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EMBEDDING CIRCLE-LIKE CONTINUA IN E^3

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1. Introduction. A space X is *locally planar* if each point of X has a neighborhood which is embeddable in the plane. If X is a closed, locally planar subset of $\mathbf{E}^3$, we will say that X is *locally tame* if each point of X has a neighborhood in X which lies on a tame disk in $\mathbf{E}^3$; if every *cell-like subset* of X has such a neighborhood, we say that X is *strongly* locally tame.

Our principal result is that every circularly chainable continuum has a strongly locally tame embedding in $\mathbf{E}^3$. (It follows from the argument for Theorem 8 of [**6**] that every circularly chainable continuum is locally planar.)

As an application, we show that for any pseudosolenoid X, the hyperspace $C(X)$ of subcontinua of X has a particularly nice embedding in $\mathbf{E}^4$, and that (X) is embeddable in $\mathbf{E}^3$ if and only if X is embeddable in $\mathbf{E}^2$.

2. Definitions and conventions. Much of our terminology is standard and will not be repeated here. We use the terms *chainable* and *circularly chainable* as synonymous with "snake-like" and "circle-like" as defined in [**6**], and adopt the usual definitions and notations relating to upper semicontinuous decompositions.

We use the term *pseudosolenoid*, suggested by C. E. Burgess (MR 41, #9213), for any hereditarily indecomposable, circularly chainable continuum which is not chainable; we do not require that a pseudosolenoid be non-planar. (It follows from [**11**], however, that the only planar pseudosolenoid is the pseudocircle.)

A subset of $\mathbf{E}^n$ is said to be *cellular in* $\mathbf{E}^n$ if it is the intersection of a sequence $\{C_i\}$ of n-cells in $\mathbf{E}^n$ with $C_{i+1} \subset \operatorname{Int} C_i$ for each i; a continuum is *cell-like* if it can be embedded in some $\mathbf{E}^n$ so as to be cellular there. A *map* is cell-like if the preimage of each point is cell-like, and a *decomposition* is cell-like if each of its elements is cell-like. Several useful characterizations and many of the basic properties of cell-like spaces and maps are given in [**17**].

If X is a closed subset of a metric space M and G_0 is an upper semicontinuous decomposition of X, then the *trivial extension* of G_0 (obtained by adding to G_0 all singletons in $M - X$) is called the decomposition of M *generated by* G_0. As in [**1**], a cell-like upper semicontinuous decomposition G of a metric space M is said to be *simple* if $M/G \approx M$, and a closed subset X of M is said to be *simply embedded* in M if every simple decomposition of X generates a simple

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decomposition of M; if *every* cell-like upper semicontinuous decomposition of X generates a simple decomposition of M, then X is said to be *strongly* simply embedded in M.

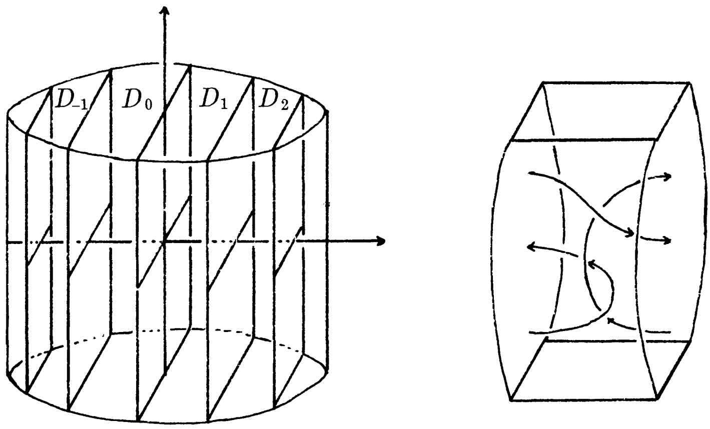
3. Preliminary remarks and examples. There are two commonly used definitions of "tame" for closed subsets X of $\mathbf{E}^3$: (1) if X is homeomorphic to a polyhedron, then X is tame if there is a homeomorphism $h:\mathbf{E}^3 \twoheadrightarrow \mathbf{E}^3$ such that $h(X)$ is a (geometric) polyhedron in $\mathbf{E}^3$, and (2) if X is a subset of a compact 2-manifold with boundary in $\mathbf{E}^3$, then X is tame if it lies on a tame 2-manifold with boundary (see [**10**, pp. 266, 333]). If X is a closed subset of $\mathbf{E}^3$ satisfying the hypothesis of (1) or (2) above, then X is locally tame if each point of X has a neighborhood in X whose closure is tame in the appropriate sense.

Since every subset of a 2-manifold with boundary is locally planar, our definition of *locally tame*, as given in the introduction, is an extension of the second of the above definitions of "locally tame" (we do not propose a definition of "tame" applicable to all closed, locally planar subsets of $\mathbf{E}^3$).

It follows from a result due to J. W. Cannon [**10**, Theorem 11.1.1] that if X is a closed, locally planar subset of $\mathbf{E}^3$ which is locally tame (in our sense), then every closed subset of X which lies on a compact 2-manifold with boundary in $\mathbf{E}^3$ is tame. In particular, if X itself lies on a compact 2-manifold with boundary, then X is strongly locally tame, since every cell-like subset of a tame 2-manifold with boundary lies on a tame disk. It is shown below, however, that locally tame closed subsets of $\mathbf{E}^3$ need not in general be strongly locally tame.

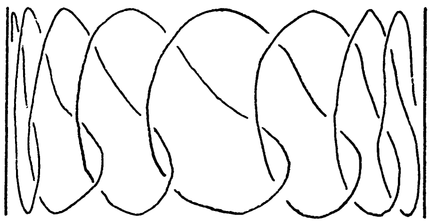
It follows from [**1**, Theorem 4.2] that every strongly locally tame closed subset of $\mathbf{E}^3$ is strongly simply embedded in $\mathbf{E}^3$. Although a locally tame subset of $\mathbf{E}^3$ need not be simply embedded (Example 3.2), it is true that for each compact locally tame set X there is a positive number ϵ such that every cell-like upper semicontinuous ϵ-decomposition of X generates a simple decomposition of $\mathbf{E}^3$; it is only necessary to cover X with a finite number of open subsets of X each of which lies in a tame disk, choose ϵ to be a Lebesgue number for this cover, and apply Theorem 4.2 of [**1**].

3.1 *Example. A locally planar continuum in* $\mathbf{E}^3$ *which is locally tame but not strongly locally tame.* Let Y denote the continuum obtained by modifying the construction of the arc of Example 1.1 of [**12**] by using, in place of the cylinder C, the set C' defined by $y^2 + z^2 \leqq 2$, $-1 \leqq x \leqq 1$, $-1 \leqq z \leqq 1$ and, instead of the ellipsoid $x^2 + 4y^2 + 4z^2 \leqq 4$, the solid K defined by $x^2 + 4y^2 \leqq 4$, $-1 \leqq z \leqq 1$. The remainder of the construction is carried out exactly as in [**12**], with the additional stipulation that for each n, the homeomorphism f_n of C' onto D_n is required to preserve z-coordinates. A comparison of Figures 1 and 2 with the corresponding figures of [**12**] should make the construction apparent.



K C'

FIGURE 1



A B

Y

FIGURE 2

If p is a point of one of the limit intervals A, B of Figure 2, there is a small closed neighborhood of p in Y which consists of a sequence of tame arcs converging nicely to an interval, and which may be taken into the xz-plane by a space homeomorphism. It follows that Y is locally tame.

If a cell-like subset of $\mathbf{E}^3$ lies on a tame disk, it is cellular in $\mathbf{E}^3$; since Y is homeomorphic to a "double sin $(1/x)$" curve, it is cell-like, and since $\mathbf{E}^3 - Y$ is homeomorphic to the complement of the arc of Example 1.1 of [**12**], Y is not cellular in $\mathbf{E}^3$. It follows that Y is not strongly locally tame.

We note that Y is not strongly simply embedded in $\mathbf{E}^3$ since the cell-like decomposition of Y whose only element is Y itself does not generate a simple decomposition of $\mathbf{E}^3$. If G_0 is any *simple* decomposition of Y, however, then G_0 has only a countable number of nondegenerate elements, each of which is a tame arc, and it follows from [**5**, Theorem 3] that G_0 generates a simple decomposition of $\mathbf{E}^3$. Hence Y is simply embedded in $\mathbf{E}^3$; using the continuum Y, however, it is easy to construct a non-simply embedded example.

3.2 *Example. A locally planar continuum in* $\mathbf{E}^3$ *which is locally tame but not simply embedded.* Let $K_1, K_2, \ldots$ be a sequence of disjoint copies of the solid K used in the construction of Y, each having its upper and lower bases on the planes $z = 1$ and $z = -1$, respectively, such that $\{K_i\}$ converges to a vertical interval Z. For each i, let Y_i be a continuum constructed in K_i exactly as Y was constructed in K, and let α_i be a horizontal interval on the x-axis irreducible from K_i to K_{i+1}. Let

$$X = Z \cup \bigcup_{i=1}^{\infty} Y_i \cup \bigcup_{i=1}^{\infty} \alpha_i.$$

It is not difficult to see that X is locally tame, and it is clear that X/Y_2 is homeomorphic to X. If G_0 is the decomposition of X generated by $\{Y_2\}$, then G_0 is a simple decomposition of X which does not generate a simple decomposition of $\mathbf{E}^3$ (since Y_2 is not cellular), and hence X is not simply embedded in $\mathbf{E}^3$.

Since X and Y are embeddable in the plane, it is clear that these continua can be strongly simply embedded in $\mathbf{E}^3$ [**1**, Lemma 4.1]. It would be interesting to know whether every locally planar continuum which is embeddable in $\mathbf{E}^3$ has a simple embedding, or a locally tame embedding. We do not know the answer even for locally planar tree-like continua. In the next section, however, it is shown that every circularly chainable continuum has a strongly locally tame (and therefore a strongly simple) embedding in $\mathbf{E}^3$.

4. Embedding circle-like continua in $\mathbf{E}^3$. A *solid torus* is a homeomorphic image of $B^2 \times S^1$. Whenever we speak of a solid torus T we shall always assume given a particular homeomorphism $h: B^2 \times S^1 \twoheadrightarrow T$. The *core* of T is $h(\{0\} \times S^1)$ and a *cross-section* of T is a 2-cell of the form $h(B^2 \times \{p\})$ for some $p \in S^1$. Notice that the core and cross-sections depend on the choice of h and that distinct cross-sections are disjoint. A *section* of T is a 3-cell of the form $h(B^2 \times A)$ where A is an arc in S^1. If p and q are the endpoints of A, then $B^2 \times \{p\}$ and $B^2 \times \{q\}$ are the *ends* of the section. A choice of $n \geq 3$ distinct points $a_1, a_2, \ldots, a_n$ of S^1 determines a collection $L_1, L_2, \ldots, L_n$ of sections of T whose union is T and such that if $1 \leqq i < j \leqq n$, then $L_i \cap L_j$ is either empty or an end of each of Li and Lj. Such a choice is called a *sectioning* of T into $L_1, L_2, \ldots, L_n$. We shall usually deal with sectioned solid tori and, when a sectioning of T into $L_1, L_2, \ldots, L_n$ has been given, we shall simply refer to $L_1, L_2, \ldots, L_n$ as "the sections" of T. Suppose T has been assigned a metric and that ϵ is a positive number. Then a sectioning of T is said to be an *ϵ-sectioning* provided each of its sections has diameter less than ϵ. Finally, an *annular web* of T is an annulus A in T such that if $D = h(B^2 \times \{p\})$ is a cross-section of T, $A \cap D$ is an arc spanning D and $h((0, p)) \in A \cap D$.

The proof of the following fact is implicit in [**6**]. (In particular, see [**6**, Theorems 4 and 8 and the remarks in the first paragraph on p. 120].)

4.1 THEOREM (Bing). *If X is a circularly chainable continuum, then there exists a homeomorphic image X' of X in* $\mathbf{E}^3$ *such that* $X' = \bigcap_{i=1}^{\infty} T_i$ *where*

(1) *if* $i = 1, 2, \ldots$, *then* T_i *is a smooth solid torus whose interior contains* T_{i+1},

(2) *if* $i = 1, 2, \ldots$, *then* T_i *has an* ϵ_i*-sectioning where* $\lim_{i\to\infty}\epsilon_i = 0$, *and*

(3) *if* $i = 1, 2, \ldots$ *and* L *is a section of* T_i, *then* $L \cap T_{i+1}$ *is a union of sections of* T_{i+1}.

We now state the main result of this section.

4.2 THEOREM. *Every circularly chainable continuum can be embedded in* $\mathbf{E}^3$ *as a strongly locally tame subset; in fact, any such continuum can be embedded so that every closed proper subset lies on a tame disk.*

Remarks. It is well-known that every chainable continuum is embeddable in $\mathbf{E}^2$, and hence it is sufficient to consider only those circularly chainable continua X which are not chainable; we shall construct, for each such continuum, a homeomorphic image X' of X in $\mathbf{E}^3$ as the intersection of a sequence of solid tori having properties (1)-(3) of Theorem 4.1. However, this in itself will not be enough to guarantee that the theorem is true. For example, the simple closed curve J of [**4**] is constructed as the intersection of such a sequence, yet *no* subcontinuum of J can be pushed into the xy-plane by a homeomorphism of $\mathbf{E}^3$ onto itself. Thus, we shall need to require much more of the sequence defining X'. We also note that Bing showed [**6**, Theorem 8] that there is a homeomorphic image of X lying in Z, where Z is the union of the xy-plane and the upper half of the xz-plane. But this is again insufficient to obtain the conclusion of Theorem 4.2 since there is, for example, a simple closed curve J which lies in Z and contains a wild arc.

Proof of Theorem 4.2. Suppose X is a circularly chainable continuum which is not chainable and let $\mathscr{C}_1, \mathscr{C}_2, \ldots$ be a sequence of circular chains defining X whose meshes converge rapidly to 0 (see the proof of Theorem 4 of [**6**]). Let n_i denote the number of links in $\mathscr{C}_i$.

(1) *The construction of* T_1. Let J_1 be a smooth simple closed curve in the xy-plane which is the union of n_1 arcs, pairwise disjoint except possibly for endpoints and each of diameter less than 1. To simplify the construction at later stages, we require that J_1 contain a straight segment parallel to the x-axis. T_1 is a small tubular neighborhood of J_1, chosen so that there is a circular chain $\mathscr{T}_1$ having n_1 links, each of diameter less than 1, such that the union of the links of $\mathscr{T}_1$ is Int T_1. The core of T_1 is J_1 and the cross-sections of T_1 are circular disks lying in planes normal to J_1. The intersection of T_1 with the xy-plane is an annular web of T_1, which we denote by A_1. We further suppose that T_1 has a 1-sectioning and that some section, which we denote by

K_1, intersects J_1 in a straight segment parallel to the x-axis and has the property that $K_1 \cap \text{Int } T_1$ lies in a link of $\mathscr{T}_1$.

Remarks. The section K_1 singled out in the above construction is the "crossing-section" of T_1. Each T_i we construct will have a smooth core J_i, a smooth annular web A_i, and a crossing-section K_i such that $K_i \cap A_i$ lies in the xy-plane and $K_i \cap J_i$ is a straight segment parallel to the x-axis. Furthermore, we will have $K_i \supset \text{Int } K_i \supset K_{i+1}$, mesh $K_i \to 0$, and $X' - K_i \subset A_i$.

(2) *The construction of* T_2. Let T_1' denote a small tubular neighborhood of J_1 lying in Int T_1 and having a $\frac{1}{2}$-sectioning such that if L is a section of T_1, then $L \cap T_1'$ is a union of sections of T_1'. J_1 is the core of T_1' and the cross-sections of T_1' are the intersections of T_1' with the cross-sections of T_1. $A_1 \cap T_1' = A_1'$ is an annular web of T_1'.

We will construct J_2, a smooth simple closed curve circling $\mathscr{T}_1$ just as $\mathscr{C}_2$ circles $\mathscr{C}_1$. J_2 will be the union of n_2 arcs, pairwise disjoint except possibly for endpoints and each of diameter less than $\frac{1}{2}$. J_2 will lie in Int T_1' and will pierce the ends of the sections of T_1' normally. Also, $J_2 - K_1$ will lie in A_1'.

Establish polar coordinates (r, θ) in A_1' so that $1 \leqq r \leqq 3$, $0 \leqq \theta < 2\pi$, J_1 is the set $\{(r, \theta) | r = 2\}$, and $K_1 \cap A_1'$ is the set

$$\{(r, \theta) | 1 \leqq r \leqq 3, \pi/4 \leqq \theta \leqq 3\pi/4\}.$$

We can suppose that for each $k \in [1, 3]$, the simple closed curve $\{(r, \theta)|r = k\}$ intersects K_1 in a segment parallel to the x-axis and that for each $k \in [\pi/4, 3\pi/4]$, the arc $\theta = k$ intersects K_1 in a segment perpendicular to the x-axis. We begin the construction of J_2 by constructing the smooth arc $J_2 \cap A_1'$. $J_2 \cap A_1'$ circles $\mathscr{T}_1$ as $\mathscr{C}_2$ circles $\mathscr{C}_1$, and has its endpoints in Int K_1. The construction of J_2 will be completed by joining the endpoints of $J_2 \cap A_1'$ by an arc in $T_1' \cap \text{Int } K_1$. $J_2 \cap A_1'$ is constructed as the image of a smooth non-singular path $\alpha:[0, 1] \to A_1'$.

We construct the path so that if $0 \leqq s < t \leqq 1$, then $r(\alpha(s))$, the r-coordinate of $\alpha(s)$, is not smaller than $r(\alpha(t))$. We also require that

$$r(\alpha(0)) = r(\alpha(1/20)) = 5/2, r(\alpha(1/10)) < 5/2, r(\alpha(9/10)) > 2,$$

and $r(\alpha(19/20)) = r(\alpha(1)) = 2$. In addition, if $\theta(p)$ denotes the θ-coordinate of p, then $\theta\alpha$ is increasing on $[0, 1/10]$ and on $[9/10, 1]$ with $\theta\alpha(0) = 5\pi/8$, $\theta(\alpha(1/10)) = 3\pi/4$, $\theta(\alpha(9/10)) = \pi/4$, and $\theta(\alpha(1)) = 3\pi/8$ (See Figure 3). We then complete the construction of J_2 by adding the arc B as shown in Figure 3. The interior of B lies above the xy-plane and the projection π of $\mathbf{E}^3$ onto the xy-plane carries B homeomorphically onto $\pi(B)$. These special properties are possible to obtain since $K_1 \cap \text{Int } T_1'$ lies in a link of $\mathscr{T}_1$.

T_2 will be a small tubular neighborhood of J_2. J_2 is the core of T_2, and the cross-sections of T_2 lie in planes normal to J_2. We also choose T_2 so that if p is a point in the closure of $T_2 \cap A_1' - K_1$,then $2 <$ r-coordinate of $p < 5/2$. Also, there is a circular chain $\mathscr{T}_2$ having n_2 links, each of diameter less than $\frac{1}{2}$,

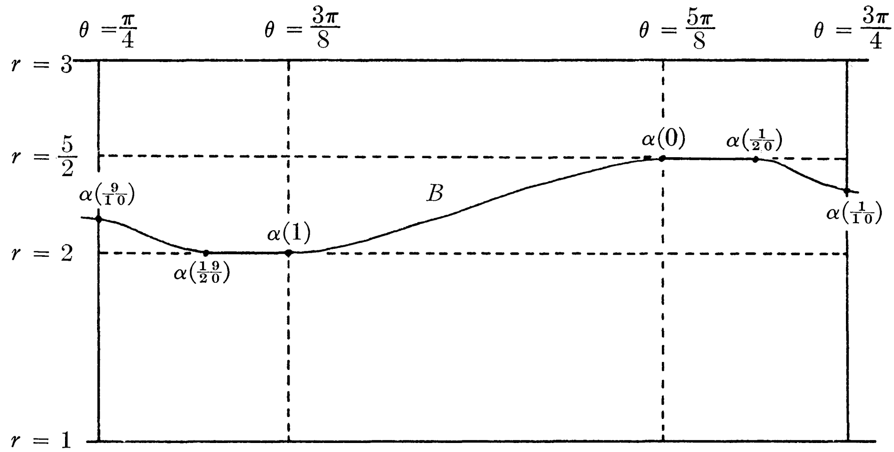


FIGURE 3

and circling $\mathscr{T}_1$ as $\mathscr{C}_2$ circles $\mathscr{C}_1$, such that the union of the links of $\mathscr{T}_2$ is Int T_2.

(3) *The construction of* A_2. We section T_2 so that the intersection of T_2 with any section of T_1' is a union of sections of T_2. This is possible since J_2 pierces the ends of the sections of T_1' normally. Note that this gives us a $\frac{1}{2}$-sectioning of T_2. We may suppose that the sectioning is such that there are four adjacent sections K_2, M_1, N, and M_2 such that

$$K_2 \cap J_2 = \alpha([19/20, 39/40]),\ M_1 \cap J_2 = \alpha([39/40, 1]),\ N \cap J_2 = B,$$

and $M_2 \cap J_2 = \alpha([0, 1/20])$.

Now we are ready to define A_2. A_2 is chosen so that

$$A_2 \cap (T_2 - M_1 \cup N \cup M_2) = A_1' \cap (T_2 - M_1 \cup N \cup M_2).$$

We may suppose that K_2 lies in a link of $\mathscr{T}_2$, so that K_2 becomes the crossing-section of T_2. Now, $A_2 \cap (M_1 \cup N \cup M_2)$ is constructed as in Figure 4.

A_2 is twisted inside M_1 and M_2 and the part of A_2 inside N is constructed so that π carries $A_2 \cap N$ homeomorphically into the xy-plane. The reason for twisting A_2 inside M_1 and M_2 will become clear in (4) below.

(4) *The homeomorphisms* h_L. Let L be a section of T_2 which does not lie in K_1. Then there is a homeomorphism h_L of $\mathbf{E}^3$ onto itself such that h_L is fixed outside T_1', h_L carries each cross-section of T_1' onto itself, and h_L carries the closure of $A_2 - L$ into A_1. The construction of h_L is perhaps best indicated by Figures 5-8.

$L \cup M_2$ separates T_2 into two components, R_1 and R_2, one of which, say R_1,

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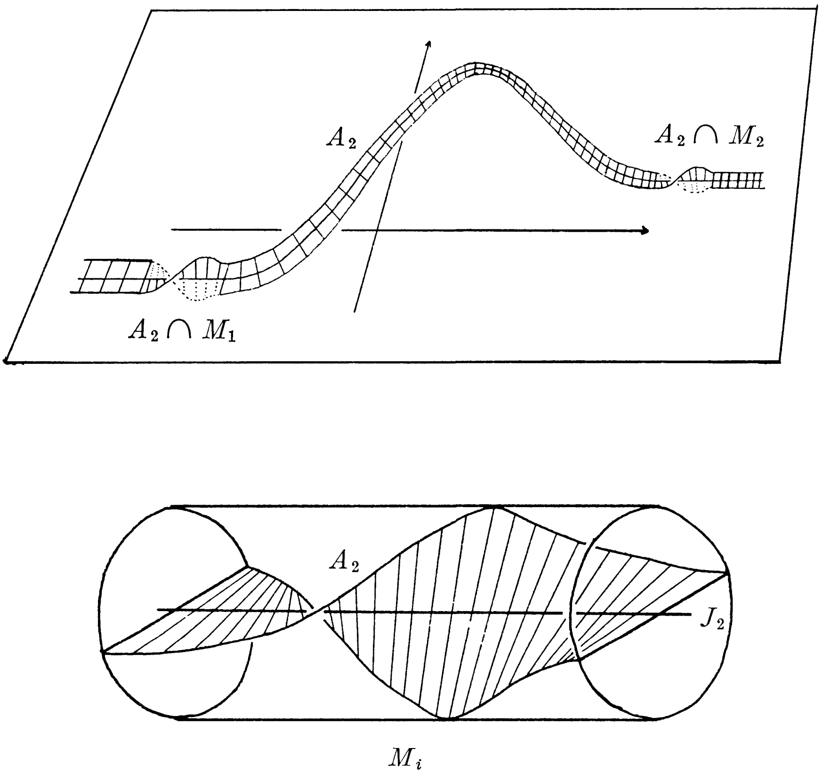

FIGURE 4

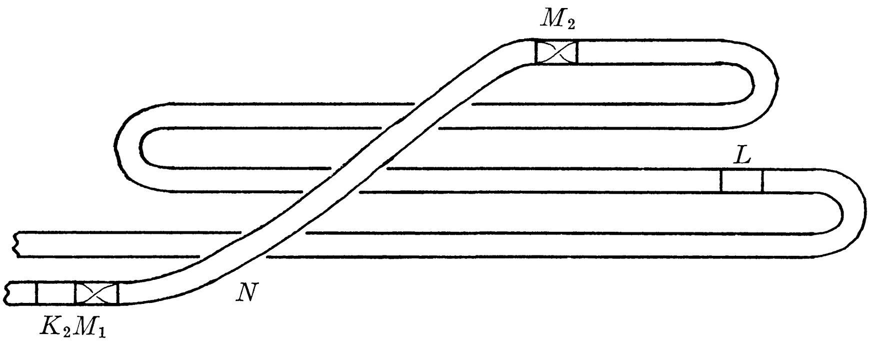

FIGURE 5

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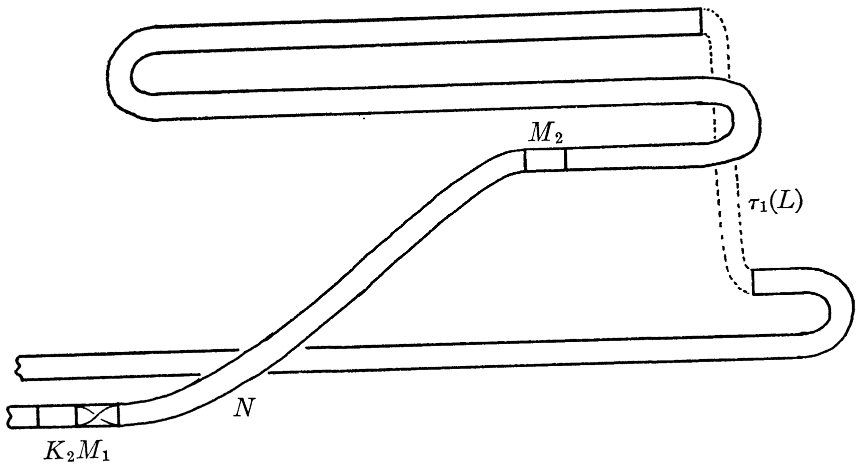

FIGURE 6

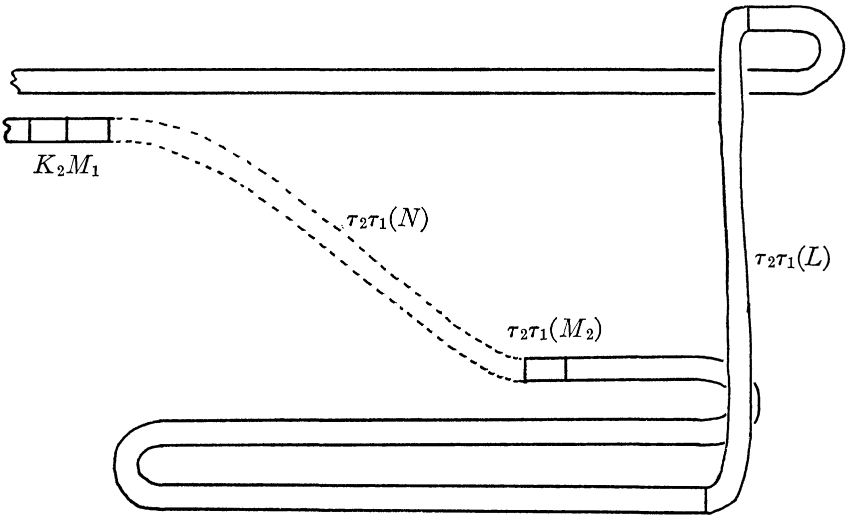

FIGURE 7

fails to contain K_2. We first move R_1 "out from under B" by "untwisting" at M_2 via a homeomorphism τ_1 of $\mathbf{E}^3$ onto itself fixed outside T_1' and carrying each cross-section of T_1' onto itself. In each cross-section Q of T_1', $R_1 \cap Q$ is rotated by τ_1 about the point of $A_1' \cap Q$ with r-coordinate 5/2 while $R_2 \cap Q$ remains fixed. τ_1 carries $A_2 \cap (R_1 \cup M_2)$ into A_1'; See Figure 6; the homeomorphism τ_1 introduces a half-twist in the part of A_2 lying inside L, but this is not indicated in the figure. The next move is similar. $M_1 \cup \tau_1(L)$ separates $\tau_1(T_2)$ into two components, S_1 and S_2, one of which, say S_1, fails to intersect

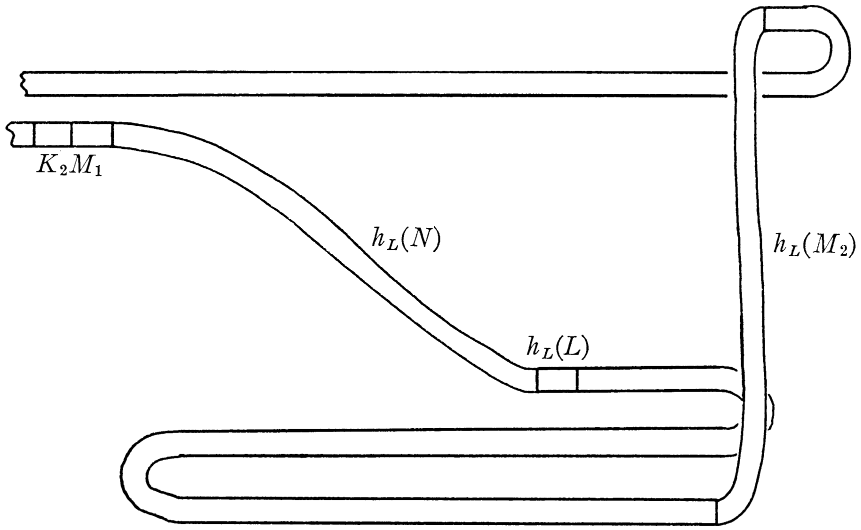

FIGURE 8

K_2. We "untwist" at M_1 via a homoemorphism τ_2 which is fixed on S_2 and carries $\tau_1(A_2 - (L \cup N))$ into A_1'. τ_2 carries $A_2 \cap N$ below the xy-plane; see Figure 7. τ_2 introduces another half-twist in the part of $\tau_1(A_2)$ lying inside $\tau_1(L)$, so that $\tau_2\tau_1(A_2)$ has two half-twists inside $\tau_2\tau_1(L)$. There are now no twists in $\tau_2\tau_1(A_2)$ outside $\tau_2\tau_1(L)$, however, and hence we can press $\tau_2\tau_1(A_2 \cap N)$ into the xy-plane via a homeomorphism τ_3 which is fixed outside T_1', fixed on $\tau_2\tau_1(T_2 - N)$, and carries cross-sections of T_1' onto themselves; see Figure 8. Then $h_L = \tau_3\tau_2\tau_1$. We note that the image, under h_L, of any section of T_2 lies in a section of T_1', and hence has diameter less than $\frac{1}{2}$.

Remarks. Had we begun with a δ-sectioning of T_1' we would have obtained a δ-sectioning of T_2 so that the image under h_L of any section of T_2 has diameter less than δ. This observation is needed for the construction of $T_3, T_4, \ldots$. We also note that any homeomorphism of $\mathbf{E}^3$ onto itself which is fixed outside T_2, and which carries each cross-section of T_2 onto itself, also carries each section of T_1 onto itself.

(5) *Completing the construction.* We continue the process begun above to construct solid tori $T_1, T_2, T_3, \ldots$ such that $T_i \supset \operatorname{Int} T_i \supset T_{i+1}$ and $X' = \bigcap_{i=1}^{\infty} T_i \approx X$. Each T_i has an associated smooth annular web A_i and a $(1/i)$-sectioning into $K_i, L_1^i, L_2^i, \ldots, L_{m_i}^i$. If L is a section of T_i, then $L \cap T_{i+1}$ is a union of sections of T_{i+1}.

If $i \geqq 1$ and L_j^{i+1} is a section of T_{i+1} not lying in K_i, then there is a homeomorphism h_j^{i+1} of $\mathbf{E}^3$ onto itself which is the identity outside T_i, which carries cross-sections of T_i onto themselves, and which carries the closure of

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$A_{i+1} - L_j^{i+1}$ into A_i. Using the remark at the end of (4), we may section T_{i+1} finely enough that if $2 \leqq s \leqq i$ and $J_s, j_{s+1}, \ldots, j_i$ are integers such that $h_{j_s}{}^s, h_{j_{s+1}}{}^{s+1}, \ldots, h_{j_i}{}^i$ are defined, then the image of every section of T_{i+1} under the composition $h_{j_s}{}^s h_{j_{s+1}}{}^{s+1} \ldots h_{j_i}{}^i h_j{}^{i+1}$ has diameter less than $1/i + 1$. We also note that if $1 \leqq k \leqq i$, then the image of each section of T_k under this composition is the same as its image under $h_{j_s}{}^s h_{j_{s+1}}{}^{s+1} \ldots h_{j_i}{}^i$.

(6) *The strong local tameness of* X'. It will be shown that each closed proper subset of X' lies on a tame disk, and this will imply, in particular, that X' is strongly locally tame in $\mathbf{E}^3$. (If K is a cell-like subset of X′, then $K \neq X'$, since X' is not chainable, and hence some closed neighborhood of K in X' is a proper subset of X'.)

Suppose Y is a closed proper subset of X'. Then there exists an integer $i \geqq 2$ such that not every section of T_i intersects Y. If K_i fails to intersect Y, then there exists a homeomorphism of $\mathbf{E}^3$ onto itself carrying $A_i - K_i$, and hence Y, into the xy-plane.

Otherwise, let $L_{j_i}{}^i$ be a section of T_i which fails to intersect Y. Let $L_{j_i}{}^i \supset L_{j_{i+1}}{}^{i+1} \supset \ldots$ be sections. If $r \geqq i + 1$, let f_r denote the homeomorphism $h_{j_{i+1}}{}^{i+1} h_{j_{i+2}}{}^{i+2} \ldots h_{j_r}{}^r$. Then the sequence $\{f_r\}_{r=i+1}^{\infty}$ converges to a homeomorphism f of $\mathbf{E}^3$ onto itself which carries $X' - L_{j_i}{}^i$ into $A_i - L_{j_i}{}^i$. But $A_i - L_{j_i}{}^i$ can be carried into the xy-plane by a homeomorphism of $\mathbf{E}^3$ onto itself, and the proof is complete.

5. An application to hyperspaces. For any continuum X, the hyperspace of subcontinua of X (with the Hausdorff metric) will be denoted by $C(X)$. It has been shown recently that if X is a chainable continuum [**13**] or a circularly chainable plane continuum [**19**], then $C(X)$ is embeddable in $\mathbf{E}^3$. Earlier, Transue [**22**] had given a very nice, explicit embedding of $C(X)$ into $\mathbf{E}^3$ when X is a pseudoarc (or any hereditarily indecomposable plane continuum which does not separate the plane).

It follows from known results [**20**; **15**] that $C(X)$ is embeddable in $\mathbf{E}^4$ if X is any circularly chainable continuum. We show below that if X is a pseudosolenoid, Theorem 4.2 can be used to give an explicit embedding of $C(X)$ in $\mathbf{E}^4$, completely analogous to Transue's embedding into $\mathbf{E}^3$ of the hyperspace of a pseudoarc. It is also shown that $C(X)$ is not embeddable in $\mathbf{E}^3$ unless X is embeddable in $\mathbf{E}^2$.

Let X be a pseudosolenoid and let $\mu: C(X) \to [0, 1]$ be defined as in [**23**]. Since $\mu(A) < \mu(B)$ whenever A is a proper subset of B, it follows that $\mu(\{x\}) = 0$ for each $x \in X$; clearly it may be assumed that $\mu(X) = 1$. Since X is hereditarily indecomposable, if $\mu(A) = \mu(B)$ and $A \cap B \neq \emptyset$, then $A = B$; hence for each $t \in [0, 1)$, $\mu^{-1}(t)$ is a collection of disjoint proper subcontinua of X which, as shown in [**16**], forms a continuous decomposition of X. (It is easy to show that $\mu^{-1}(t)$, with the topology it inherits as a subspace of $C(X)$, is homeomorphic to $X/\mu^{-1}(t)$, with the decomposition topology.)

We regard $\mathbf{E}^4$ as $\mathbf{E}^3 \times \mathbf{E}^1$, with $\mathbf{E}^3$ identified with $\mathbf{E}^3 \times \{0\}$, and we denote the projection onto the second coordinate by π_2.

5.1. THEOREM. *If X is a pseudosolenoid, there is an embedding $\varphi: C(X) \to \mathbf{E}^4$ such that the diagram*

$$\begin{array}{ccc} C(X) & \xrightarrow{\varphi} & \mathbf{E}^3 \times \mathbf{E}^1 \\ & {\scriptstyle\mu}\searrow & \downarrow{\scriptstyle \pi_2} \\ & & \mathbf{E}^1 \end{array}$$

is commutative.

Proof. By Theorem 4.2 and Theorem 4.2 of [**1**], it may be assumed that X is strongly simply embedded in $\mathbf{E}^3$. Let

$$F = \{(x, t) \in \mathbf{E}^3 \times \mathbf{E}^1 | x \in X, t \in [0, 1]\},$$

and let B be a (spherical) ball in $\mathbf{E}^3 \times \{1\}$ which contains $X \times \{1\}$. For each $t \in [0, 1]$, let $G_t = \{g \times \{t\} | g \in \mu^{-1}(t)\}$ and let $G_1 = \{B\}$. For $t \in [0, 1)$, G_t is cell-like decomposition of $X \times \{t\}$ and hence, since X is strongly simply embedded in $\mathbf{E}^3$, G_t generates a simple decomposition of $E^3 \times \{t\}$. Let $G = \cup \{G_t | t \in [0, 1])\}$; it is clear that G is an upper semicontinuous decomposition of $F \cup B$. Since for each $t \in [0, 1]$, G_t generates a simple decomposition of $\mathbf{E}^3 \times \{t\}$, it follows from the Addendum to Corollary 4 of [**21**] that G generates a simple decomposition $\tilde{G}$ of $\mathbf{E}^4$ and, in fact, there is a map $f: \mathbf{E}^4 \twoheadrightarrow \mathbf{E}^4$ such that $\tilde{G} = \{f^{-1}(p) | p \in \mathbf{E}^4\}$ and such that for each $t \in \mathbf{E}^1, f(\mathbf{E}^3 \times \{t\}) = \mathbf{E}^3 \times \{t\}$.

Define $\varphi: C(X) \twoheadrightarrow f(F \cup B)$ by setting $\varphi(X) = f(B)$ and $\varphi(g) = f(g \times \{t\})$ if $\mu(g) = t < 1$. It follows exactly as in [**22**] that φ is a homeomorphism, and it is clear that the desired commutativity condition holds.

It is shown in [**19**] that if X is a nonplanar solenoid, then $C(X)$ is homeomorphic to $K(X)$, the cone over X, and hence [**2**] $C(X)$ is not embeddable in $\mathbf{E}^3$. We will show that the hyperspace of a nonplanar pseudosolenoid is also not embeddable in $\mathbf{E}^3$; the method of [**19**] does not apply here since if X is a pseudosolenoid, or any hereditarily indecomposable continuum, then $C(X)$ and $K(X)$ are not homeomorphic. (Let X be a nondegenerate hereditarily indecomposable continuum and define $\pi: X \times I \twoheadrightarrow X$ by $\pi(x, t) = x$. If A is an arc in $X \times I$, then $\pi(A)$ is a locally connected continuum in X and hence is a single point. Thus every arc in $X \times I$ lies in $\{p\} \times I$ for some $p \in X$ and it follows that every simple triod in $K(X)$ has the vertex of $K(X)$ as its emanation point. On the other hand, suppose g_0 is a nondegenerate proper subcontinuum of X and let x_1, x_2 be points of different composants of g_0. If $A_1 = \{g \in C(X)|x_1 \in g \subset g_0\}$, $A_2 = \{g \in C(X)|x_2 \in g \subset g_0\}$ and $A_3 = \{g \in C(X)|g_0 \subset g\}$, then $A_1 \cup A_2 \cup A_3$ is a simple triod in $C(X)$ with emanation point g_0. Hence $C(X) \not\approx K(X)$.)

The proof of the next lemma is a straightforward modification of the argument for Theorem 3 of [**3**].

5.2. LEMMA. *If X is a circularly chainable continuum and G is a monotone upper semicontinuous decomposition of X, then X/G is circularly chainable.*

Proof. Let $Y = X/G$ and let $P:X \twoheadrightarrow Y$ be the projection map. Let ρ and $\bar{\rho}$ be metrics for X and Y, respectively. We will show that for each $\epsilon > 0$, Y can be covered by a circular ϵ-chain of open subsets of Y.

Suppose $\epsilon > 0$ and let δ be a positive number such that if $A, B \subset X$ and $\rho(A, B) < \delta$, then $\bar{\rho}(P(A), P(B)) < \epsilon/10$. Let $\mathscr{C} = [C(1), C(2), \ldots C(m)]$ be a circular chain of mesh $< \delta$ covering X; it may be assumed that no element of G intersects every link of $\mathscr{C}$. For every integer n, define $C(n)$ to be $C(i)$, where $1 \leqq i \leqq m$ and $n \equiv i \pmod m$, and for every pair (i, j) of integers with $i \leqq j$ and $j - i < m$, let $\mathscr{C}(i, j)$ denote the (linear) chain

$$[C(i), C(i + 1), \ldots C(j)].$$

Let $1 = n_1 < n_2 < \ldots < n_j = m$ be a sequence of integers such that for $i = 1, 2, \ldots j - 1$, n_{i+1} is the largest integer $n \leqq m$ such that some element of G intersects every link of the chain $\mathscr{C}(n_i, n)$. If $j \leqq 7$, let $k = 0$ and let $\mathscr{U}_0 = \mathscr{U}_k = \mathscr{C}$. If $j > 7$, let k be a positive integer such that $j - 6 \leqq 4k + 1 \leqq j - 3$ and let $\mathscr{U}_i = \mathscr{C}(n_{4i+1}, n_{4i+7})$, $i = 0, 1, \ldots k - 1$, and $\mathscr{U}_k = \mathscr{C}(n_{4k+1}, m + n_3)$. For $0 \leqq i \leqq k$, let U_i denote the union of the links of $\mathscr{U}_i$ and let $D_i = \{g \in G|g \subset U_i\}$. Then each D_i is an open subset of Y having diameter less than ϵ, and $[D_0, D_1, \ldots D_k]$ is a circular chain which covers Y.

The next lemma involves the notions of the *shape* of a compactum [**7**; **8**] and of *movable* compacta [**9**]. Since we will not make explicit use of the definitions of these terms but will rely on cited theorems concerning them, the definitions will not be repeated here.

5.3. LEMMA. *If X is a nonplanar circularly chainable continuum and G is a cell-like upper semicontinuous decomposition of X, then X/G is a nonplanar circularly chainable continuum.*

Proof. It follows immediately from the statement and proof of Theorem 19 of [**18**] that X has the shape of a nonplanar solenoid, and since movability is a shape invariant [**9**, Corollary 3.11] and nonplanar solenoids are not movable [**9**, p. 138], it follows that X is not movable.

By Lemma 5.2, X/G is circularly chainable; hence dim $(X/G) \leqq 1$ and it follows from Theorem 11 of [**21**] that X/G has the shape of X. Thus X/G is

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not movable, and since every plane compactum is movable [**9**, Corollary 5.5], it follows that X/G is not embeddable in the plane.

5.4. THEOREM. *If X is a pseudosolenoid, then $C(X)$ is embeddable in* $\mathbf{E}^3$ *if and only if X is embeddable in* $\mathbf{E}^2$.

Proof. That $C(X)$ is embeddable in $\mathbf{E}^3$ if X is embeddable in $\mathbf{E}^2$ follows immediately from Theorem 1 of [**19**].

Suppose then that X is a nonplanar pseudosolenoid. We will show that $C(X)$ cannot be embedded in $\mathbf{E}^3$ by an argument closely parallel to that given in [**2**] to show that the cone over a solenoid cannot be so embedded.

Let $\mu: C(X) \twoheadrightarrow [0, 1]$ be the Whitney function described earlier. For each $p \in X$, there is a unique arc A_p from $\{p\}$ to X in $C(X)$ [**16**]. Let $F = X \times [0, 1]$ and for each $p \in X$, let $F_p = \{p\} \times [0, 1]$. Let $\hat{X}$ denote the subset of $\mathscr{C}(X)$ consisting of the singleton subsets of X.

We note first that

(1) there is a map $\varphi: F \twoheadrightarrow C(X)$ such that for each $p \in X$, $\varphi(F_p) = A_p$, and

(2) for each $t_0 \in [0, 1]$, there is a retraction $r_{t_0}: \mu^{-1}([0, t_0]) \twoheadrightarrow \mu^{-1}(t_0)$.

To see that (1) is true, it is sufficient to let $\varphi(p, t)$ denote the unique subcontinuum g of X for which $p \in g$ and $\mu(g) = t$. Condition (2) may be obtained by defining $r_{t_0}(g)$, for $g \in \mu^{-1}([0, t_0])$, to be the unique subcontinuum g' of X for which $g \subset g'$ and $\mu(g') = t_0$.

Now suppose $h: C(X) \to \mathbf{E}^3$ is an embedding, and let S be a 2-sphere in $\mathbf{E}^3$ which separates the point $h(X)$ from the closed set $h(\hat{X})$. Since $\varphi^{-1}(h^{-1}(S))$ is a closed subset of F which separates $X \times \{1\}$ from $X \times \{0\}$ in F, it follows from the lemma proved in [**2**] that $\varphi^{-1}h^{-1}(S)$ contains a continuum B which intersects each F_p, $p \in X$. Then $\varphi(B) = B'$ is a continuum in $C(X)$ which intersects each A_p, $p \in X$. There is a $t_0 \in [0, 1)$ such that $B' \subset \mu^{-1}([0, t_0])$; since B' intersects each A_p, the retraction r_{t_0} maps B' *onto* $\mu^{-1}(t_0)$. If G is the decomposition of X whose elements are the continua belonging to $\mu^{-1}(t_0)$, then the decomposition space X/G is homeomorphic to the subspace $\mu^{-1}(t_0)$ of $C(X)$; hence by Lemma 5.3, $\mu^{-1}(t_0)$ is a nonplanar pseudosolenoid. Since $\mu^{-1}(t_0)$ is not locally connected, it follows that $h(B') \not\supset S$. But this implies that B' is homeomorphic to a plane continuum and therefore [**14**, Theorem 5] cannot be mapped onto $\mu^{-1}(t_0)$.

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