} \frac{df_0}{d\rho} + p^2 f_0 = 0$$

and

$$(3.2) \qquad \frac{d^2 f_n}{d\rho^2} + \frac{1}{\rho} \frac{df_n}{d\rho} + p^2 f_n = 2p(1-\rho^2) f_{n-1} .$$

Equation (3.1) has the obvious solution $f_0(\rho, p) = J_0(\rho p)$, and substituting

$$(3.3) \qquad f_n(\rho, p) = \sum_{m=0}^{n} F_{nm} J_{n+m}(\rho p)$$

in equation (3.2) gives (after a tedious but otherwise straightforward calculation)

$$(3.4) \qquad F_{nm} = \frac{2^m}{3^m p^n} \sum_{s=0}^{m} (-\tfrac{1}{3})^s \gamma_{ms} \rho^{2m+3s} \phi_{n-m-s}(\rho) ,$$

where

https://doi.org/10.1017/S0004972700044294 Published online by Cambridge University Press

(3.5) $$\phi_t(\rho) = (\rho-\rho^3/3)^t/t!$$

and the γ_{ms} satisfy the recurrence relation

(3.6) $$\gamma_{ms} = \left(\gamma_{m-1,s-1}+(2s+1)\gamma_{m-1,s}+(s+1)^2\gamma_{m-1,s+1}\right)\Big/(2r+3s)$$

with $\gamma_{00} = 1$, $\gamma_{ms} = 0$ if $s > m$, $s < 0$ or $m < 0$.

It can be shown that

$$\gamma_{rr} = \frac{1}{5^r r!}$$

$$\gamma_{r+1,r} = \frac{1}{5^r r!}\left(\frac{2}{7}r + \frac{1}{2}\right)$$

$$\gamma_{r+2,r} = \frac{1}{5^r r!}\left(\frac{2}{49}r^2 + \frac{46}{245}r + \frac{7}{40}\right)$$

and in general

$$\gamma_{r+t,r} = \frac{1}{5^r r!}\left(a_{tt}r^t + \ldots + a_{t0}\right) ,$$

where $a_{tt} = (\frac{2}{7})^t.\frac{1}{t!}$. Hence

(3.7) $f(\rho;\ p;\ P\acute{e}) =$

$$\sum_{n=0}^{\infty} (P\acute{e})^n \sum_{m=0}^{n} \sum_{s=0}^{m} (-\tfrac{1}{3})^s \gamma_{ms}\rho^{2m+3s}\phi_{n-m-s}(\rho)\,\frac{2^m}{3^m p^m}\,J_{n+m}(\rho p) .$$

Rearranging the order of summation and using Lommel's expansion

(3.8) $$(z+h)^{-\frac{1}{2}\nu}J_\nu\left((z+h)^{\frac{1}{2}}\right) = \sum_{m=0}^{\infty} (-\tfrac{1}{2}h)^m z^{-\frac{1}{2}(\nu+m)}J_{\nu+m}\left(z^{\frac{1}{2}}\right)/m! ,$$

we obtain finally

(3.9) $$f(\rho;\ p,\ P\acute{e}) = \sum_{m=0}^{\infty} 2^m \sum_{s=0}^{m} (-1)^s (P\acute{e}.p.\rho^4/3)^{m+s}\gamma_{ms}J_{2m+s}(y)/y^{2m+s} ,$$

where

(3.10) $$y^2 = p^2\rho^2 - 2pP\acute{e}(\rho^2-\rho^4/3) .$$

https://doi.org/10.1017/S0004972700044294 Published online by Cambridge University Press

Since $\left|J_n(y)/y^n\right|$ has a maximum of $(\frac{1}{2})^n/n!$ and is $O\left(y^{-n-\frac{1}{2}}\right)$ as $y \to \infty$, it is apparent that this series solution converges rapidly.

4. Zeros of $f(1;\, p,\, P\acute{e})$

When $\rho = 1$, the expression (3.9) gives

$$f(1;\, p,\, P\acute{e}) \sim J_0\left((p - \tfrac{4}{3}\, pP\acute{e})^{\frac{1}{2}}\right) . \tag{4.1}$$

Writing c_n for the n-th positive zero of $J_0(x)$ we obtain immediately the asymptotic expression

$$\alpha_n,\ \beta_n \sim \pm\, c_n\left(1+\left(2P\acute{e}/3c_n\right)^2\right)^{\frac{1}{2}} + 2P\acute{e}/3 . \tag{4.2}$$

Alternately, substituting

$$\alpha_n = \alpha_{n0} + P\acute{e}.\alpha_{n1} + (P\acute{e})^2\alpha_{n2} + \ldots \tag{4.3}$$

in (3.7) and equating powers of $P\acute{e}$, we obtain

$$\begin{aligned} \alpha_{n0} &= c_n \\ \alpha_{n1} &= \tfrac{3}{4} + J_3\left(c_n\right)/12J_1\left(c_n\right) \\ &\sim \tfrac{2}{3} \quad \text{as} \quad n \to \infty , \end{aligned} \tag{4.4}$$

with similar but more complicated expressions for the following terms.

The equivalent expansion for β_n is obviously

$$\beta_n = -\,\alpha_{n0} + P\acute{e}\alpha_{n1} - (P\acute{e})^2\alpha_{n2} + \ldots . \tag{4.5}$$

These formulae have been used to calculate the values of $\alpha_{n1}, \alpha_{n2}, \alpha_{n3}$ for $n = 1, 8$. It will be seen from these results that for small values of $P\acute{e}$, these first four terms in the expansion appear to be adequate.

Putting $\rho = 1$, $y^2 = p^2 - \frac{4}{3}\, pP\acute{e}$ in (3.9) and differentiating with respect to p, we obtain

$$\left.\frac{\partial f}{\partial p}\right|_{\rho=1} = \sum_{m=0}^{\infty} 2^m \sum_{s=0}^{m} (-1)^s \gamma_{m,s}\left(\frac{pP\acute{e}}{3}\right)^{m+s}\left[\frac{m+s}{p}\,\frac{J_{2m+s}(y)}{y^{2m+s}} - (p - \tfrac{2}{3}\, P\acute{e})\,\frac{J_{2m+s+1}(y)}{y^{2m+s+1}}\right] , \tag{4.7}$$

from which the terms up to $m = 5$ were used to calculate the coefficients in (2.8) for the first eight positive and negative eigenvalues corresponding to $P\acute{e} = \frac{1}{2}$ and $P\acute{e} = 1$.

Differentiating (3.9) with respect to ρ we obtain

$$(4.8)\quad \left.\frac{\partial f}{\partial \rho}\right|_{\rho=1} =$$

$$p \sum_{m=0}^{\infty} 2^m \sum_{s=0}^{m} (-1)^s \gamma_{m,s} \left(\frac{pP\acute{e}}{3}\right)^{m+s} \left[\frac{4(m+s)}{p} \frac{J_{2m+s}(y)}{y^{2m+s}} - (p - \tfrac{2}{3} P\acute{e}) \frac{J_{2m+s+1}(y)}{y^{2m+s+1}}\right]$$

from which $\left.\frac{\partial f}{\partial \rho}\right|_{\rho=1}$ can be calculated in a similar fashion. Since the dominant term in the expansion is given by $m = 0$, $s = 0$, we see that for large p ,

$$\left.\frac{\partial f}{\partial \rho}\right|_{\rho=1} \sim p \left.\frac{\partial f}{\partial p}\right|_{\rho=1} .$$

The eigenfunctions were calculated by rewriting equation (2.5) in the form

$$(4.9)\qquad f(\rho) = 1 - \int_0^{\rho} s \log(\rho/s)\left(p^2 - 2pP\acute{e}(1-s^2)\right) f(s)ds$$

and integrating numerically.

The integral $\int_0^1 \rho(1-\rho^2)f(\rho)d\rho$ was calculated simultaneously. This integral is required for evaluating the mean mixed temperature

$$\theta_M = 4 \int_0^1 \rho(1-\rho^2)\theta(\xi, \rho)d\rho$$

from which the Nusselt number

$$Nu = \frac{2\frac{\partial\theta}{\partial\rho}}{\theta_W - \theta_M}$$

is calculated, θ_W being the appropriate wall temperature.

The results of these computations are given in the tables and graphs

https://doi.org/10.1017/S0004972700044294 Published online by Cambridge University Press

below. The limiting Nusselt numbers as $\xi \to \infty$ agree with those given by Pahor and Strnad [4].

In conclusion, it will be noted that the incoming fluid is significantly pre-heated and hence that the assumption that the temperature is constant for $x < 0$ is not valid for low Péclet numbers.

Table 1. Coefficients in the expansion $\alpha_n = \alpha_{n0} + Pé\alpha_{n1} + \ldots$

n	α_{n1}	α_{n2}	α_{n3}
1	.78194	.12112	-.00405
2	.68855	.04697	.00172
3	.67557	.03039	.00072
4	.67146	.02244	.00038
5	.66966	.01777	.00024
6	.66871	.01471	.00016
7	.66815	.01254	.00011
8	.66779	.01093	.00009

https://doi.org/10.1017/S0004972700044294 Published online by Cambridge University Press

Table 2. Eigenvalues and coefficients for $P\acute{e} = 1$

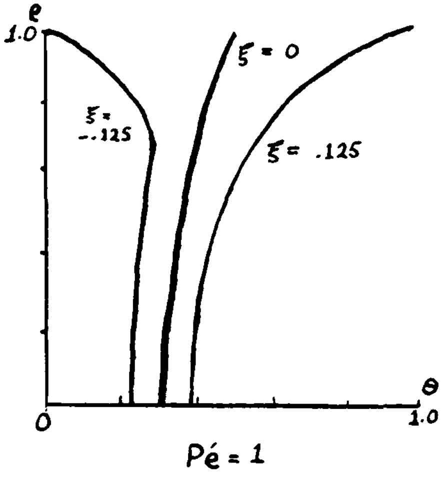
p	$\frac{\partial f}{\partial p}$	$1/p\ \frac{\partial f}{\partial p}$	$\frac{\partial f}{\partial \rho}$	$\int_0^1 \rho(1-\rho^2)f(\rho)d\rho$
3.29957	-.41890	-.58406	-1.5136	.16056
6.27554	.32531	.49126	1.9986	.00094
9.36040	-.26256	-.40689	-2.4425	.00072
12.4858	.22592	.35451	2.7886	-.00032
15.6186	-.20229	-.31651	-3.1539	.00016
18.7546	.18448	.28903	3.4559	-.00009
21.8924	-.17068	-.26762	-3.7337	.00005
25.0313	.15958	.25034	3.9923	-.00003
-1.74386	.56511	-1.01474	-1.1299	.14403
-4.87694	-.36041	.56895	1.7732	-.01433
-8.00782	.28227	-.44240	-2.2618	.00289
-11.1421	-.24054	.37312	2.6758	-.00090
-14.2788	.21134	-.33138	-3.0163	.00036
-17.4169	-.19132	.30011	3.3307	-.00018
-20.5559	.17608	-.27629	-3.6182	.00010
-23.6955	-.16398	.25736	3.8845	-.00006

Table 3. Eigenvalues and coefficients for $P\acute{e} = .5$

p	$\frac{\partial f}{\partial p}$	$1/p \frac{\partial f}{\partial p}$	$\frac{\partial f}{\partial \rho}$	$\int_0^1 \rho(1-\rho^2)f(\rho)d\rho$
2.82530	-.51374	-.68895	-1.3550	.15392
5.87631	.33201	.51256	1.9368	-.00398
8.99920	-.26675	-.41658	-2.3951	.00115
12.1329	.22942	.35926	2.7806	-.00043
15.2702	-.20435	-.32047	-3.1186	.00020
18.4091	.18606	.29195	3.4240	-.00011
21.5489	-.17195	-.26989	-3.7043	.00006
24.6891	.16062	.25217	3.9649	-.00004
-2.04437	.56319	-.91227	-1.1773	.14616
-5.18733	-.34979	.55113	1.8236	-.01151
-8.32345	.27664	-.43430	-2.3046	.00227
-11.4614	-.23496	.37134	2.6535	-.00072
-14.6005	.20888	-.32790	-3.0498	.00030
-17.7404	-.18948	.29749	3.3614	-.00015
-20.8807	.17464	-.27422	-3.6465	.00008
-24.0213	-.16282	.25568	3.9110	-.00005

Table 4. Mean mixed temperatures and Nusselt numbers

$Pé = 1$				$Pé = .5$			
ξ	θ_M	$\frac{\partial\theta}{\partial\rho}$	Nu	ξ	θ_M	$\frac{\partial\theta}{\partial\rho}$	Nu
-10.0	.0000	-.0000	4.7135	-10.0	.0000	-.0000	4.4016
-1.0	.0138	-.0346	4.9995	-1.0	.0252	-.0583	4.6291
-.5	.0720	-.2244	6.2358	-.5	.1037	-.2939	5.6662
-.1	.2694	-2.3903	17.7484	-.1	.3254	-2.5342	15.5779
0	.3754	∞	∞	0	.4355	∞	∞
.1	.4860	+2.9958	11.6569	.1	.5480	2.8386	12.5595
.5	.7526	+.5905	4.7791	.5	.8061	.4812	4.9643
1.0	.8995	.2085	4.0697	1.0	.9308	.1449	4.1888
10.0	1.0000	.0000	3.9224	10.0	1.0000	.0000	4.0027



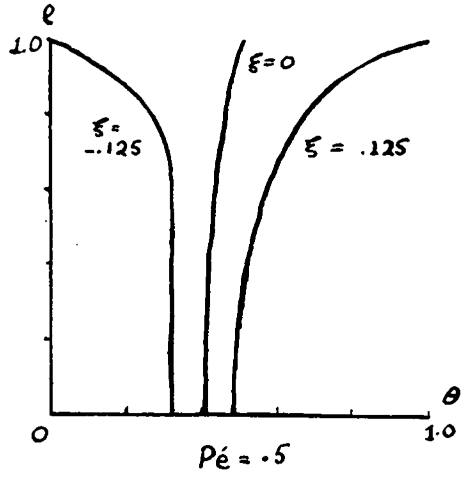


Figure 1. Thermal profiles near the origin. The profiles for $\xi = 0$ are the means of the limits $\xi \to 0+$ and $\xi \to 0-$.

https://doi.org/10.1017/S0004972700044294 Published online by Cambridge University Press

References

[1] M. Abramowitz, W.F. Cahill and C. Wade, Jr, "Heat transfer in laminar flow through a tube", *J. Res. Nat. Bur. Standards* 62 (1959), 101-105.

[2] H.C. Agrawal, "Heat transfer in laminar flow between parallel plates at small Péclét numbers", *Appl. Sci. Res.* A9 (1960), 177-189.

[3] Knox Millsaps and Karl Pohlhausen, "Heat transfer to Hagen-Poiseuille flows", *Proc. Conf. Differential Equations, Univ. Maryland, Maryland*, 1955, 271-294 (University of Maryland Bookstore, College Park, Maryland, 1956).

[4] Sergej Pahor and Janez Strnad, "Heat transfer from laminar flow through cylindrical tubes", *Z. angew. Math. Phys.* 12 (1961), 80-81.

[5] S.N. Singh, "Heat transfer by laminar flow in a cylindrical tube", *Appl. Sci. Res.* A7 (1958), 325-340.

Department of Mathematics,
University of Queensland,
St Lucia,
Queensland.