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Almost-Free E-Rings of Cardinality $\aleph_1$

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Abstract. An E-ring is a unital ring R such that every endomorphism of the underlying abelian group R^+ is multiplication by some ring element. The existence of almost-free E-rings of cardinality greater than $2^{\aleph_0}$ is undecidable in ZFC. While they exist in Gödel's universe, they do not exist in other models of set theory. For a regular cardinal $\aleph_1 \leq \lambda \leq 2^{\aleph_0}$ we construct E-rings of cardinality λ in ZFC which have $\aleph_1$-free additive structure. For $\lambda = \aleph_1$ we therefore obtain the existence of almost-free E-rings of cardinality $\aleph_1$ in ZFC.

1 Introduction

Recall that a unital ring R is an E-ring if the evaluation map $\varepsilon\colon \operatorname{End}_{\mathbb{Z}}(R^+) \to R$ given by $\varphi \mapsto \varphi(1)$ is a bijection. Thus every endomorphism of the abelian group R^+ is multiplication by some element $r \in R$. E-rings were introduced by Schultz [20] and easy examples are subrings of the rationals $\mathbb{Q}$ or pure subrings of the ring of p-adic integers. Schultz characterized E-rings of finite rank and the books by Feigelstock [9, 10] and an article [18] survey the results obtained in the eighties, see also [8, 19]. In a natural way the notion of E-rings extends to modules by calling a left R-module M an $E(R)$-module or just E-module if $\operatorname{Hom}_{\mathbb{Z}}(R, M) = \operatorname{Hom}_R(R, M)$ holds, see [1]. It turned out that a unital ring R is an E-ring if and only if it is an E-module.

E-rings and E-modules have played an important role in the theory of torsion-free abelian groups of finite rank. For example Niedzwecki and Reid [17] proved that a torsion-free abelian group G of finite rank is cyclically projective over its endomorphism ring if and only if $G = R \oplus A$, where R is an E-ring and A is an $E(R)$-module. Moreover, Casacuberta and Rodríguez [2] noticed the role of E-rings in homotopy theory.

It can be easily seen that every E-ring has to be commutative and hence can not be free as an abelian group except when $R = \mathbb{Z}$. But it was proved in [6] and extended in [4, 5], using a Black Box argument from [3], that there exist arbitrarily large E-rings R which are $\aleph_1$-free as abelian groups, which means that every countable subgroup of R^+ is free. The smallest candidate in [4, 5, 6] has size $2^{\aleph_0}$. This implies the existence of $\aleph_1$-free E-rings of cardinality $\aleph_1$ under the assumption of the continuum hypothesis. Moreover, it was shown in [16] that there exist almost-free E-rings for any regular not weakly compact cardinal $\kappa > \aleph_0$ assuming diamond, a prediction principle which

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holds for example in Gödel's constructible universe. Here, a group of cardinality λ is called *almost-free* if all its subgroups of smaller cardinality than λ are free.

Since the existence of $\aleph_2$-free E-rings of cardinality $\aleph_2$ is undecidable in ordinary set theory ZFC (see [15, Theorem 5.1] and [16]) it is hopeless to conjecture that there exist almost-free E-rings of cardinality κ in ZFC for cardinals κ larger than $2^{\aleph_0}$. However, we will prove in this paper that there are $\aleph_1$-free E-rings in ZFC of cardinality λ for every regular cardinal $\aleph_1 \leq \lambda \leq 2^{\aleph_0}$. Thus the existence of almost-free E-rings of size $\aleph_1$ in ZFC follows.

The construction of $\aleph_1$-free E-rings R in ZFC is much easier if $|R| = 2^{\aleph_0}$, because in case $|R| = \aleph_1$ we are closer to freeness, a property which tries to prevent endomorphisms from being scalar multiplication. Thus we need more algebraic arguments and will utilize a combinatorial prediction principle similar to the one used by the first two authors in [14] for constructing almost-free groups of cardinality $\aleph_1$ with prescribed endomorphism rings.

The general method for such constructions is very natural and it will be explained in full detail in Shelah [21, Chapter VII, Section 5]. Our notations are standard and for unexplained notions we refer to [11, 12, 13] for abelian group theory and to [7] for set-theory. All groups under consideration are abelian.

2 Topology, Trees and a Forest

In this section we explain the underlying geometry of our construction which was used also in [14], see there for further details.

Let F be a fixed countable principal ideal domain with $1 \neq 0$ with a fixed infinite set $S = \{s_n : n \in \omega\}$ of pair-wise coprime elements, that is $s_nF + s_mF = F$ for all $n \neq m$. For brevity we will say that F is a *p-domain*, which certainly cannot be a field. We choose a sequence of elements

$$q_0 = 1 \text{ and } q_{n+1} = s_n q_n \quad \text{for all } n \in \omega \tag{2.1}$$

in F, hence the descending chain q_nF $(n \in \omega)$ of principal ideals satisfies $\bigcap_{n\in\omega} q_nF = 0$ and generates the Hausdorff S-topology on F. Thus F is a dense and S-pure subring of its S-adic completion $\hat{F}$ satisfying $q_nF = q_n\hat{F} \cap F$ for all $n \in \omega$.

Now let $T = {}^{\omega>}2$ denote the tree of all finite branches $\tau\colon n \to 2$ $(n \in \omega)$. Moreover, ${}^{\omega}2 = \mathrm{Br}(T)$ denotes all infinite branches $\eta\colon \omega \to 2$ and clearly $\eta\restriction_n \in T$ for all $\eta \in \mathrm{Br}(T)$ $(n \in \omega)$. If $\eta \neq \mu \in \mathrm{Br}(T)$ then

$$\mathrm{br}(\eta, \mu) = \inf\{n \in \omega : \eta(n) \neq \mu(n)\}$$

denotes the *branch point* of η and μ. If $C \subset \omega$ then we collect the subtree

$$T_C = \{\tau \in T : \text{ if } e \in l(\tau) \setminus C \text{ then } \tau(e) = 0\}$$

of T where $l(\tau) = n$ denotes the *length* of the finite branch $\tau\colon n \to 2$.

Similarly,

$$\mathrm{Br}(T_C) = \{\eta \in \mathrm{Br}(T) : \text{ if } e \in \omega \setminus C \text{ then } \eta(e) = 0\}$$

and hence $\eta\restriction_n\in T_C$ for all $\eta\in \mathrm{Br}(T_C)$ $(n\in\omega)$.

Now we collect some trees to build a forest. Let $\aleph_1\leq\lambda\leq 2^{\aleph_0}$ be a regular cardinal and choose a family $\mathfrak{C}=\{C_\alpha\subset\omega:\alpha<\lambda\}$ of pair-wise almost disjoint infinite subsets of ω. Let $T\times\alpha=\{v\times\alpha: v\in T\}$ be a disjoint copy of the tree T and let $T_\alpha=T_{C_\alpha}\times\alpha$ for $\alpha<\lambda$. For simplicity we denote the elements of T_α by τ instead of $\tau\times\alpha$ since it will always be clear from the context to which α the finite branch τ refers to. By [14, Observation 2.1] we may assume that each tree T_α is perfect for $\alpha<\lambda$, *i.e.* if $n\in\omega$ then there is at most one finite branch $\eta\restriction_n$ such that $\eta\restriction_{(n+1)}\neq\mu\restriction_{(n+1)}$ for some $\mu\in T_\alpha$. We build a forest by letting

$$T_\Lambda=\bigcup_{\alpha<\lambda}T_\alpha.$$

Now we define our *base algebra* as $B_\Lambda=F[z_\tau:\tau\in T_\Lambda]$ which is a pure and dense subalgebra of its S-adic completion $\widehat{B_\Lambda}$ taken in the S-topology on B_Λ.

For later use we state the following definition which allows us to view the algebra B_Λ as a module generated over F by monomials in the "variables" z_τ $(\tau\in T_\Lambda)$.

Definition 2.1 Let X be a set of commuting variables and R an F-algebra. If $Y\subseteq R$ then $M(Y)$ will denote the set of all products of elements from Y, the Y-monomials.

Then any map $\sigma\colon X\to R$ extends to a unique epimorphism $\sigma\colon F[X]\to F[\sigma(X)]$. Thus any $r\in F[\sigma(X)]$ can be expressed by a polynomial $\sigma_r\in F[X]$, which is a preimage under σ: There are $l_1,\dots,l_n$ in $\sigma(X)$ such that

$$r=\sigma_r(l_1,\dots,l_n)=\sum_{m\in M(\{l_1,\dots,l_n\})}f_m m\quad\text{with } f_m\in F$$

becomes a polynomial-like expression.

In particular, if $Z_\alpha=\{z_\tau:\tau\in T_\alpha\}$ $(\alpha<\lambda)$ and $Z_\Lambda=\{z_\tau:\tau\in T_\Lambda\}$, then as always the polynomial ring B_Λ can be viewed as a free F-module over the basis of monomials, we have $B_\Lambda=\bigoplus_{z\in M(Z_\Lambda)}zF$ and a subring $B_\alpha=\bigoplus_{z\in M(Z_\alpha)}zF$.

Since $\aleph_1\leq\lambda\leq 2^{\aleph_0}=|\mathrm{Br}(T_{C_\alpha})|$ we can choose a family $\{V_\alpha\subseteq\mathrm{Br}(T_{C_\alpha}):\alpha<\lambda\}$ of subsets V_α of $\mathrm{Br}(T_{C_\alpha})$ with $|V_\alpha|=\lambda$ for $\alpha<\lambda$. Note that for $\alpha\neq\beta<\lambda$ the infinite branches from V_α and V_β branch at almost disjoint sets since $C_\alpha\cap C_\beta$ is finite, thus the pairs V_α, V_β are disjoint. Moreover, we may assume that for any $m\in\omega$, λ pairs of branches in V_α branch above m.

3 The Construction

Following [14] we use the

Definition 3.1 Let $x\in\widehat{B_\Lambda}$ be any element in the completion of the base algebra B_Λ. Moreover, let $\eta\in V_\alpha$ with $\alpha<\lambda$. We define the *branch like elements* $y_{\eta nx}$ for $n\in\omega$ as follows: $y_{\eta nx}:=\sum_{i\geq n}\frac{q_i}{q_n}(z_{\eta\restriction_i})+x\sum_{i\geq n}\frac{q_i}{q_n}\eta(i)$.

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Note that each element $y_{\eta n x}$ connects an infinite branch $\eta \in \mathrm{Br}(T_{C_\alpha})$ with finite branches from the tree T_α. Furthermore, the element $y_{\eta n x}$ encodes the infinite branch η into an element of $\widehat{B_\Lambda}$. We have a first observation which describes this as an equation and which is crucial for the rest of this paper.

$$y_{\eta n x} = s_{n+1} y_{\eta(n+1)x} + z_{\eta\restriction_n} + x\eta(n) \quad \text{for all } \alpha < \lambda, \eta \in V_\alpha. \tag{3.1}$$

Proof We calculate the difference

$$\begin{aligned} q_n y_{\eta n x} - q_{n+1} y_{\eta(n+1)x} &= \sum_{i \geq n} q_i(z_{\eta\restriction_i}) + x \sum_{i \geq n} q_i \eta(i) - \sum_{i \geq n+1} q_i(z_{\eta\restriction_i}) - x \sum_{i \geq n+1} q_i \eta(i) \\ &= q_n z_{\eta\restriction_n} + q_n x \eta(n). \end{aligned}$$

Dividing by q_n yields $y_{\eta n x} = s_{n+1} y_{\eta(n+1)x} + z_{\eta\restriction_n} + x\eta(n)$. ■

The elements of the polynomial ring B_Λ are unique finite sums of monomials in Z_λ with coefficients in F. Thus, by S-adic topology, any $0 \neq g \in \widehat{B_\Lambda}$ can be expressed uniquely as a sum

$$g = \sum_{z \in [g]} g_z,$$

where z runs over an at most countable subset $[g] \subseteq M(Z_\Lambda)$ of monomials and $0 \neq g_z \in z\hat{F}$. We put $[g] = \varnothing$ if $g = 0$. Thus any $g \in \widehat{B_\Lambda}$ has a unique *support* $[g] \subseteq M(Z_\Lambda)$, and support extends naturally to subsets of $\widehat{B_\Lambda}$ by taking unions of the support of its elements. It follows that

$$[y_{\eta n o}] = \{z_{\eta\restriction_j \times \alpha} : j \in \omega, j \geq n\}$$

for any $\eta \in V_\alpha$, $n \in \omega$ and $[z] = \{z\}$ for any $z \in M(Z_\Lambda)$.

Support can be used to define the norm of elements. If $X \subseteq M(Z_\Lambda)$ then

$$\|X\| = \inf\Big\{ \beta < \lambda : X \subseteq \bigcup_{\alpha < \beta} M(Z_\alpha) \Big\}$$

is the *norm* of X. If the infimum is taken over an unbounded subset of λ, we write $\|X\| = \infty$. However, since $\mathrm{cf}(\lambda) > \omega$, the *norm of an element* $g \in B_\Lambda$ is $\|g\| = \|[g]\| < \infty$ which is an ordinal $< \lambda$ hence either a successor or cofinal to ω. Norms extend naturally to subsets of B_Λ. In particular $\|y_{\eta n o}\| = \alpha + 1$ for any $\eta \in V_\alpha$.

We are ready to define the final F-algebra R as a F-subalgebra of the completion of B_Λ. Therefore choose a transfinite sequence b_α $(\alpha < \lambda)$ which runs λ times through the non-zero pure elements

$$b = \sum_{m \in M} m \in B_\Lambda \quad \text{with finite } M \subseteq M(T_\Lambda). \tag{3.2}$$

We call these b's *special pure* elements which have the property that B_Λ/Fb is a free F-module.

Definition 3.2 Let F be a p-domain and let $B_\Lambda := F[z_\tau : \tau \in T_\Lambda]$ be the polynomial ring over Z_Λ as above. Then we define the following smooth ascending chain of F-subalgebras of $\widehat{B_\Lambda}$.

(1) $R_0 = \{0\}$; $R_1 = F$;
(2) $R_\alpha = \bigcup_{\beta<\alpha} R_\beta$, for α a limit ordinal;
(3) $R_{\alpha+1} = R_\alpha[y_{\eta n x_\alpha}, z_\tau : \eta \in V_\alpha, \tau \in T_\alpha, n \in \omega]$;
(4) $R = R_\lambda = \bigcup_{\alpha<\lambda} R_\alpha$.

We let $x_\alpha = b_\alpha$ if $b_\alpha \in R_\alpha$ with $\|b_\alpha\| \le \alpha$ and $x_\alpha = 0$ otherwise.

For the rest of this paper purification is F-purification and properties like freeness, linear dependence or rank are taken with respect to F. First we prove some properties of the rings R_α ($\alpha \le \lambda$). It is easy to see that $R_\alpha = F[y_{\eta n x_\beta}, z_\tau : \eta \in V_\beta, \tau \in T_\beta, n \in \omega, \beta < \alpha]$ is not a polynomial ring: the set $\{y_{\eta n x_\alpha}, z_\tau : \eta \in V_\beta, \tau \in T_\beta, n \in \omega, \beta < \alpha\}$ is not algebraically independent over F. Nevertheless we have the following

Lemma 3.3 *For any fixed $n \in \omega$ and $\alpha < \lambda$ the set $\{y_{\eta n x_\alpha}, z_\tau : \eta \in V_\alpha, \tau \in T_\alpha\}$ is algebraically independent over R_α. Thus $R_\alpha[y_{\eta n x_\alpha}, z_\tau : \eta \in V_\alpha, \tau \in T_\alpha]$ is a polynomial ring.*

Proof Assume that the set of monomials $M(y_{\eta n x_\alpha}, z_\tau : \eta \in V_\alpha, \tau \in T_\alpha)$ is linearly dependent over R_α for some $\alpha < \lambda$ and $n \in \omega$. Then there exists a non-trivial linear combination of the form

$$\sum_{y\in Y} \sum_{z \in E_y} g_{y,z} yz = 0 \tag{3.3}$$

with $g_{y,z} \in R_\alpha$ and finite sets $Y \subset M(y_{\eta n x_\alpha} : \eta \in V_\alpha)$ and $E_y \subset M(Z_\alpha)$. We have chosen $V_\beta \cap V_\gamma = \varnothing$ for all $\beta \ne \gamma$ and $M(Z_\alpha) \cap R_\alpha = \varnothing$. Moreover $\|R_\alpha\| < \|R_{\alpha+1}\|$ and hence there exists a basal element $z_y \in B_\Lambda$ (high enough in an infinite branch) for any $1 \ne y \in Y$ with the following properties

(i) $z_y \notin E_{\tilde y}$ for all $\tilde y \in Y$;
(ii) $z_y \notin [\tilde y]$ for all $y \ne \tilde y \in Y$;
(iii) $z_y \notin [g_{\tilde y, z}]$ for all $\tilde y \in Y, z \in E_{\tilde y}$;
(iv) $z_y \in [y]$.

Now we restrict the equation (3.3) to the basal element z_y and obtain $g_{y,z} z_y z = 0$ for all $z \in E_y$. Since $z_y \notin [g_{y,z}]$ we derive $g_{y,z} = 0$ for all $1 \ne y \in Y$ and $z \in E_y$. Therefore equation (3.3) reduces to $\sum_{z\in E_1} g_{1,z} z = 0$. We apply $M(Z_\alpha) \cap R_\alpha = \varnothing$ once more. Since each z is a basal element from the set $M(Z_\alpha)$ we get that $g_{1,z} = 0$ for all $z \in E_1$. Hence $g_{y,z} = 0$ for all $y \in Y$, $z \in E_y$, contradicting the assumption that (3.3) is a non-trivial linear combination. ■

The following lemma shows that the F-algebras $R_\delta / s_{n+1} R_\delta$ are also polynomial rings over $F/s_{n+1}F$ for every $n < \omega$. For $\delta < \lambda$ and $n \in \omega$ we can choose a set $U_{n\delta} \subseteq V_\delta$ such that for any $\eta \in V_\delta$ there is $\eta' \in U_{n\delta}$ with $\operatorname{br}(\eta, \eta') > n$ and if $\eta, \eta' \in U_{n\delta}$, then $\operatorname{br}(\eta, \eta') \le n$. Obviously $|U_{n\delta}| \le 2^n$. Moreover, let $T'_\delta = T_\delta \setminus \{z_{\eta\restriction_n} : \eta \in U_{n\delta}\}$.

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Lemma 3.4 *If* $n < \omega$*, then the set* $X^\delta_{n+1} = \{y_{\eta n x_\beta}, y_{\eta(n+1)x_\beta}, z_\tau : \eta \in U_{n\beta}, \tau \in T'_\beta, \beta < \lambda\}$ *is algebraically independent over* $F/s_{n+1}F$ *and generates the algebra* $R_\delta/s_{n+1}R_\delta$*. Thus* $R_\delta/s_{n+1}R_\delta = F/s_{n+1}F[X^\delta_{n+1}]$ *is a polynomial ring.*

Remark Here we identify the elements in $X^\delta_{n+1} \subseteq R_\delta$ with their canonical images *modulo* $s_{n+1}R_\delta$.

Proof First we show that X^δ_{n+1} is algebraically independent over $F/s_{n+1}F$. Suppose

$$\sum_{y\in Y}\sum_{z\in E_y} f_{y,z}yz \equiv 0 \mod s_{n+1}R \tag{3.4}$$

with $f_{y,z} \in F$ and finite sets $Y \subseteq M(y_{\eta n x_\beta}, y_{\eta(n+1)x_\beta} : \eta \in U_{n\beta}, \beta < \delta)$ and $E_y \subseteq M(\bigcup_{\beta<\delta} T'_\beta)$.

Choose a basal element $z_y \in [y]$ for any $1 \neq y \in Y$ which is a product of basal element z_τ with $l(\tau) = n$ and $z_y \notin [y']$ for any $y \neq y' \in Y$ and moreover require $z_y \notin E_{y'}$ for all $y' \in Y$. This is possible by the choice of $U_{n\beta}$ and T'_β. Restricting (3.4) to z_y yields

$$\sum_{z\in E_y} f_{y,z}z_y z \equiv 0 \mod s_{n+1}R$$

hence $f_{yz} \equiv 0 \mod s_{n+1}R$. Therefore (3.4) reduces to $\sum_{z\in E_1} f_{1,z}z \equiv 0 \mod s_{n+1}F$ and thus also $f_{1,z} \equiv 0 \mod s_{n+1}F$ is immediate. This shows that the set X^δ_{n+1} is algebraically independent over $F/s_{n+1}F$.

Finally we must show that $R_\delta/s_{n+1}R_\delta = (F/s_{n+1}F)[X^\delta_{n+1}]$. We will show by induction on $\alpha < \delta$ that

$$(R_\alpha + s_{n+1}R_\delta)/s_{n+1}R_\delta \subseteq (F/s_{n+1}F)[X^\delta_{n+1}].$$

If $\alpha = 0$ or $\alpha = 1$ then the claim is trivial, hence assume that $\alpha > 1$ and for all $\beta < \alpha$ we have

$$(R_\beta + s_{n+1}R_\delta)/s_{n+1}R_\delta \subseteq (F/s_{n+1}F)[X^\delta_{n+1}].$$

If α is a limit ordinal, then $(R_\alpha + s_{n+1}R_\delta)/s_{n+1}R_\delta \subseteq (F/s_{n+1}F)[X^\delta_{n+1}]$ is immediate. Thus assume that $\alpha = \beta+1$. By assumption and $x_\beta \in R_\beta$ we know that $(x_\beta+s_{n+1}R_\delta) \in (F/s_{n+1}F)[X^\delta_{n+1}]$. Hence equation (3.1) shows that the missing elements $z_{\eta\restriction_n} + s_{n+1}R_\delta$ $(\eta \in U_{n\beta})$ are in $(F/s_{n+1}F)[X^\delta_{n+1}]$.

For $\eta \in V_\beta$ we can choose $\eta' \in U_{n\beta}$ such that $\mathrm{br}(\eta, \eta') > n$. Then using (3.1) we obtain $y_{\eta n x_\beta} - y_{\eta' n x_\beta} \equiv 0 \mod s_{n+1}R$ and therefore $y_{\eta n x_\beta} + s_{n+1}R \in (F/s_{n+1}F)[X^\delta_{n+1}]$. By induction on $m < \omega$ using again (3.1) it is now easy to verify $y_{\eta m x_\beta} + s_{n+1}R_\delta \in (F/s_{n+1}F)[X^\delta_{n+1}]$ for every $m < \omega, \eta \in U_{n\beta}$ and hence $R_\alpha + s_{n+1}R_\delta \subseteq (F/s_{n+1}F)[X^\delta_{n+1}]$ which finishes the proof. ■

Now we are able to prove that the members R_α of the chain $\{R_\sigma : \sigma < \lambda\}$ are F-pure submodules of R and that R is an $\aleph_1$-free domain.

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Lemma 3.5 *R is a commutative F-algebra without zero-divisors and R_α as an F-module is pure in R for all $\alpha < \lambda$.*

Proof By definition each R_α is a commutative F-algebra and hence R is commutative. To show that R has no zero-divisors it is enough to show that each member R_α of the chain $\{R_\sigma : \sigma < \lambda\}$ is an F-algebra without zero-divisors. Since F is a domain we can assume, by induction, that R_β has no zero-divisors for all $\beta < \alpha$ and some $1 < \alpha < \lambda$. If α is a limit ordinal then it is immediate that R_α has no zero-divisors. Hence $\alpha = \gamma + 1$ is a successor ordinal and R_γ is a domain. If $g, h \in R_\alpha$ with $gh = 0 \neq g$, then we must show that $h = 0$. Write g in the form

$$g = \sum_{y \in Y_g} \sum_{z \in E_{g,y}} g_{y,z} yz \tag{g}$$

with $0 \neq g_{y,z} \in R_\gamma$ and finite sets $E_{g,y} \subset M(Z_\gamma)$ and $Y_g \subset M(y_{\eta n x_\gamma} : \eta \in V_\gamma)$ for some $n \in \omega$. By (3.1) and $x_\gamma \in R_\gamma$ we may assume n is fixed. Similarly, we write

$$h = \sum_{y \in Y_h} \sum_{z \in E_{h,y}} h_{y,z} yz \tag{h}$$

with $h_{y,z} \in R_\gamma$ and finite sets $Y_h \subset M(y_{\eta n x_\gamma} : \eta \in V_\gamma)$ and $E_{h,y} \subset M(Z_\gamma)$.

Next we want $h_{y,z} = 0$ for all $y \in Y_h, z \in E_{h,y}$. The proof follows by induction on the number of $h_{y,z}$'s. If $h = h_{w,z'} wz'$, then

$$gh = \sum_{y \in Y_g, z \in E_{g,y}} g_{y,z} h_{w,z'} yzwz'$$

and from Lemma 3.3 follows $g_{y,z} h_{w,z'} = 0$ for all $y \in Y_g$, $z \in E_{g,y}$. Since R_γ has no zero-divisors we obtain $h_{w,z'} = 0$ and thus $h = 0$. Now assume that $k + 1$ coefficients $h_{y,z} \neq 0$ appear in (h). We fix an arbitrary coefficient $h_{w,z'}$ and write $h = h_{w,z'} wz' + h'$ so that wz' does not appear in the representation of h'. Therefore the product gh is of the form

$$gh = \sum_{y \in Y_g} \sum_{z \in E_{g,y}} g_{y,z} h_{w,z'} yzwz' + gh'. \tag{gh}$$

If the monomial wz' appears in the representation of (g) then the monomial $w^2(z')^2$ appears in the representation of (gh) only once with coefficient $g_{w,z'} h_{w,z'}$. Using Lemma 3.3 and the hypothesis that R_γ has no zero-divisors we get $h_{w,z'} = 0$.

If the monomial wz' does not appear in the representation of (g) then $g_{y,z} h_{w,z'} = 0$ for all appearing coefficients $g_{y,z}$ is immediate by Lemma 3.3. Thus $h_{w,z'} = 0$ and $h = h'$ follows. By induction hypothesis also $h = 0$ and R has no zero-divisors.

It remains to show that R_α is a pure F-submodule of R for $\alpha < \lambda$. Let $g \in R \setminus R_\alpha$ such that $fg \in R_\alpha$ for some $0 \neq f \in F$ and choose $\beta < \lambda$ minimal with $g \in R_\beta$. Then $\beta > \alpha$ and it is immediate that $\beta = \gamma + 1$ for some $\gamma \geq \alpha$, hence $fg \in R_\alpha \subset R_\gamma$. Now we can write

$$g = \sum_{y \in Y_g} \sum_{z \in E_{g,y}} g_{y,z} yz \tag{g}$$

with $g_{y,z} \in R_\gamma$ and finite sets $Y_g \subset M(y_{\nu k x_\gamma} : \nu \in V_\gamma)$ for some fixed $k \in \omega$ and $E_g \subset M(Z_\gamma)$ and clearly

$$fg = \sum_{y\in Y_g} \sum_{z\in E_{g,y}} fg_{y,z} yz \in R_\gamma.$$

Hence there exists $g_\gamma \in R_\gamma$ such that

$$fg - g_\gamma = \sum_{y\in Y_g} \sum_{z\in E_{g,y}} fg_{y,z} yz - g_\gamma = 0.$$

From Lemma 3.3 follows $fg_{y,z} = 0$ for all $1 \neq y \in Y_g$, $1 \neq z \in E_{g,y}$, thus $g_{y,z} = 0$ because R is a torsion-free F-module. Hence (g) reduces to the summand with $y = z = 1$, but $g = g_{1,1} \in R_\gamma$ contradicts the minimality of β. Thus $g \in R_\alpha$ and R_α is pure in R. ■

From the next theorem follows for $\alpha = 0$ that R is an $\aleph_1$-free F-module. We say that R is *polynomial* $\aleph_1$*-free* if every countable F-submodule of R can be embedded into a polynomial subring over F of R. Clearly, polynomial $\aleph_1$-freeness implies $\aleph_1$-freeness.

Theorem 3.6 *If F is a p-domain and $R = \bigcup_{\alpha<\lambda} R_\alpha$ is the F-algebra constructed above, then R is a domain of size λ with R/R_α is polynomial $\aleph_1$-free for all $\alpha < \lambda$.*

Proof $|R| = \lambda$ is immediate by construction and R is a domain by Lemma 3.5. It remains to show that R is an polynomial $\aleph_1$-free ring. Therefore let $U \subseteq R$ be a countable pure submodule of R. There exist elements $u_i \in R$ such that

$$U = \langle u_1, \dots, u_n, \dots \rangle_* \subseteq R.$$

Here the suffix $*$ denotes purification as an F-module. Let $U_n := \langle u_1, \dots, u_n\rangle_*$ for $n \in \omega$. Hence there is a minimal $\alpha_n < \lambda$ such that $u_i \in R_{\alpha_n}$ for $i \leq n$ and $n \in \omega$, which obviously is a successor ordinal $\alpha_n = \gamma_n + 1$. Moreover, $U_n \subseteq R_{\alpha_n}$ since R_{α_n} is pure in R and by induction we may assume that R_{γ_n} is polynomial $\aleph_1$-free. Fix $n \in \omega$. Using $R_{\alpha_n} = R_{\gamma_n+1} = R_{\gamma_n}[y_{\eta m x_{\gamma_n}}, z_\tau : \eta \in V_{\gamma_n}, \tau \in T_{\gamma_n}, m \in \omega]$ from Definition 3.2 we can write

$$u_i = \sum_{y\in Y_i} \sum_{z\in E_{i,y}} g_{y,z,i} yz$$

with $g_{y,z,i} \in R_{\gamma_n}$ and finite sets $Y_i \subset M(y_{\eta m x_{\gamma_n}} : \eta \in V_{\gamma_n})$ for some fixed $m \in \omega$ and $E_{i,y} \subset M(Z_{\gamma_n})$. Choose the pure submodule $R_{U_n} := \langle g_{y,z,i} : y \in Y_i, z \in E_{i,y}, 1 \leq i \leq n\rangle_* \subseteq R_{\gamma_n}$ of R_{γ_n} and let

$$U_n' := \{y, z : y \in Y_i, z \in E_{i,y}, 1 \leq i \leq n\}.$$

By induction there is a polynomial subring $L_n \subseteq R_{\gamma_n}$ of R_{γ_n} which contains R_{U_n} purely. Again by induction we may assume that L_{n+1} is a polynomial ring over L_n

for all $n \in \omega$. Hence $U''_n := L_n[U'_n] \subseteq_* R_{\alpha_n}$ is a polynomial ring by Lemma 3.3 and purity of R_{U_n} in R_{γ_n}. Thus $U_n \subseteq_* U''_n \subseteq_* R_{\alpha_n}$. By construction $L_{n+1}[U'_{n+1}]$ is a polynomial ring over $L_n[U'_n]$ and thus the union $U'' = \bigcup_{n\in\omega} U''_n$ is a polynomial ring containing U. Similar arguments show that R/R_α is polynomial $\aleph_1$-free for every $\alpha < \lambda$. ■

4 Main Theorem

In this section we will prove that the F-algebra R from Definition 3.2 is an $E(F)$-algebra, hence every F-endomorphism of R viewed as an F-module is multiplication by some element r from R. Every endomorphism of R is uniquely determined by its action on B_Λ which is an S-dense submodule of R. It is therefore enough to show that a given endomorphism φ of R acts as multiplication by some $r \in R$ when restricted to B_Λ. It is our first aim to show that such φ acts as multiplication on each special pure element x_α for $\alpha < \lambda$. Therefore we need the following

Definition 4.1 A set $W \subseteq \lambda$ is *closed* if

$$x_\alpha \in R^\alpha_W := F[y_{\eta n x_\beta}, z_\tau : \eta \in V_\beta, \tau \in T_\beta, \beta \in W, \beta < \alpha, n \in \omega]$$

for every $\alpha \in W$. Moreover let $R_W := F[y_{\eta n x_\beta}, z_\tau : \eta \in V_\beta, \tau \in T_\beta, \beta \in W, n \in \omega]$.

We have a first lemma.

Lemma 4.2 *Any finite subset of λ is a subset of a finite and closed subset of λ.*

Proof If $\varnothing \neq W \subseteq \lambda$ is finite then let $\gamma = \max(W)$. We prove the claim by induction on γ. If $\gamma = 0$, then $W = \{0\}$, $R_W = F$, $x_0 = 0$ and there is nothing to prove. If $\gamma > 0$, then $x_\gamma \in R_\gamma = F[y_{\eta n x_\beta}, z_\tau : n \in \omega, \eta \in V_\beta, \tau \in T_\beta, \beta < \gamma]$ and there exists a finite set $Q \subseteq \gamma$ such that

$$x_\gamma \in F[y_{\eta n x_\beta}, z_\tau : n \in \omega, \eta \in V_\beta, \tau \in T_\beta, \beta \in Q].$$

If $Q_1 = Q \cup (W \setminus \{\gamma\})$ then $\max(Q_1) < \gamma$. Thus by induction there exists a closed and finite $Q_2 \subseteq \lambda$ containing Q_1. It is now easy to see that $W' = Q_2 \cup \{\gamma\}$ is as required. ■

Closed and finite subsets W of λ give rise to nice presentations of elements in R_W.

Lemma 4.3 *Let W be a closed and finite subset of λ and $r \in R_W$. Then there exists $m^r_* \in \mathbb{N}$ such that $r \in F[y_{\eta n x_\beta}, z_\tau : \eta \in V_\beta, \tau \in T_\beta, \beta \in W]$ for every $n \geq m^r_*$.*

Proof We apply induction on $|W|$. If $|W| = 0$, then $R_W = R_\varnothing = F$ and Lemma 4.3 holds. If $|W| > 0$ then $\gamma = \max(W)$ is defined. It is easy to see that $W' = W \setminus \{\gamma\}$ is still closed and finite. Thus $x_\delta \in R_{W'}$ for all $\delta \in W$. By induction there is m^δ_* such

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that $x_\delta \in F[y_{\eta n x_\beta}, z_\tau : \eta \in V_\beta, \tau \in T_\beta, \beta \in W']$ for every $n \geq m_*^\delta$ $(\delta \in W)$. Any $r \in R_W$ can be written as a polynomial

$$r = \sigma\big(\{y_{\eta_{r,l} k_{r,l} x_{\beta_{r,l}}}, z_{\tau_{r,j}} : \eta_{r,l} \in V_{\beta_{r,l}}, \tau_{r,j} \in T_{\beta_{r,j}}, l < l_r, j < j_r\}\big)$$

for some $l_r, j_r \in \mathbb{N}, \beta_{r,l}, \beta_{r,j} \in W$ and $\eta_{r,l} \in V_{\beta_{r,l}}, \tau_{r,j} \in T_{\beta_{r,j}}$. Let $m_*^r = \max\big(\{m_*^\delta, k_{r,l} : l < l_r, \delta \in W\}\big)$. Using (3.1) now it follows easily that $r \in F[y_\eta n x_\beta, z_\tau : \eta \in V_\beta, \tau \in T_\beta, \beta \in W]$ for every $n \geq m_*^r$. ■

We are ready to show that every endomorphism of R acts as multiplication on each of the special pure elements x_α.

Definition 4.4 If R_α is as above, then let $G_\alpha = \langle y_{\eta n x_\alpha}, z_\tau : \eta \in V_\alpha, \tau \in T_\alpha, n \in \omega \rangle_F$ be the F-submodule of R_α for any $\alpha < \lambda$.

From (3.1) we note that $x_\alpha \in G_\alpha$ and our claim will follow if we can show that every homomorphism from G_α to R^+ maps x_α to a multiple of itself.

Proposition 4.5 *If* $h \colon G_\alpha \to R$ *is an F-homomorphism, then* $h(x_\alpha) \in x_\alpha R$.

Proof Let $h \colon G_\alpha \to R$ be an F-homomorphism and assume towards contradiction that $h(x_\alpha) \notin x_\alpha R$. For a subset $V \subseteq V_\alpha$ of cardinality λ we define the F-submodule

$$G_V = \langle x_\alpha, y_{\eta n x_\alpha} : \eta \in V, n \in \omega \rangle_* \subseteq G_\alpha$$

and note that $\{z_{\eta\restriction_n} : \eta \in V, n \in \omega\} \subseteq G_V$ from $x_\alpha \in G_V$ and (3.1). Also $G_{V_\alpha} \in \mathfrak{H} =: \{G_V : V \subseteq V_\alpha, |V| = \lambda\} \neq \varnothing$ and we can choose $\beta_* = \min\{\beta \leq \lambda : \exists G_V \in \mathfrak{H}$ and $h(G_V) \subseteq R_\beta\}$. There is $G_V \in \mathfrak{H}$ such that $h(G_V) \subseteq R_{\beta_*}$.

We first claim that $\beta_* < \lambda$ and assume towards contradiction that $\beta_* = \lambda$ and we can choose inductively a minimal countable subset $U =: U_V \subseteq V$ such that

$$(4.1) \qquad (\forall \eta \in V)(\forall n \in \omega)(\exists \rho \in U_V) \quad \text{such that } \eta\restriction_n = \rho\restriction_n .$$

For each $\eta \in V$ we define the countable set $Y_\eta = \{y_{\eta n x_\alpha} : n < \omega\}$. Using $\mathrm{cf}(\lambda) = \lambda > \aleph_0$ we can find a successor ordinal $\beta < \lambda$ such that $h(x_\alpha) \in R_\beta$ and $h(Y_\rho) \subseteq R_\beta$ for all $\rho \in U$. If $n_* \in \omega$ and $\eta \in V$ choose $n_* < n \in \omega$ and $\rho_n \in U$ by (4.1) such that $\eta\restriction_n = \rho_n\restriction_n$. From Definition 3.1 and (2.1) we see that

$$\begin{aligned}(4.2)\quad & y_{\eta n_* x_\alpha} - y_{\rho_n n_* x_\alpha}\\ &= \sum_{i \geq n_*} \frac{q_i}{q_{n_*}}(z_{\eta\restriction_i}) + x_\alpha \sum_{i \geq n_*} \frac{q_i}{q_{n_*}}\eta(i) - \sum_{i \geq n_*} \frac{q_i}{q_{n_*}}(z_{\rho_n\restriction_i}) - x_\alpha \sum_{i \geq n_*} \frac{q_i}{q_{n_*}}\rho_n(i)\\ &= \sum_{i \geq n+1} \frac{q_i}{q_{n_*}}(z_{\eta\restriction_i}) + x_\alpha \sum_{i \geq n} \frac{q_i}{q_{n_*}}\eta(i) - \sum_{i \geq n+1} \frac{q_i}{q_{n_*}}(z_{\rho_n\restriction_i}) - x_\alpha \sum_{i \geq n} \frac{q_i}{q_{n_*}}\rho_n(i)\end{aligned}$$

is divisible by s_{n-1}. Thus s_{n-1} divides $h(y_{\eta n_* x_\alpha} - y_{\rho_n n_* x_\alpha})$ for $n_* < n < \omega$. From $h(y_{\rho_n n_* x_\alpha}) \in R_\beta$ and the choice of $\rho_n \in U$ it follows that $h(y_{\eta n_* x_\alpha}) + R_\beta \in R/R_\beta$

is divisible by infinitely many s_n. Hence $h(y_{\eta n_* x_\alpha}) \in R_\beta$ since R/R_β is $\aleph_1$-free by Lemma 3.6. However n_* was chosen arbitrarily, we therefore have $h(Y_\eta) \subseteq R_\beta$ for all $\eta \in V$ and $h(G_V) \subseteq R_\beta$ follows, which contradicts the minimality of β_*. Therefore $\beta_* \neq \lambda$.

Since $h(G_V) \subseteq R_{\beta_*}$ we can write $h(y_{\eta o x_\alpha}) = \sigma_\eta(\{y_{\nu_{\eta,l} m_{\eta,l} x_{\beta_{\eta,l}}}, z_{\tau_{\eta,k}} : l < l_\eta, k < k_\eta\})$ for every $\eta \in V$ and suitable $\beta_{\eta,l}, \beta_{\eta,k} < \beta_*$, $\nu_{\eta,l} \in V_{\beta_{\eta,l}}$ and $\tau_{\eta,k} \in T_{\beta_{\eta,k}}$. Recall that polynomials σ_η depend on $\eta \in V$. For notational simplicity we shall assume that all pairs $(\beta_{\eta,l}, \beta_{\eta,k})$ are distinct. For obvious cardinality reasons we may assume without loss of generality that $l_\eta = l_*$ and $k_\eta = k_*$ for some fixed $l_*, k_* \in \mathbb{N}$ for all $\eta \in V$. Moreover, since F is countable, we may assume that the polynomials σ_η are independent of η and thus we can write $\sigma_\eta = \sigma$. Hence

$$h(y_{\eta o x_\alpha}) = \sigma\big(\{y_{\nu_{\eta,l} m_{\eta,l} x_{\beta_{\eta,l}}}, z_{\tau_{\eta,k}} : l < l_*, k < k_*\}\big).$$

We put $W_\eta = \{\beta_{\eta,l}, \beta_{\eta,k} : l < l_*, k < k_*\}$, which is a finite subset of λ for every $\eta \in V$. By Lemma 4.2 we may assume that W_η is closed. Moreover, possibly enlarging W_η, we also may assume that $h(x_\alpha) \in R_{W_\eta}$ for all $\eta \in V$. Since $\beta_* < \lambda$ and λ is regular the ordinal β_* is a set of cardinality $< \lambda$ with $W_\eta \subseteq \beta_*$ for all $\eta \in V$. By cardinality arguments it easily follows that there is $W = \{\beta_l, \beta_k : l < l_*, k < k_*\} \subseteq \beta_*$ such that $W_\eta = W$ for all $\eta \in V'$ for some $V' \subseteq V$ of cardinality λ. We rename $V = V'$. Let $m_\eta \in \mathbb{N}$ such that $m_\eta > l(\tau_{\eta,k})$ for all $\eta \in V$ and $k < k_*$. Again, passing to an equipotent subset (of) V we may assume that $m_\eta = m_1$ is fixed for all $\eta \in V$. Now we apply Lemma 4.3 to obtain $h(y_{\eta o x_\alpha}) \in F[y_{\eta n_\eta x_\beta}, z_\tau : \eta \in V_\beta, \tau \in T_\beta, \beta \in W]$ for $\eta \in V$ and some $n_\eta \in \mathbb{N}$. Since $|V| > \aleph_0$ we may assume that $n_\eta = n_*$ does not depend on $\eta \in V$ anymore. If $m_* = \max\{n_*, m_1\}$ we find new presentations

$$h(y_{\eta o x_\alpha}) = \sigma\big(\{y_{\nu_{\eta,l} m_* x_{\beta_l}}, z_{\tau_{\eta,k}} : l < l_*, k < k_*\}\big) \tag{4.3}$$

for every $\eta \in V$ and $\beta_l, \beta_k \in W$, $\nu_{\eta,l} \in V_{\beta_l}$ and $\tau_{\eta,k} \in T_{\beta_k}$. Moreover, $l(\tau_{\eta,k}) \leq m_*$ for all $\eta \in V$ and $k < k_*$. The reader may notice that when obtaining equation (4.3) the polynomial σ and the natural number k_* may become dependent on η again but a cardinality argument allows us to unify them again and for notational reasons we stick to σ and k_*. Using that T_α is countable, we are allowed to assume that $\tau_{\eta,k} = \tau_k$ for all $\eta \in V$ and $k < k_*$, hence $h(y_{\eta o x_\alpha}) = \sigma\big(\{y_{\nu_{\eta,l} m_* x_{\beta_l}}, z_{\tau_k} : l < l_*, k < k_*\}\big)$.

Finally, increasing m_* (and unifying σ and k_* again) we may assume that all $\nu_{\eta,l} \restriction_{m_*}$ are different ($l < l_*$) and that

$$\nu_{\eta,l} \restriction_{m_*} \neq \tau_k \tag{4.4}$$

for all $\eta \in V$ and $l < l_*, k < k_*$. Using a cardinality argument and the countability of the trees T_{β_l} we may assume that $\nu_{\eta,l} \restriction_{m_*}$ does not dependent on $\eta \in V$ for all $l < l_*$. Thus

$$\nu_{\eta,l} \restriction_{m_*} =: \bar{\tau}_l \in T_{\beta_l} \tag{4.5}$$

and $\tau_k \neq \bar{\tau}_l$ for all $l < l_*$, $k < k_*$ from (4.4). Since W is closed and $h(x_\alpha) \in R_W$ we can finally write

$$h(x_\beta) = \sigma_\beta\big(\{y_{\nu_{\beta,l} m_* x_{\beta_l}}, z_{\tau_{\beta,k}} : l < l_\beta, k < k_\beta\}\big)$$

for every $\beta \in W \cup \{\alpha\}$ and suitable $l_\beta, k_\beta \in \mathbb{N}$, $\beta_l, \beta_k \in W$. Obviously, increasing m_* once more, we may assume that

$$\nu_{\beta,l}\restriction_{m_*} \neq \nu_{\beta',l'}\restriction_{m_*} \quad \text{and} \quad \nu_{\beta,l}\restriction_{m_*} \neq \bar{\tau}_j \tag{4.6}$$

for all $\beta, \beta' \in W \cup \{\alpha\}$, $l < l_\beta$, $l' < l_{\beta'}$, $j < l_*$.

Now choose any $n_* > m_*$ such that

(i) $n_* > \sup(C_\beta \cap C_{\beta'})$ for all $\beta \neq \beta' \in W \cup \{\alpha\}$;
(ii) s_{n_*} is relatively prime to all coefficients in σ;
(iii) s_{n_*} is relatively prime to all coefficients in σ_β for all $\beta \in W \cup \{\alpha\}$.

Using $\aleph_0 < |V|$ we can choose pairs of branches $\eta_1, \eta_2 \in V$ with arbitrarily large branch point $\operatorname{br}(\eta_1, \eta_2) = n + 1 \geq n_*$. Let U be the infinite set of all such n's. An easy calculation using (3.1) shows

$$y_{\eta_1 o x_\alpha} - y_{\eta_2 o x_\alpha} = \Big(\prod_{l \leq n} s_l\Big)(y_{\eta_1 n x_\alpha} - y_{\eta_2 n x_\alpha})$$

and as $\operatorname{br}(\eta_1, \eta_2) = n + 1$ we obtain

$$y_{\eta_1 o x_\alpha} - y_{\eta_2 o x_\alpha} \equiv \Big(\prod_{l \leq n} s_l\Big) x_\alpha \mod s_{n+1}R. \tag{4.7}$$

We now distinguish three cases.

Case 1 If $\operatorname{br}(\nu_{\eta_1,l}, \nu_{\eta_2,l}) > n + 1$ for some $l < l_*$ then from (3.1) follows

$$y_{\nu_{\eta_1,l} m_* x_{\beta_l}} - y_{\nu_{\eta_2,l} m_* x_{\beta_l}} \equiv 0 \mod s_{n+1}R.$$

Case 2 If $\operatorname{br}(\nu_{\eta_1,l}, \nu_{\eta_2,l}) = n + 1$ for some $l < l_*$ then from (3.1) follows

$$y_{\nu_{\eta_1,l} m_* x_{\beta_l}} - y_{\nu_{\eta_2,l} m_* x_{\beta_l}} + s_{n+1}R \in x_{\beta_l}R + s_{n+1}R.$$

We have chosen $\operatorname{br}(\eta_1, \eta_2) = n+1 > n_* > \sup(C_\beta \cap C_{\beta'})$ for all $\beta \neq \beta' \in W \cup \{\alpha\}$. Hence $n+1$ can not be the splitting point of pairs of branches from different levels α and β_l. Thus $\beta_l = \alpha$ and the last displayed expression becomes

$$y_{\nu_{\eta_1,l} m_* x_\alpha} - y_{\nu_{\eta_2,l} m_* x_\alpha} + s_{n+1}R \in x_\alpha R + s_{n+1}R.$$

Case 3 If $k = \operatorname{br}(\nu_{\eta_1,l}, \nu_{\eta_2,l}) < n+1$ for some $l < l_*$ then $m_* < k$ by (4.5). From (3.1) and the choice of n we see that $y_{\nu_{\eta_1,l} n x_\alpha}$ appears in some monomial of $h(y_{\eta_1 o x_\alpha} - y_{\eta_2 o x_\alpha})$ with coefficient relatively prime to s_{n+1}. By an easy support argument (restricting to

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$\nu_{\eta_1,l}\restriction_k$ and using (4.4), (4.5) and (4.6)) this monomial can not appear in $h(x_\alpha)$. From Lemma 3.4 now follows

$$h(y_{\eta_1 o x_\alpha} - y_{\eta_2 o x_\alpha}) - \Bigl(\prod_{l\le n} s_l\Bigr) h(x_\alpha) \not\equiv 0 \mod s_{n+1}R$$

which contradicts (4.7).

Therefore, for all $n \in U$ we obtain

$$\Bigl(\prod_{l\le n} s_l\Bigr) h(x_\alpha) \in s_{n+1}R + x_\alpha R.$$

The elements $\prod_{l\le n} s_l$ and s_{n+1} are co-prime, thus

$$h(x_\alpha) \in \bigcap_{n\in U} s_{n+1}R + x_\alpha R.$$

Using that U is infinite, we claim

$$\bigcap_{n\in U} s_n R + x_\alpha R = x_\alpha R,$$

which then implies $h(x_\alpha) \in x_\alpha R$ and finishes the proof of Proposition 4.5.

The special pure elements are of the form (3.2), thus $x_\alpha = \sum_{m\in M} m$ for some finite subset M of $M(T_\Lambda)$. Choose $y \in \bigcap_{n\in U} s_n R + x_\alpha R$. Then there are $f_n, r_n \in R$ for $n \in U$ such that

$$y - s_n f_n = x_\alpha r_n. \tag{4.8}$$

Put $R' = \langle [x_\alpha], y, f_n, r_n : n \in U\rangle_*$ and let L be the pure polynomial subring of R that contains R' and exists by Theorem 3.6. Hence equation (4.8) holds in L. We may assume that the finite support M of x_α is contained in a basis of L and hence the quotient $L/x_\alpha L$ is free and therefore S-reduced. This contradicts

$$y \equiv s_n f_n \mod x_\alpha L \tag{4.9}$$

which follows from equation (4.8) for every $n \in U$ unless $y \in x_\alpha L$ and hence $y \in x_\alpha R$. ■

We are now ready to prove that R is an $E(F)$-algebra.

Main Theorem 4.6 *Let F be a countable principal ideal domain with $1 \neq 0$ and infinitely many pair-wise coprime elements. If $\aleph_1 \le \lambda \le 2^{\aleph_0}$ is a regular cardinal, then the F-algebra R in Definition 3.2 is an $\aleph_1$-free $E(F)$-algebra of cardinality λ.*

Proof If h is a F-endomorphism of R viewed as F-module, then we must show that h is scalar multiplication by some element $b \in R$. From Proposition 4.5 for $h \restriction G_\alpha$ there exists an element $b_\alpha \in R$ such that $h(x_\alpha) = x_\alpha b_\alpha$ for any $\alpha < \lambda$, where the x_α's run through all special pure elements.

Now let U_α be a countable subset of V_α for every $\alpha < \lambda$ as in (4.1). Then

$$R^*_\alpha = F[y_{\eta n x_\beta}, z_\tau : \eta \in U_\beta, \tau \in T_\beta, \beta < \alpha, n \in \omega]$$

is a countable subalgebra of R_α. Since λ is regular uncountable there exists for every $\alpha < \lambda$ an ordinal $\gamma_\alpha < \lambda$ such that $h(R^*_\alpha) \subseteq R_{\gamma_\alpha}$. We put $C = \{\delta < \lambda : \forall(\alpha < \delta)(\gamma_\alpha < \delta)\}$ which is a cub in λ. Intersecting with the cub of all limit ordinals we may assume that C consists of limit ordinals only. If $\delta \in C$, then similar arguments as in the proof of Proposition 4.5 after equation (4.1), using the fact that R/R_δ is $\aleph_1$-free show that $h(R_\beta) \subseteq R_\delta$ for every $\beta < \delta$ and taking unions $h(R_\delta) \subseteq R_\delta$.

Let us assume for the moment that there is some $\delta_* \in C$ such that for every special pure element $r \in B_\Lambda$ we have $b_r \in R_{\delta_*}$. Suppose r_1 and r_2 are two distinct pure elements with $b_{r_1} \neq b_{r_2}$. Then choose $\delta_* < \delta \in C$ such that $r_1, r_2 \in R_\delta$ and $\tau \in T_\delta$ with $\tau \notin ([r_1] \cup [r_2])$. Then

$$(4.10) \quad b_\tau \tau + b_{r_1} r_1 = h(\tau) + h(r_1) = h(\tau + r_1) = b_{\tau+r_1}(\tau + r_1) = b_{\tau+r_1}\tau + b_{\tau+r_1} r_1.$$

Now note that R_δ is an R_{δ_*}-module and that R/R_δ is torsion-free as an R_{δ_*}-module. Moreover, b_τ, b_{r_1} and $b_{\tau+r_1}$ are elements of R_{δ_*}, hence τ is not in the support of either of them. Thus restricting equation (4.10) to τ we obtain

$$b_\tau \tau = b_{\tau+r_1} \tau$$

and therefore $b_\tau = b_{\tau+r_1}$. Now equation (4.10) reduces to $b_{r_1} r_1 = b_{\tau+r_1} r_1$ and since R is a domain we conclude $b_{r_1} = b_{\tau+r_1}$. Hence $b_{r_1} = b_\tau$ and similarly $b_{r_2} = b_\tau$, therefore $b_{r_1} = b_{r_2}$ which contradicts our assumption. Thus $b_r = b$ does not depend on the special pure elements $r \in B_\Lambda$ and therefore h acts as multiplication by b on the special pure elements of B_Λ. Thus h is scalar multiplication by b on B_Λ and using density also on R.

It remains to prove that there is $\delta_* < \lambda$ such that for every $r \in B_\Lambda$ we have $b_r \in R_{\delta_*}$.

Assume towards contradiction that for every $\delta \in C$ there is some element $r_\delta \in B_\Lambda$ such that $b_\delta = b_{r_\delta} \notin R_\delta$. We may write r_δ and also b_{r_δ} as elements in some polynomial ring over R_δ, hence $r_\delta = \sigma_{r_\delta}(x^\delta_i : i < i_{r_\delta})$ and $b_\delta = \sigma_{b_\delta}(\tilde{x}^\delta_i : i < i_{b_\delta})$. Thus σ_{r_δ} and σ_{b_δ} are polynomials over R_δ and the x^δ_i's and $\tilde{x}^\delta_i$ are independent elements over R_δ. For cardinality reasons we may assume that for all $\delta \in C$ we have $i_{r_\delta} = i_r$ and $i_{b_\delta} = i_b$ for some fixed $i_r, i_b \in \mathbb{N}$. Now choose $n < \omega$ and note that canonical identification $\varphi\colon \bigcup_{\alpha<\lambda} R_\alpha/s_n R_\alpha \to \bigcup_{\alpha<\lambda}(R^*_\alpha + s_n R)/s_n R$ is an epimorphism. Let $\bar{\sigma}_{r_\delta}$ and $\bar{\sigma}_{b_\delta}$ be the images of the polynomials σ_{r_δ} and σ_{b_δ} under φ. Since $|\bigcup_{\alpha<\delta}(R^*_\alpha + s_n R)/s_n R| < \lambda$ for every $\delta < \lambda$ and C consists of limit ordinals the mapping $\phi\colon C \to R/s_n R$, $\delta \mapsto (\bar{\sigma}_{r_\delta}, \bar{\sigma}_{b_\delta})$ is regressive on C. Thus application of Fodor's lemma shows that ϕ is constant on some stationary subset C' of C and without loss of generality we may assume that $C = C'$.

For $\delta \in C$ choose $\delta_1, \delta_2 \in C$ such that $\delta_1 < \delta_2$ and $x_i^\delta, \tilde{x}_j^\delta \in R_{\delta_1}$ for all $i < i_r$, $j < i_b$. Let R' be the smallest polynomial ring over R_δ generated by at least the elements $x_i^{\delta_1}, x_i^{\delta_2}$ and $\tilde{x}_i^{\delta_1}, \tilde{x}_i^{\delta_2}$ such that $a_1a_2 = a_3$ and $a_2, a_3 \in R'$ implies $a_1 \in R'$. We may choose $R' = R_\delta[H]$ as the polynomial ring where $H \subseteq R \setminus R_\delta$ contains the set $\{x_i^{\delta_1}, x_i^{\delta_2}, \tilde{x}_j^{\delta_1}, \tilde{x}_j^{\delta_2} : i < i_r, j < i_b\}$. We now consider

$$b_{r_\delta + r_{\delta_2}}(r_\delta + r_{\delta_2}) = h(r_\delta + r_{\delta_2}) = h(r_\delta) + h(r_{\delta_2}) = b_\delta r_\delta + b_{\delta_2} r_{\delta_2}. \tag{4.11}$$

By choice of R' and $r_\delta, r_{\delta_2}, b_\delta, b_{\delta_2} \in R'$ follows $b_{r_\delta + r_{\delta_2}} \in R'$. If some x_i^δ appears in the support of $b_{r_\delta + r_{\delta_2}}$, then the product $x_i^\delta x_j^{\delta_2}$ appears on the left side (for some $j < i_b$) of (4.11) but not on the right side—a contradiction. Similarly, no $x_i^{\delta_2}$ can appear in the support of $b_{r_\delta + r_{\delta_2}}$. Thus $(b_{r_\delta + r_{\delta_2}} - b_\delta)r_\delta = -(b_{r_\delta + r_{\delta_2}} - b_{\delta_2})r_{\delta_2}$ and therefore $b_{r_\delta + r_{\delta_2}} = b_\delta = b_{\delta_2}$. Hence $b_{\delta_2} \in R_{\delta_2}$. But this contradicts the choice of r_{δ_2}. The existence of δ^* such that all elements b_r related to special pure elements are in R_{δ^*} is established. ■

Corollary 4.7 *There exists an almost-free E-ring of cardinality* $\aleph_1$.

Remark 4.8 We note that the Main Theorem could also be proved for cardinals $\aleph_1 \le \lambda \le 2^{\aleph_0}$ which are not regular. The proof for $\mathrm{cf}(\lambda) = \omega$ would be much more technical and complicated.

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