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LIMITS OF UNBOUNDED SEQUENCES OF CONTINUED FRACTIONS

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Let $X = \{x_k\}_{k\geqslant 1}$ be a sequence of positive integers. Let $Q_k = [0; x_k, x_{k-1}, \dots, x_1]$ be the finite continued fraction with partial quotients $x_i (1 \leqslant i \leqslant k)$. Denote the set of the limit points of the sequence $\{Q_k\}_{k\geqslant 1}$ by $\Lambda(X)$. In this note a necessary and sufficient condition is given for $\Lambda(X)$ to contain no rational numbers other than zero.

Let $X = \{x_k\}_{k\geqslant 1}$ be a sequence of positive integers. Let $Q_k = [0; x_k, x_{k-1}, \dots, x_1]$ be the finite continued fraction with partial quotients $x_i (1 \leqslant i \leqslant k)$. We denote the set of limit points of the sequence $\{Q_k\}_{k\geqslant 1}$ by $\Lambda(X)$. Recently, Angell [**1**] proved an interesting result on $\Lambda(X)$: $\Lambda(X)$ contains no rational numbers if the sequence $\{x_k\}_{k\geqslant 1}$ is bounded.

It is easily seen that $0 \in \Lambda(X)$ if and only if X is unbounded. In this note, using the idea in [**3**], we prove that Angell's result holds for a large family of unbounded sequences if 0 is excluded from $\Lambda(X)$.

We first introduce some new notions.

DEFINITION 1: Let $X = \{x_k\}_{k\geqslant 1}$ be a sequence of positive integers and N be a positive integer. An infinite subsequence $\{x_{k_i}\}_{i\geqslant 1}$ is said to be an N-subsequence if $x_{k_i} = N$ for all sufficiently large i.

DEFINITION 2: Let $X = \{x_k\}_{k\geqslant 1}$ be a sequence of positive integers. Then X is said to be an $\mathcal{N}$-sequence if for each N-subsequence $\{x_{k_i}\}_{i\geqslant 1}$, the subsequence $\{x_{k_i-1}\}_{i\geqslant 1}$ is bounded, that is, there is a positive number $I(N)$ such that $x_{k_i-1} \leqslant I(N)$ for $i = 1, 2, \dots$.

Obviously a bounded sequence is an $\mathcal{N}$-sequence. The converse is not true. The following example is an unbounded $\mathcal{N}$-sequence.

EXAMPLE 1: $X = \{x_k\}_{k\geqslant 1} = \{1, 1, 2, 1, 8, 4, 2, 1, 512, 256, 128, 64, 8, 4, 2, 1, \dots\}$, where $x_k = 1$ for $k = 2^n$, $(n = 0, 1, 2, \dots)$ and $x_k = 2^i$ for $k = 2^n - i$, $(n = 2, 3, \dots$ and $1 \leqslant i < 2^{n-1})$. Because for each 2^i-subsequence $\{x_{k_i}\}$, the subsequence $\{x_{k_i-1}\}$ is bounded by $I(2^i) = 2^{i+1}$, X is an $\mathcal{N}$-sequence.

Now we give the main result.

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Theorem 1. *$\Lambda(X)$ contains no rational numbers other than zero if and only if X is an $\mathcal{N}$-sequence.*

Proof: Necessity. Suppose X is not an $\mathcal{N}$-sequence. Then there is an N-subsequence $\{x_{k_i}\}_{i\geqslant 1}$ such that $\{x_{k_i-1}\}_{i\geqslant 1}$ is not bounded. Hence there is a subsequence $\{x_{k_{i_m}-1}\}_{m\geqslant 1}$ of $\{x_{k_i-1}\}_{i\geqslant 1}$, satisfying $x_{k_{i_m}-1} \to \infty$ as $m \to \infty$. Since $\{x_{k_{i_m}}\}_{m\geqslant 1}$ is a subsequence of the N-subsequence $\{x_{k_i}\}_{i\geqslant 1}$, we have $x_{k_{i_m}} = N$ for all sufficiently large m. Hence $Q_{k_{i_m}} \to 1/N$ as $m \to \infty$. Thus $\Lambda(X)$ contains a rational number other than zero.

Sufficiency. Let a be an arbitrary rational number other than 0. Since $0 < Q_k < 1$, without loss of generality, we may assume $0 < a \leqslant 1$.

We first prove $1 \notin \Lambda(X)$. Suppose there is a subsequence $Q_{k_i} \to 1$ as $i \to \infty$. Then since $Q_{k_i} = [0; x_{k_i}, \dots, x_1] < 1/x_{k_i}$, $\{x_{k_i}\}_{i\geqslant 1}$ must be a 1-subsequence. Hence $\{x_{k_i-1}\}_{i\geqslant 1}$ is bounded by $I(1)$ and $Q_{k_i} < [0; 1, I(1)] = 1/(1+1/I(1)) < 1$, a contradiction to the assumption $Q_{k_i} \to 1$. Therefore $1 \notin \Lambda(X)$.

Suppose there is a rational number $a \neq 0$ and $a \in \Lambda(X)$. Since $0 < a < 1$, a can be expanded as a finite continued fraction: $a = [0; a_1, \dots, a_r]$. Let $Q_{k_i} \to a$. If $\{x_{k_i}\}_{i\geqslant 1}$ is not an a_1-subsequence, there are infinitely many i such that $x_{k_i} \neq a_1$. We discuss the following possible cases.

(1) There are infinitely many i with $x_{k_i} \geqslant a_1 + 2$. For these i, we have

$$\begin{aligned} a - Q_{k_i} &> [0; a_1, a_2, \dots, a_r] - [0; x_{k_i}, x_{k_i-1}, \dots, x_1] \\ &> [0; a_1, 1] - [0; a_1 + 2] > 1/(a_1+2)^2. \end{aligned}$$

Hence $Q_{k_i} \not\to a$.

(2) There are infinitely many i with $x_{k_i} \leqslant a_1 - 2$. For these i, we have

$$Q_{k_i} - a > [0; a_1 - 2, 1] - [0; a_1] > 1/a_1^2.$$

Hence $Q_{k_i} \not\to a$.

(3) There are infinitely many i with $x_{k_i} = a_1 + 1$, that is, there is an (a_1+1)-subsequence $\{x_{k_{i_m}}\}_{m\geqslant 1}$. Then $\{x_{k_{i_m}-1}\}_{m\geqslant 1}$ is bounded by $I(a_1+1)$ and

$$a - Q_{k_{i_m}} > [0; a_1, 1] - [0; a_1 + 1, I(a_1+1)] > 1/(1 + I(a_1+1))(a_1+1)^2.$$

Hence $Q_{k_i} \not\to a$.

(4) There are infinitely many i with $x_{k_i} = a_1 - 1$, that is, there is an (a_1-1)-subsequence $\{x_{k_{i_n}}\}_{n\geqslant 1}$. Then there are two possibilities:

(i) There are infinitely many n with $x_{k_{i_n}-1} \geqslant 2$. For these n, we have

$$Q_{k_{i_n}} - a > [0; a_1 - 1, 2] - [0; a_1] > 1/2a_1^2.$$

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Hence $Q_{k_i} \not\to a$.

(ii) The subsequence $\{x_{k_{i_n}-1}\}_{n\geqslant 1}$ is a 1-subsequence. Then $\{x_{k_{i_n}-2}\}_{n\geqslant 1}$ is bounded by $I(1)$ and

$$Q_{k_{i_n}} - a > [0; a_1 - 1, 1, I(1)] - [0; a_1] > 1/a_1^2(I(1)+1).$$

Hence $Q_{k_i} \not\to a$.

□

From the discussion above, we know that $\{x_{k_i}\}_{i\geqslant 1}$ must be an a_1-subsequence.

Now we prove $x_{k_i-(j-1)} = a_j \quad (1 \leqslant j \leqslant r)$ for all sufficiently large i.

Suppose j_0 is the smallest index j such that for each j with $1 \leqslant j \leqslant j_0$, $\{x_{k_i-(j-1)}\}_{i\geqslant 1}$ is an a_j-subsequence, but $\{x_{k_i-j_0}\}_{i\geqslant 1}$ is not an a_{j_0+1}-subsequence. Then for sufficiently large i, we have

$$x_{k_i-(j-1)} = a_j \qquad (1 \leqslant j \leqslant j_0),$$
$$a = [0; a_1, \ldots, a_{j_0}, a_{j_0+1}, \ldots, a_r],$$
$$Q_{k_i} = [0; a_1, \ldots, a_{j_0}, x_{k_i-j_0}, \ldots, x_1].$$

Let

$$\alpha_{j_0+1} = [a_{j_0+1}; \ldots a_r],$$
$$\beta_{j_0+1}(i) = [x_{k_i-j_0}; \ldots, x_1],$$
$$p_{j_0}/q_{j_0} = [0; a_1, \ldots, a_{j_0}].$$

By a well known fact ([**2**, Theorem 7.3] or [**3**, Lemma 1]), we have

$$a = \frac{\alpha_{j_0+1}p_{j_0} + p_{j_0-1}}{\alpha_{j_0+1}q_{j_0} + q_{j_0-1}},$$
$$Q_{k_i} = \frac{\beta_{j_0+1}(i)p_{j_0} + p_{j_0-1}}{\beta_{j_0+1}(i)q_{j_0} + q_{j_0-1}},$$
$$|Q_{k_i} - a| = \frac{|\alpha_{j_0+1} - \beta_{j_0+1}(i)|}{(\alpha_{j_0+1}q_{j_0} + q_{j_0-1})(\beta_{j_0+1}(i)q_{j_0} + q_{j_0-1})}$$
$$> \left|\alpha_{j_0+1}^{-1} - \beta_{j_0+1}^{-1}\right| / (4q_{j_0}^2).$$

Consider $D = \left|\alpha_{j_0+1}^{-1} - \beta_{j_0+1}^{-1}(i)\right| = \left|[0; a_{j_0+1}, \ldots, a_r] - [0; x_{k_i-j_0}, \ldots, x_1]\right|$. Since $\{x_{k_i-j_0}\}_{i\geqslant 1}$ is not an a_{j_0+1}-subsequence, there are infinitely many i such that $x_{k_i-j_0} \neq a_{j_0+1}$. Similar to the discussion of x_{k_i} being an a_1-subsequence, we may discuss the four cases for infinitely many i: (1) $x_{k_i-j_0} \geqslant a_{j_0+1} + 2$, (2) $x_{k_i-j_0} \leqslant a_{j_0+1} - 2$, (3) $x_{k_i-j_0} = a_{j_0+1} + 1$, (4) $x_{k_i-j_0} = a_{j_0+1} - 1$, and obtain the conclusion $Q_{k_i} \not\to a$. Hence $\{x_{k_i-j_0}\}_{i\geqslant 1}$ must be an a_{j_0+1}-subsequence, a contradiction to the assumption. Therefore $x_{k_i-(j-1)} = a_j \quad (1 \leqslant j \leqslant r)$ for sufficiently large i, and

$$Q_{k_i} = [0; a_1, \ldots, a_r, x_{k_i-r}, \ldots, x_1].$$

Again we show that $Q_{k_i} \not\to a$. We discuss two cases:

(1) r is odd. Then $Q_{k_i} > [0; a_1, \ldots, a_r, 1] = (p_r + p_{r-1})/(q_r + q_{r-1})$, and

$$\left|Q_{k_i} - a\right| > \frac{p_r}{q_r} - \frac{p_r + p_{r-1}}{q_r + q_{r-1}} > \frac{1}{2q_r^2}.$$

(2) r is even. Then $Q_{k_i} < [0; a_1, \ldots, a_r, 1]$ and

$$\left|Q_{k_i} - a\right| > \frac{p_r + q_{r-1}}{p_r + q_{r-1}} - \frac{p_r}{q_r} > \frac{1}{2q_r^2}.$$

In both cases, $Q_{k_i} \not\to a$. Therefore $\Lambda(X)$ contains no rational number other than 0. The proof is completed.

References

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