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THE LAMBDA-PROPERTY FOR GENERALISED DIRECT SUMS OF NORMED SPACES

ROBERT H. LOHMAN AND THADDEUS J. SHURA

This paper considers direct sums of normed spaces with respect to a Banach space with a normalised, unconditionally strictly monotone basis. Necessary and sufficient conditions are given for such direct sums to have the λ-property. These results are used to construct examples of reflexive Banach spaces U and V such that U has the uniform λ-property but U^* fails to have the λ-property, while V and V^* fail to have the λ-property.

If X is a normed space and x is in the closed unit ball B_X of X, a triple (e, y, λ) is said to be amenable to x in case $e \in \text{ext}\,(B_X)$, $y \in B_X$, $0 < \lambda \leqslant 1$ and $x = \lambda e + (1-\lambda)y$. In this case, the number $\lambda(x)$ is defined by

$$\lambda(x) = \sup\{\lambda : (e, y, \lambda) \text{ is amenable to } x\}.$$

X is said to have the λ-property if each $x \in B_X$ admits an amenable triple. If X has the λ-property and $\lambda(X) \equiv \inf\{\lambda(x) : x \in B_X\} > 0$, then X is said to have the uniform λ-property .

General facts and geometric ramifications concerning the λ-property can be found in [**1**] and [**4**]. It is now known that many different types of classical sequence and function spaces have the λ-property or uniform λ-property (see [**1-3, 5, 7, 10**]). In this paper, our goal is to consider generalised direct sums of the form $\left(\oplus \sum_{k=1}^{\infty} X_k\right)_Z$, where Z is a Banach space with a normalised, unconditionally strictly monotone basis and (X_k) is a sequence of normed spaces. We give necessary and sufficient conditions for such spaces to have the λ-property (Theorem 8, Corollary 10) or to have the uniform λ-property (Theorem 9, Corollary 11). In particular, our results generalise Theorem 3 of [**5**], which considered $\left(\oplus \sum_{k=1}^{\infty} X_k\right)_{\ell_1}$. Using our results, we are able to give examples of reflexive Banach spaces U and V such that U has the uniform λ-property but U^* fails to have the λ-property , while V and V^* both fail to have the λ-property .

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0. Notation

Throughout the paper, Z denotes a Banach space with a normalised unconditional basis (z_k). The basis (z_k) is said to be unconditionally monotone in case $\sum_{k=1}^{\infty} a_k z_k, \sum_{k=1}^{\infty} b_k z_k \in Z$ and $|a_k| \leqslant |b_k|$ for all k imply

$$\left\| \sum_{k=1}^{\infty} a_k z_k \right\| \leqslant \left\| \sum_{k=1}^{\infty} b_k z_k \right\|. \tag{1}$$

If, in addition, $|a_k| < |b_k|$ for some k implies strict inequality in (1), then (z_k) is said to be unconditionally strictly monotone. For example, the standard unit vector bases of ℓ_p, $1 \leqslant p < \infty$, are unconditionally strictly monotone, while the standard unit vector basis of c_0 is not. *Throughout the paper, it is assumed that (z_k) is unconditionally strictly monotone.* S_X denotes the unit sphere of a normed space X. If X and Y are normed spaces, we write $X \cong Y$ if X is isometrically isomorphic to Y.

Given a sequence (X_k) of normed spaces, $X = \left(\oplus \sum_{k=1}^{\infty} X_k\right)_Z$ denotes the normed space

$$\{x = (x_k) : x_k \in X_k \text{ for all } k \text{ and } \sum_{k=1}^{\infty} \|x_k\| z_k \in Z\}$$

with $\|x\|$ defined by

$$\|x\| = \left\| \sum_{k=1}^{\infty} \|x_k\| z_k \right\|.$$

If each space X_k has the λ-property , we denote its λ-function by λ_k. In this case, if $x = (x_k) \in B_X$ and $x \neq 0$, we write

$$\Lambda(x) = \inf\{\lambda_k \left(\frac{x_k}{\|x_k\|} \right) : x_k \neq 0\}.$$

1. The λ-Property in $\left(\oplus \sum_{k=1}^{\infty} X_k\right)_Z$

In order to investigate the λ-property for the normed space $X = \left(\oplus \sum_{k=1}^{\infty} X_k\right)_Z$, it is necessary to have a description of the extreme points of B_X. We wish to thank Professor Pei-Kee Lin for pointing out such a description (Lemma 1) and for suggesting investigation of the λ-property in general spaces X as above. The proof of the following lemma is essentially the same as that for the case $Z = \ell_p$ (see [9]) and is omitted.

LEMMA 1. *If $x = (x_k) \in B_X$, the following are equivalent:*

(a) $x \in \operatorname{ext}(B_X)$

(b) $\sum_{k=1}^{\infty} \|x_k\| z_k \in \operatorname{ext}(B_Z)$ *and* $x_k/\|x_k\| \in \operatorname{ext}(B_{X_k})$, *if* $x_k \neq 0$.

REMARK 2. Let (ε_k) be a sequence of scalars with $|\varepsilon_k| = 1$ for all k.

The mapping $T : Z \to Z$, defined by

$$T\left(\sum_{k=1}^{\infty} a_k z_k\right) = \sum_{k=1}^{\infty} \varepsilon_k a_k z_k,$$

is a linear isometry of Z onto Z. In particular, $z \in \operatorname{ext}(B_Z)$ if and only if $T(z) \in \operatorname{ext}(B_Z)$.

LEMMA 3. *Assume that each summand X_k has the λ-property and $x = (x_k) \in S_X$. If $\sum_{k=1}^{\infty} \|x_k\| z_k$ admits an amenable triple $\left(\sum_{k=1}^{\infty} a_k z_k, \sum_{k=1}^{\infty} b_k z_k, \lambda\right)$ and $\Lambda(x) > 0$, then x admits an amenable triple and $\lambda(x) \geqslant \lambda \Lambda(x)$.*

PROOF: Let $0 < \alpha < \Lambda(x)$. If $x_k \neq 0$, then $\alpha < \Lambda(x) \leqslant \lambda_k(x_k/\|x_k\|)$. By Proposition 1.2 of [1], there is a triple (e_k, y'_k, α) amenable to $x_k/\|x_k\|$. If $x_k = 0$, define $e_k = y'_k = 0$. Then for all k, we have

$$x_k = \alpha\|x_k\| e_k + (1-\alpha)\|x_k\| y'_k. \tag{2}$$

Since $\sum_{k=1}^{\infty} \|x_k\| z_k = \sum_{k=1}^{\infty} [\lambda a_k + (1-\lambda) b_k] z_k$, we obtain

$$\|x_k\| = \lambda a_k + (1-\lambda) b_k \tag{3}$$

for all k. By (2) and (3), write $x_k = (\alpha\lambda) a_k e_k + (1-\alpha\lambda) y_k$, where

$$y_k = \frac{\alpha(1-\lambda) b_k e_k + (1-\alpha)\lambda a_k y'_k + (1-\alpha)(1-\lambda) b_k y'_k}{1-\alpha\lambda}.$$

Next, observe that all of the sequences $(a_k e_k)$, $(b_k e_k)$, $(a_k y'_k)$, $(b_k y'_k)$ are in B_X (for example, $\|(a_k e_k)\| = \|\sum_{k=1}^{\infty} \|a_k e_k\| z_k\| \leqslant \|\sum_{k=1}^{\infty} a_k z_k\| = 1$). Thus, if $y = (y_k)$, we have

$$\|y\| \leqslant \frac{\alpha(1-\lambda)\|(b_k e_k)\| + (1-\alpha)\lambda\|(a_k y'_k)\| + (1-\alpha)(1-\lambda)\|(b_k y'_k)\|}{1-\alpha\lambda} \leqslant 1.$$

Letting $e = (a_k e_k)$, we see that $\|e\| \leqslant 1$. Since $x = \alpha\lambda e + (1-\alpha\lambda) y$, $0 < \alpha\lambda < 1$ and $\|x\| = 1$, we must have $\|e\| = \|y\| = 1$. Therefore,

$$1 = \|e\| = \|\sum_{k=1}^{\infty} \|a_k e_k\| z_k\| \leqslant \|\sum_{k=1}^{\infty} |a_k| z_k\| = 1.$$

By strict monotonicity, $\|a_k e_k\| = |a_k|$ for all k. Consequently, $\sum_{k=1}^{\infty} \|a_k e_k\| z_k = \sum_{k=1}^{\infty} |a_k| z_k$. By hypothesis $\sum_{k=1}^{\infty} a_k z_k \in \operatorname{ext}(B_Z)$. Remark 2 yields $\sum_{k=1}^{\infty} |a_k| z_k \in \operatorname{ext}(B_Z)$. Also, if $a_k e_k \neq 0$, then $e_k \in \operatorname{ext}(B_{X_k})$ and $a_k e_k / \|a_k e_k\| = \varepsilon_k e_k$, where $|\varepsilon_k| = 1$. Therefore, $a_k e_k / \|a_k e_k\| \in \operatorname{ext}(B_{X_k})$ whenever $a_k e_k \neq 0$. By Lemma 1, $e \in \operatorname{ext}(B_X)$. This shows $(e, y, \alpha\lambda)$ is amenable to x and establishes the fact that $\lambda(x) \geqslant \alpha\lambda$. Taking the supremum over all such α establishes $\lambda(x) \geqslant \lambda\Lambda(x)$.

Theorem 4. *Assume that $X_k, k = 1, 2, \dots$, and Z have the λ-property. If there exists a subset $\mathbb{N}_0$ of $\mathbb{N}$, with finite complement, such that $\inf_{k \in \mathbb{N}_0} \lambda_k(X_k) > 0$, then*

(i) $X = \left(\oplus \sum_{k=1}^{\infty} X_k\right)_Z$ *has the λ-property.*

(ii) *If $0 \neq x = (x_k) \in B_X$,*

$$\lambda(x) \geqslant \frac{1 + \|x\|}{2} \Lambda(x) \lambda \left(\sum_{k=1}^{\infty} \frac{\|x_k\|}{\|x\|} z_k \right).$$

Proof: Let $0 \neq x = (x_k) \in B_X$. Then $x/\|x\| = (x_k)/\|x\| \in S_X$ and $\sum_{k=1}^{\infty} (\|x_k\|/\|x\| z_k)$ admits an amenable triple $\left(\sum_{k=1}^{\infty} a_k z_k, \sum_{k=1}^{\infty} b_k z_k, \lambda\right)$, where $\lambda \leqslant \lambda\left(\sum_{k=1}^{\infty} (\|x_k\|/\|x\|) z_k\right)$. Since $\inf_{k \in \mathbb{N}_0} \lambda_k(X_k) > 0$, it follows that $\Lambda(x) > 0$. By Lemma 3, $x/\|x\|$ admits an amenable triple and

$$\lambda(x/\|x\|) \geqslant \lambda\Lambda(x/\|x\|) = \lambda\Lambda(x).$$

Taking the supremum over all such λ shows

$$\lambda\left(\frac{x}{\|x\|}\right) \geqslant \Lambda(x) \lambda \left(\sum_{k=1}^{\infty} \frac{\|x_k\|}{\|x\|} z_k \right).$$

By the proof of Lemma 2.1 of [1], x admits an amenable triple, establishing (i), and $\lambda(x) \geqslant ((1 + \|x\|)/2) \lambda(x/\|x\|)$, establishing (ii). ∎

In order to obtain a converse of Theorem 4, we need $\operatorname{ext}(B_Z)$ to have a diversity of extreme points.

Definition 5. The extreme points of B_Z are said to be diversified if for each increasing sequence (k_n) in $\mathbb{N}$, B_Z has an extreme point of the form $\sum_{n=1}^{\infty} a_n z_{k_n}$, where $a_n \neq 0$ for all n.

REMARK 6. There are many different conditions under which the extreme points of B_Z are diversified. Let (k_n) be an increasing sequence in $\mathbb{N}$. If Z is strictly convex, let $w = \sum_{n=1}^{\infty} (z_{k_n})/(2^n)$. Then $w/\|w\|$ is an extreme point of B_Z of the form required in Definition 5. Also, if Z is a symmetric space (see [8]) and B_Z contains an extreme point $\sum_{k=1}^{\infty} a_k z_k$ with infinite support, let the nonzero a_k's be indexed by $j_1 < j_2 < \cdots$. Then the vector $\sum_{n=1}^{\infty} a_{j_n} z_{k_n}$ is an extreme point of B_Z. Finally, if the extreme points of B_Z are diversified, then for each increasing sequence (k_n) in $\mathbb{N}$, Remark 2 guarantees that there exists $\sum_{n=1}^{\infty} a_n z_{k_n} \in \operatorname{ext}(B_Z)$ with $a_n > 0$ for all n.

THEOREM 7. *Assume* $X = \left(\oplus \sum_{k=1}^{\infty} X_k\right)_Z$ *has the* λ*-property. Then:*

(i) *Each summand* X_k *has the* λ*-property.*
(ii) Z *has the* λ*-property.*
(iii) *If, in addition, the extreme points of* B_Z *are diversified, there exists a subset* $\mathbb{N}_0$ *of* $\mathbb{N}$*, with finite complement, such that* $\inf_{k \in \mathbb{N}_0} \lambda_k(X_k) > 0$.

PROOF: (i) We show that X_1 has the λ-property (the proof for other indices is the same). If $x_1 \in S_{X_1}$, define $x = (x_1, 0, 0, \ldots)$. By hypothesis, we can write $x = \lambda e + (1-\lambda)y$, where $e = (e_k) \in \operatorname{ext}(B_X)$, $y = (y_k) \in B_X$, $0 < \lambda \leqslant 1$. If $\lambda = 1$, then $x = e$ and $x_1 = e_1 \in \operatorname{ext}(B_{X_1})$ by Lemma 1. If $0 < \lambda < 1$, then $x_1 = \lambda e_1 + (1-\lambda) y_1$ forces $\|e_1\| = \|y_1\| = 1$. By strict monotonicity, $e_k = y_k = 0$ for $k \geqslant 2$. Then $e_1 \in \operatorname{ext}(B_{X_1})$ and (e_1, y_1, λ) is amenable to x_1. Since unit vectors in X_1 admit amenable triples, the proof of Lemma 2.1 of [1] shows that X_1 has the λ-property.

(ii) It suffices to show that each $z = \sum_{k=1}^{\infty} a_k z_k \in S_Z$ admits an amenable triple. By Remark 2, we may assume $a_k \geqslant 0$ for all k. For each k, choose $e_k \in \operatorname{ext}\left(B_{X_k}\right)$ and define $x = (a_k e_k)$. Then $\|x\| = \|z\| = 1$. We can write $x = \lambda e + (1-\lambda)y$, where $e = (v_k) \in \operatorname{ext}(B_X)$, $y = (y_k) \in S_X$ and $0 < \lambda \leqslant 1$. If $\lambda = 1$, then $x = e$ and Lemma 1 yields $z \in \operatorname{ext}(B_Z)$. Thus, we may assume $0 < \lambda < 1$. For all k, we have

$$a_k e_k = \lambda v_k + (1-\lambda) y_k.$$

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Therefore,

$$\begin{aligned} 1 = \|x\| = \left\|\sum_{k=1}^{\infty} a_k z_k\right\| &= \left\|\sum_{k=1}^{\infty} \|\lambda v_k + (1-\lambda) y_k\| z_k\right\| \\ &\leqslant \left\|\sum_{k=1}^{\infty} [\lambda\|v_k\| + (1-\lambda)\|y_k\|]\, z_k\right\| \\ &\leqslant \lambda \left\|\sum_{k=1}^{\infty} \|v_k\| z_k\right\| + (1-\lambda)\left\|\sum_{k=1}^{\infty} \|y_k\| z_k\right\| \\ &= 1. \end{aligned}$$

By strict monotonicity,

$$a_k = \|a_k e_k\| = \lambda\|v_k\| + (1-\lambda)\|y_k\|$$

for all k. Consequently,

$$z = \lambda\left(\sum_{k=1}^{\infty} \|v_k\| z_k\right) + (1-\lambda)\left(\sum_{k=1}^{\infty} \|y_k\| z_k\right).$$

Since $\sum_{k=1}^{\infty} \|v_k\| z_k \in \operatorname{ext}(B_Z)$ by Lemma 1, it follows that $\left(\sum_{k=1}^{\infty} \|v_k\| z_k, \sum_{k=1}^{\infty} \|y_k\| z_k, \lambda\right)$ is amenable to z.

(iii) Assume, to the contrary, that no such set $\mathbb{N}_0$ exists. Then there exist $k_1 < k_2 < \cdots$ with $\lambda_{k_n}(X_{k_n}) \to 0$. Therefore, we can choose $u_{k_n} \in S_{X_{k_n}}$ such that $\lambda_{k_n}(u_{k_n}) \to 0$. By hypothesis, there exists $\sum_{n=1}^{\infty} a_{k_n} z_{k_n} \in \operatorname{ext}(B_Z)$ with $a_{k_n} > 0$ for all n. If $k \notin \{k_1, k_2, \dots\}$, define $a_k = 0$, $u_k = 0$. Then $x \equiv (a_k u_k) \in S_X$.

We can write $x = \lambda e + (1-\lambda) y$, where $0 < \lambda \leqslant 1$, $e = (v_k) \in \operatorname{ext}(B_X)$, $y = (y_k) \in S_X$. If $\lambda = 1$, then $x = e$ and, by Lemma 1, $u_{k_n} \in \operatorname{ext}\left(B_{X_{k_n}}\right)$ for all n, which contradicts $\lambda_{k_n}(u_{k_n}) \to 0$. Thus, $0 < \lambda < 1$ and, as in the proof of (ii), we obtain

$$a_k = \lambda\|v_k\| + (1-\lambda)\|y_k\|$$

for all k. In particular, $v_k = y_k = 0$ for $k \notin \{k_1, k_2, \dots\}$. Therefore,

$$\sum_{n=1}^{\infty} a_{k_n} z_{k_n} = \lambda\left(\sum_{n=1}^{\infty} \|v_{k_n}\| z_{k_n}\right) + (1-\lambda)\left(\sum_{n=1}^{\infty} \|y_{k_n}\| z_{k_n}\right).$$

Since $\sum_{n=1}^{\infty} a_{k_n} z_{k_n} \in \operatorname{ext}(B_Z)$, we must have

$$\sum_{n=1}^{\infty} a_{k_n} z_{k_n} = \sum_{n=1}^{\infty} \|v_{k_n}\| z_{k_n} = \sum_{n=1}^{\infty} \|y_{k_n}\| z_{k_n},$$

or

$$a_{k_n} = \|v_{k_n}\| = \|y_{k_n}\|$$

for all n. Therefore, for all n

$$u_{k_n} = \lambda \frac{v_{k_n}}{\|v_{k_n}\|} + (1-\lambda) \frac{y_{k_n}}{\|y_{k_n}\|}.$$

But $v_{k_n}/\|v_{k_n}\| \in \operatorname{ext}\left(B_{X_{k_n}}\right)$ implies $\lambda_{k_n}(u_{k_n}) \geqslant \lambda$ for all n, a contradiction. ∎

Combining Theorems 4 and 7, we obtain

THEOREM 8. *Assume that the extreme points of B_Z are diversified. The following are equivalent:*

(a) $X = \left(\oplus \sum_{k=1}^{\infty} X_k\right)_Z$ *has the λ-property .*

(b) *Each space X_k has the λ-property , there exists a subset $\mathbb{N}_0$ of $\mathbb{N}$, with finite complement, such that $\inf_{k \in \mathbb{N}_0} \lambda_k(X_k) > 0$, and Z has the λ-property*

We now turn our attention to the uniform λ-property .

THEOREM 9. *The following are equivalent:*

(a) $X = \left(\oplus \sum_{k=1}^{\infty} X_k\right)_Z$ *has the uniform λ-property .*

(b) *Each summand X_k has the uniform λ-property , $\Lambda \equiv \inf_k \lambda_k(X_k) > 0$ and Z has the uniform λ-property .*

In this case, we have

$$\lambda(X) \geqslant \frac{\Lambda}{2} \lambda(Z).$$

PROOF: $(a) \Rightarrow (b)$. By Theorem 7, each summand X_k has the λ-property . Moreover, the proof of Theorem 7 shows that if $x_k \in S_{X_k}$, then

$$\lambda_k(x_k) \geqslant \lambda(0, \dots, 0, x_k, 0, \dots) \geqslant \lambda(X) > 0.$$

It follows from Lemma 2.1 of [1] that $\Lambda > 0$.

$(b) \Rightarrow (a)$. This follows from Theorem 4, as does the asserted inequality. ∎

In case the summands are the same, we can sharpen our results as follows.

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Corollary 10. *Let $X_k = Y$ for all k and assume B_Z contains an extreme point with infinite support. The following are equivalent:*

(a) $X = \left(\oplus \sum_{k=1}^{\infty} Y\right)_Z$ *has the λ-property.*

(b) *Y has the uniform λ-property and Z has the λ-property.*

Proof: $(b) \Rightarrow (a)$. This follows from Theorem 7.

$(a) \Rightarrow (b)$. By Theorem 7, Y and Z have the λ-property. If Y fails to have the uniform λ-property, there is a sequence (w_n) in S_Y with $\lambda(w_n) \to 0$. By hypothesis, there exists $\sum_{n=1}^{\infty} a_{k_n} z_{k_n} \in \text{ext}\,(B_Z)$ with $a_{k_n} > 0$ for all n. Define $a_k = 0$ if $k \notin \{k_1, k_2, \ldots\}$ and let

$$u_k = \begin{cases} w_n, & \text{if } k = k_n \text{ for some } n \\ 0, & k \notin \{k_1, k_2, \ldots\}. \end{cases}$$

We can then write $x = (a_k u_k)$ and proceed as in the proof of part (iii) of Theorem 7 to obtain the same contradiction as before. ∎

An immediate consequence of Theorem 9 is

Corollary 11. *Let $X_k = Y$ for all k. The following are equivalent:*

(a) $X = \left(\oplus \sum_{k=1}^{\infty} Y\right)_Z$ *has the uniform λ-property.*

(b) *Y and Z have the uniform λ-property.*

In this case, $\lambda(X) \geqslant \lambda(Y)\lambda(Z)/2$.

Combining Corollaries 10 and 11, we obtain

Corollary 12. *Assume that Y has the uniform λ-property and that Z has the λ-property but not the uniform λ-property. If B_Z contains an extreme point with infinite support, then $\left(\oplus \sum_{k=1}^{\infty} Y\right)_Z$ has the λ-property but not the uniform λ-property*

The preceding corollary provides us with the following curiosity which one should compare to the well-known fact that $\left(\oplus \sum_{k=1}^{\infty} \ell_p\right)_{\ell_p} \cong \ell_p$, $1 \leqslant p < \infty$.

Corollary 13. *If Z has the λ-property but not the uniform λ-property and $\left(\oplus \sum_{k=1}^{\infty} Z\right)_Z \cong Z$, then all the extreme points of B_Z have finite support.*

Remark 14. In view of Corollary 12, it should be noted that there are spaces Z which have the λ-property, fail to have the uniform λ-property and for which B_Z contains extreme points with infinite support. For example, consider ℓ_1 and ℓ_2 over the reals and let $Z = (\ell_1 \oplus \ell_2)_{\ell_2}$ (that is, $Z \cong (\ell_1 \oplus \mathbf{R} \oplus \mathbf{R} \oplus \cdots)_{\ell_2}$). Then Z has a normalised

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unconditionally strictly monotone basis and, by Lemma 1, B_Z contains extreme points with infinite support. On the other hand, ℓ_1 has the λ-property but not the uniform λ-property ([**1**]). Consequently, Theorems 8 and 9 imply that Z has the λ-property but not the uniform λ-property .

2. Reflexivity and the λ-property

We close with two examples concerning reflexive Banach spaces. It has been conjectured that reflexivity might play a special role in the study of the λ-property and the uniform λ-property . We now show that the most natural questions one might pose regarding reflexive spaces and the λ-property have a negative answer. Consequently, reflexivity does not appear to play any significant role in the study of these properties.

It follows from the results of [1] that $\lambda(\ell_\infty^k) = 1/2$ and $\lambda(\ell_1^k) \leqslant 1/k$ for all k. By Theorem 9, the reflexive Banach space

$$U = \left(\oplus \sum_{k=1}^{\infty} \ell_\infty^k\right)_{\ell_2}$$

has the uniform λ-property . Since $U^* \cong \left(\oplus \sum_{k=1}^{\infty} \ell_1^k\right)_{\ell_2}$, Theorem 8 shows that U^* fails to have the λ-property (this fact was also obtained in [**6**] by means of direct calculations rather than a general theorem).

Now let $V = (U \oplus U^*)_{\ell_2}$; that is,

$$V \cong (\ell_\infty^1 \oplus \ell_1^1 \oplus \ell_\infty^2 \oplus \ell_1^2 \oplus \cdots)_{\ell_2}.$$

Then V is a reflexive Banach space which fails to have the λ-property by Theorem 8. Since $V^* \cong V, V^*$ also fails to have the λ-property . This is the first example of a reflexive Banach space with this property (a nonreflexive Banach space with this property was given in [**1**]).

Finally, it should be noted that a reflexive Banach space W with the λ-property does not necessarily have the uniform λ-property . Such an example is given in [**6**]. In fact, B_W can be constructed from B_{ℓ_2} with very slight modifications.

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Department of Mathematical Sciences
Kent State University
Kent, OH 44242
United States of America

Department of Mathematical Sciences
Kent State University at Salem
Salem OH 44460
United States of America