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THE FILLED INELASTIC MEMBRANE: A SET OF CHALLENGING PROBLEMS IN THE CALCULUS OF VARIATIONS

MICHAEL A. B. DEAKIN

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Abstract

In a wide variety of related problems, a flexible but inextensible membrane is filled to capacity with incompressible fluid. In all such cases, the resulting shape satisfies a set of three simultaneous partial differential equations. At present there are no known solutions of any generality, and the main aim of this paper is to formulate the partial differential equations with a view to stimulating further interest in this important class of problems.

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Consider a flexible but inextensible membrane of a specified shape, which is to be filled to capacity with incompressible fluid. There are many familiar examples of this situation. The circular air-cushion (Mylar balloon) has been discussed in such a context (see [3, 5] and [4, pp. 371–380]), and rectangular analogues have also attracted attention [2, 6, 7]. Other such cases give the shapes of sausages, squeeze-tubes, life-rings and the like. Professor Jim Hill (personal communication) has suggested possible applications to questions arising in nanoscientific contexts. The inverse problem, in which the final shape is specified and that of the unfilled membrane is sought, is also of interest, with applications to the design of high-altitude scientific balloons [1] and possibly to the construction of automobile airbags. The sausage and the circular air-cushion (two of the most amenable examples) can be analyzed in terms of the equations given here, but are better discussed in simpler terms because the circular symmetry allows the three partial differential equations to be replaced by a single ordinary one; see [3–5] and the analysis below.

In all of these cases, the shape of the filled membrane can be described by means of three Cartesian coordinates x, y and z, with the usual conventions. Points on the bounding surface (the membrane) are given coordinates X, Y and Z, and we may choose the coordinates so that $0 \le x \le X$, $0 \le y \le Y$ and $0 \le z \le Z$. The interior of the filled membrane will be denoted by V and its surface (the membrane) by ∂V.

Let us express the points on the membrane using orthogonal coordinates s, t that are fixed in the membrane. Set $0 \leq s \leq f(t) \leq a$ and $0 \leq t \leq g(s) \leq b$, say. The curves $s =$ constant and $t =$ constant generate the bounding surface ∂V of the volume when the membrane is filled. We have $X = X(s, t)$, $Y = Y(s, t)$ and $Z = Z(s, t)$; and, because the membrane is inextensible, the length of each of the generators remains constant throughout the filling process. Thus

$$\int_0^{f(t)} \sqrt{X_s^2 + Y_s^2 + Z_s^2}\, ds = f(t).$$

Write $S = X_s^2 + Y_s^2 + Z_s^2$ and $T = X_t^2 + Y_t^2 + Z_t^2$. Then it follows that

$$\int_0^b \int_0^{f(t)} \sqrt{S}\, ds\, dt = \iint_{\partial V} \sqrt{S}\, ds\, dt = \text{constant}. \tag{1}$$

Similarly,

$$\iint_{\partial V} \sqrt{T}\, ds\, dt = \text{constant}. \tag{2}$$

Equations (1) and (2) constitute the constraints on the isoperimetric problem posed by the maximal filling of the interior.

In order to calculate that volume, choose any one of the coordinates (without loss of generality let us choose Z) and imagine V to be sliced along the planes $z = Z =$ constant. Points in the interior of V may then be expressed as $(x, y, z) = (uX, uY, Z)$, where $0 \leq u \leq 1$.

For future reference, define

$$\begin{aligned} J_1 &= Y_s Z_t - Y_t Z_s, \\ J_2 &= Z_s X_t - Z_t X_s, \\ J_3 &= X_s Y_t - X_t Y_s. \end{aligned}$$

Then the volume element $dx\, dy\, dz$ may be given in the new coordinates as

$$\left| \frac{\partial(x, y, z)}{\partial(s, t, u)} \right| ds\, dt\, du = \begin{vmatrix} uX_s & uX_t & X \\ uY_s & uY_t & Y \\ Z_s & Z_t & 0 \end{vmatrix} ds\, dt\, du = |uXJ_1 + uYJ_2|\, ds\, dt\, du,$$

and so

$$V = \int_0^1 u\, du \iint_{\partial V} |XJ_1 + YJ_2|\, ds\, dt.$$

Thus, we seek to maximize

$$\iint_{\partial V} |XJ_1 + YJ_2|\, ds\, dt$$

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subject to the constraints (1) and (2). To this end, we find the Euler–Lagrange equations for the function

$$F = (XJ_1 + YJ_2) + \lambda\sqrt{S} + \mu\sqrt{T}, \quad \text{where } \lambda \text{ and } \mu \text{ are Lagrange multipliers.}$$

There are three Euler–Lagrange equations, which (following some straightforward manipulation and the incorporation of a factor 2) simplify to

$$\begin{aligned} J_1 &= \lambda\frac{\partial}{\partial s}\left[\frac{X_s}{\sqrt{S}}\right] + \mu\frac{\partial}{\partial t}\left[\frac{X_t}{\sqrt{T}}\right], \\ J_2 &= \lambda\frac{\partial}{\partial s}\left[\frac{Y_s}{\sqrt{S}}\right] + \mu\frac{\partial}{\partial t}\left[\frac{Y_t}{\sqrt{T}}\right], \\ J_3 &= \lambda\frac{\partial}{\partial s}\left[\frac{Z_s}{\sqrt{S}}\right] + \mu\frac{\partial}{\partial t}\left[\frac{Z_t}{\sqrt{T}}\right]. \end{aligned} \tag{3}$$

These equations apply to all situations of the sort under consideration here. Their action may be illustrated and their validity tested by applying them to the case of the circular Mylar balloon [2, 6, 7]. Because of the circular symmetry, we seek a solution of the form

$$X = R(s)\cos t, \quad Y = R(s)\sin t, \quad Z = Z(s).$$

Then $J_3 = R(s)R'(s)$ and $S = [R'(s)]^2 + [Z'(s)]^2$, and the third equation of (3) becomes

$$R(s)R'(s) = \lambda\frac{d}{ds}\left[\frac{Z'(s)}{\sqrt{[R'(s)]^2 + [Z'(s)]^2}}\right].$$

This equation integrates immediately to give

$$R^2 = 2\lambda\left[1 + \left(\frac{dR}{dZ}\right)^2\right]^{-1/2} + C, \quad \text{where } C \text{ is a constant.}$$

Without loss of generality, we can take the maximum value of R to be 1. When $R = 0$, we have $dR/dZ = -\infty$ so that $C = 0$; and when $R = 1$, we have $dR/dZ = 0$ so that $2\lambda = 1$. The resulting equation then gives rise to the elliptic integral discovered earlier by other authors [3–5]. However, it should be noted that in this special case it is simpler to proceed from first principles.

A noteworthy feature of the general equations (3) is that they are completely symmetrical in the coordinates X, Y and Z. This underscores the arbitrariness of the choice of Z in the derivation above. In most applications, one or other of the X, Y and Z will be a preferred coordinate, but this detail must be supplied (as in the case just outlined) by the boundary conditions imposed.

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MICHAEL A. B. DEAKIN, School of Mathematical Sciences, Monash University, VIC 3800, Australia
e-mail: michael.deakin@sci.monash.edu.au

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