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NONSTANDARD IDEALS IN RADICAL CONVOLUTION ALGEBRAS ON A HALF-LINE

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This note is about the interplay between two classes of radical Banach algebras, and we begin by describing the algebras in question.

A *weight sequence* is a positive sequence $w = (w_n)$ defined on $\mathbf{Z}^+$ (the non-negative integers) and satisfying $w_0 = 1$ and $w_{m+n} \leqq w_m w_n$ for all m and n in $\mathbf{Z}^+$. For such a sequence w, the Banach space

$$\ell^1(w) = \{x = (x_n):\|x\| = \sum |x_n|w_n < \infty\}$$

is a Banach algebra with respect to the convolution product, defined by

$$(x * y)_n = \sum_{i+j=n} x_i y_j.$$

If w satisfies the additional condition $w_n^{1/n} \to 0$ as $n \to \infty$, then $\ell^1(w)$ is a primary algebra, the unique maximal ideal being

$$I_1 = \{x \in \ell^1(w):x_0 = 0\}.$$

In such cases, we call w a *radical* weight sequence. Basic information about the algebras $\ell^1(w)$ may be found in [**7**], Section 19. It is easy to see that, for each k in $\mathbf{Z}^+$, the set

$$I_k = \{x \in \ell^1(w):x_n = 0 \text{ if } n < k\}$$

is a closed ideal in $\ell^1(w)$. These, together with $I_\infty = \{0\}$, are the *standard* ideals, and any other closed ideal will be called *nonstandard.*

A *weight function* on $[0, \infty)$ is a positive, Lebesgue measurable function ω such that

$$\omega(s + t) \leqq \omega(s)\omega(t) \quad \text{for } s \text{ and } t \text{ in } [0, \infty).$$

For such a function, the Banach space

$$L^1(\omega) = \left\{f:\|f\| = \int_0^\infty |f|\, \omega < \infty\right\}$$

is a Banach algebra with respect to the convolution product, defined by

$$(f * g)(t) = \int_0^t f(t - s)g(s)ds.$$

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If $\omega(t)^{1/t} \to 0$ as $t \to \infty$, the algebra is a radical algebra, and the weight function will again be called radical. For $\alpha \geqq 0$, the set

$$M_\alpha = \{f \in L^1(\omega): f = 0 \text{ a.e. on } [0, \alpha]\}$$

is easily shown to be a closed ideal in $L^1(\omega)$, and these, together with $M_\infty = \{0\}$ are the standard ideals. Again we refer to [**7**], Section 18 for basic facts about $L^1(\omega)$.

We shall deal only with radical weights, and, abusing language in the case of $\ell^1(w)$, we shall refer to the associated algebras as radical algebras. One of the main questions in this area has been whether a given radical algebra contains nonstandard ideals. The contribution of this note will be to show that $L^1(\omega)$ contains nonstandard ideals for certain weight functions ω. To provide a context for our results, we first give a brief description of the existing knowledge.

The earliest results in the area gave sufficient conditions on a weight sequence w to guarantee that $\ell^1(w)$ has only standard (closed) ideals. Results of this type are given by Grabiner ([**8**], [**9**]) and Nikolskiĭ (see Section 3.2 of [**12**]; earlier papers of Nikolskiĭ cited in [**12**] are also relevant). Nikolskiĭ also claimed to have constructed a weight w such that $\ell^1(w)$ has a nonstandard ideal, but the construction was shown to be erroneous by Thomas [**14**]. Starting from his analysis of Nikolskiĭ's construction, Thomas made significant advances in the area, first considerably enlarging the class of weights for which $\ell^1(w)$ is known to have no nonstandard ideals ([**15**], [**16**]), and then, by some difficult constructions, showing that there are weights for which $\ell^1(w)$ does contain nonstandard ideals [**17**], [**18**].

Now, radical weight sequences are (rapidly) converging to zero, and whether or not $\ell^1(w)$ contains nonstandard ideals seems related to the regularity of this convergence. For example, one of the early results was that if, for some $k \geqq 1$, w_{n+k}/w_n is eventually a decreasing function of n, then $\ell^1(w)$ has no nonstandard ideal [**9**]. Indeed, all conditions on w which are known to guarantee only standard ideals in $\ell^1(w)$, either include or imply that

$$w_{n+k}/w_n \to 0 \quad \text{as } n \to \infty,$$

for some k. On the other hand, the weights first constructed by Thomas [**17**] for which $\ell^1(w)$ has nonstandard ideals fail this condition; i.e.,

$$w_{n+k}/w_n \to 0$$

for no value of k. At one time, it seemed possible that

$$w_{n+k}/w_n \to 0 \quad \text{as } n \to \infty,$$

for some k, might imply that all closed ideals in $\ell^1(w)$ are standard. However, Thomas [**18**] has succeeded in constructing weights w which satisfy $w_{n+1}/w_n \to 0$ and for which $\ell^1(w)$ has nonstandard ideals.

Knowledge about the corresponding questions for $L^1(\omega)$ developed more slowly. The first results were partial, showing that certain elements of $L^1(\omega)$ necessarily generated standard ideals; see, for example, [**1**] and [**5**]. While the function theory employed was different, these results were similar to results known for $\ell^1(w)$. Then Domar [**6**] gave a solution, showing that certain "regularity of decrease" conditions on $\omega(t)$ (plus conditions on the rate of decrease slightly stronger than $\omega(t)^{1/t} \to 0$) imply that every closed ideal in $L^1(\omega)$ is standard. Once again, the technicalities differ, but the result itself parallels known results for $\ell^1(w)$ quite closely. The main purpose of the present note is to extend the analogy between $\ell^1(w)$ and $L^1(\omega)$ by showing that if, for some w, there is a nonstandard ideal in $\ell^1(w)$ then there is a nonstandard ideal in $L^1(\omega)$, for suitable ω. On one level, this is quite simple: if $x = (x_n)$ generates a nonstandard ideal in $\ell^1(w)$, and if we extend w and x to step functions on $[0, \infty)$, by putting $\omega(t) = w_n$ and $f(t) = x_n$ for t in $[n, n + 1)$, then f generates a nonstandard ideal in $L^1(\omega)$. Proving the latter statement is a fairly easy exercise. However, as the existence of nonstandard ideals seems to be associated with "irregularities" of the weight, the jump discontinuities of ω may be regarded with suspicion; we show that ω may be chosen to be continuous. Finally, we prove that if $\ell^1(w)$ has a nonstandard ideal, and if $w_{n+1}/w_n \to 0$ as $n \to \infty$, then the corresponding weight function ω may be constructed to satisfy the condition

$$\omega(s + t)/\omega(s) \to 0, \quad \text{as } s \to \infty,$$

for each $t > 0$. These constructions constitute the bulk of the paper. In a final section, we comment briefly on the relationship between our results and the theory of invariant subspaces for weighted translation operators on $L^p([0, \infty))$.

In addition to the notation already introduced, we shall write **N** for the natural numbers.

2. The construction. If ω is a weight function on $[0, \infty)$, then the dual space of $L^1(\omega)$ can be identified with the space

$$L^\infty(\omega^{-1}) = \{g:g\omega^{-1} \text{ is essentially bounded on } [0, \infty)\},$$

the norm being

$$\|g\| = \|g\omega^{-1}\|_\infty.$$

The action of $g \in L^\infty(\omega^{-1})$ on $f \in L^1(\omega)$ is given by

$$\langle f, g\rangle = \int_0^\infty fg.$$

For a given $f \in L^1(\omega)$, we write I_f for the closed ideal generated by f in $L^1(\omega)$, and we write $\alpha(f)$ for the infimum of the support of f:

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$$\alpha(f) = \sup\{\delta: f(t) = 0 \text{ a.e. on } [0, \delta)\}$$
$$= \sup\{\delta: f \in M_\delta\}.$$

Using the Titchmarsh convolution theorem, it is easy to see that

$$I_f \subseteq M_{\alpha(f)},$$

and that I_f is a standard ideal if and only if $I_f = M_{\alpha(f)}$. (Note that the weight function ω is bounded away from 0 on compact subsets of $[0, \infty)$; see [**11**], Section 7.4. Thus, members of $L^1(\omega)$ are locally integrable, and the use of the Titchmarsh theorem is justified.) It is well known that closed ideals in $L^1(\omega)$ are invariant under right translations, and this fact, together with a straightforward calculation involving Fubini's theorem, shows that a function $g \in L^\infty(\omega^{-1})$ annihilates I_f if and only if

(1) $$\int_0^\infty f(t)g(t + s)dt = 0$$

for each $s \geqq 0$. Thus, to show that for a particular f, the ideal I_f is nonstandard, we have to find g in $L^\infty(\omega^{-1})$ and h in $M_{\alpha(f)}$ such that (1) holds for each $s \geqq 0$ and such that $\langle h, g\rangle \neq 0$. As we have said, we shall produce a weight function ω for which such functions f, g and h exist by exploiting an analogous situation in the case of weighted sequence algebras.

Let w be a weight sequence. The dual space of $\ell^1(w)$ is

$$\ell^\infty(w^{-1}) = \{y:(y_n w_n^{-1}) \in \ell^\infty\},$$

normed by

$$\|y\| = \|(y_n w_n^{-1})\|_\infty,$$

the action of $y \in \ell^\infty(w^{-1})$ on $x \in \ell^1(w)$ being

$$\langle x, y\rangle = \Sigma\, x_n y_n.$$

For any sequence $x = (x_n)$, we write

$$n(x) = \min\{n: x_n \neq 0\}.$$

It has been shown by M. P. Thomas that there exists a radical weight sequence w and an element x such that x generates a nonstandard ideal in $\ell^1(w)$ [**17**], [**18**].

Fix such w and x. Then there are $y \in \ell^\infty(w^{-1})$ and $z \in I_{n(x)}$ such that

(2) $$\Sigma\, x_n y_{n+k} = 0 \quad (k \in \mathbf{Z}^+)$$

and

(3) $$\langle z, y\rangle \neq 0.$$

Equation (2) is one way of saying that y annihilates each right translate of

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x; cf. equation (1). Equation (3), together with (2) and the condition $z \in I_{n(x)}$, guarantees that the closed ideal generated by x is properly contained in $I_{n(x)}$.

Now defined functions f, g and h on $[0, \infty)$ by setting

$$f = \sum x_n \chi_{[n,n+1)}, \quad g = \sum y_n \chi_{[n,n+1)}, \quad h = \sum z_n \chi_{[n,n+1)}.$$

Here, as usual, χ_F denotes the characteristic function of the set F. It is clear that $\alpha(f) = n(x)$. The condition $z \in I_{n(x)}$ implies $\alpha(h) \geqq n(x)$. Finally, a simple calculation shows that for any non-negative number $s = k + r$, k an integer and $0 \leqq r < 1$,

$$\int_0^\infty f(t)g(t + s)dt = (1 - r) \sum x_n y_{n+k} + r \sum x_n y_{n+k+1} = 0,$$

by (2), while

$$\int_0^\infty gh \neq 0,$$

by (3). Thus, for a weight function ω, the function f will generate a nonstandard ideal in $L^1(\omega)$ provided that both f and h belong to $L^1(\omega)$, while g belongs to $L^\infty(\omega^{-1})$.

Now, it is very easy to produce such a weight function. If $w_1 \leqq 1$, take

$$\omega = \sum w_n \chi_{[n,n+1)},$$

while if $w_1 > 1$, take

$$\omega = w_1 \sum w_n \chi_{[n,n+1)}.$$

Then ω is easily checked to be a weight function, and it is trivial that the functions f, g and h belong to the appropriate spaces. However this particular weight function ω is not continuous and, since the existence of nonstandard ideals is suspected to be related to some kind of "irregularity" of the weight function, it is of interest that there are continuous weight functions for which the desired memberships hold.

Let ω be a weight function. Then $f \in L^1(\omega)$ if and only if

$$\int_0^\infty |f(t)|\omega(t)dt = \sum |x_n| \int_n^{n+1} \omega(t)dt < \infty.$$

A similar inequality, with f (respectively, x) replaced by h (respectively, z) specifies when h belongs to $L^1(\omega)$. Thus, a sufficient condition for f and h to belong to $L^1(\omega)$ is the existence of a constant $C_1 > 0$ such that

$$(4) \quad \int_n^{n+1} \omega(t)dt \leqq C_1 w_n \quad (n \in \mathbf{Z}^+).$$

Also

$$g(t)\omega^{-1}(t) = \sum y_n w_n^{-1}[w_n \omega(t)^{-1} \chi_{[n,n+1)}(t)],$$

so a sufficient condition for g to belong to $L^{\infty}(\omega^{-1})$ is the existence of a constant $C_2 > 0$ such that

(5) $\omega(t) \geqq C_2 w_n$ a.e. on $[n, n + 1)$ $(n \in \mathbf{Z}^+)$.

We can, in fact, satisfy (4) and (5) with a continuous weight function, thus proving the following result.

2.1 PROPOSITION. *Let the weight sequence* $w = (w_n)$ *be given. Then there is a continuous weight function* ω *on* $[0, \infty)$ *such that both* (4) *and* (5) *hold. Thus,* $L^1(\omega)$ *has nonstandard ideals for some continuous radical weight* ω.

We shall not prove this result because it is subsumed in Theorem 2.3, below. However, we point out that it can be obtained using fewer technicalities than are required for that result. For convenience, assume that $w_1 < 1$. Then (4) and (5) can be satisfied by a continuous weight function with $C_2 = 1$ and C_1 any constant with $C_1 > 1$. If the sequence (η_n) is defined by

$$\exp(-\eta_n) = w_n,$$

then a continuous, non-negative, non-decreasing function η can be constructed such that η is underneath the step function $\sum \eta_n \chi_{[n,n+1)}$, such that

$$\eta(s) + \eta(t) \leqq \eta(s + t) \quad (s, t \in [0, \infty)),$$

and such that $\eta(k) = \eta_{k-1}$ and

$$\int_{k-1}^{k} \exp(-\eta(t))dt \leqq C_1 \exp(-\eta_{k-1}) \quad \text{for } k \in \mathbf{N}.$$

Then the function given by

$$\omega(t) = \exp(-\eta(t))$$

satisfies the requirements of Proposition 2.1. It follows immediately that there is a continuous, radical weight function ω such that $L^1(\omega)$ has nonstandard ideals, for clearly, if ω and w are related as above, then ω is radical precisely when w is radical. The result also remains true if we require an infinitely differentiable weight function.

There is a further consideration concerning the algebras $L^1(\omega)$, which we shall now discuss. Recall that an element g of $L^1(\omega)$ is *compact* if the map

$$f \rightarrow g * f : L^1(\omega) \rightarrow L^1(\omega),$$

is a compact linear operator. The set of compact elements is a closed ideal in $L^1(\omega)$. The following result (together with a characterization of the compact elements of $L^1(\omega)$) is given in [**3**], Corollary 2.8.

2.2 LEMMA. *Let ω be a weight function on* $[0, \infty)$. *Then every element of* $L^1(\omega)$ *is compact if and only if*

$$(6) \quad \lim_{s\to\infty} \frac{\omega(s + t)}{\omega(s)} = 0 \quad \textit{for each } t > 0.$$

For each radical weight function ω, it is necessarily the case that

$$\liminf_{s\to\infty} \omega(s + t)/\omega(s) = 0 \quad \text{for each } t > 0$$

(see [**3**], Lemma 1.2 (i)). Thus, (6) requires just that

$$\lim_{s\to\infty} \omega(s + t)/\omega(s)$$

exists for each $t > 0$; that is, each element of $L^1(\omega)$ is compact if and only if ω is "regular", in the sense that the limits in (6) exist. On the other hand, the existence of nonstandard ideals in the above construction is associated with some "irregularity" that ω inherits from w. In fact, it can be shown that a weight function ω cannot satisfy all of (4), (5) and (6), so again it might be suspected that if each element of $L^1(\omega)$ is compact, then $L^1(\omega)$ has no nonstandard ideals.

Regarding compact elements, there is again a close analogy between $L^1(\omega)$ and $\ell^1(w)$, although in this case the $L^1(\omega)$ results were known first, and actually motivated some of the early investigations concerning compact elements of $\ell^1(w)$. We refer to [**4**], Section 1, for information about compact elements of $\ell^1(w)$ and related matters, and mention here only the following analogue to Lemma 2.2. Every non-invertible element of $\ell^1(w)$ is compact if and only if

$$(7) \quad \frac{w_{n+1}}{w_n} \to 0 \text{ as } n \to \infty.$$

As we have mentioned, Thomas [**18**] has shown that there are weights w satisfying (7) and for which $\ell^1(w)$ has nonstandard ideals. Our main theorem will be that the corresponding property (6), along with continuity, can be achieved in a weight function ω such that $L^1(\omega)$ has nonstandard ideals. Since (4), (5) and (6) cannot all hold for ω, the construction outlined earlier must be modified in order to prove the result.

2.3 THEOREM. *There exists a continuous weight function ω satisfying* (6), *so that every element of* $L^1(\omega)$ *is compact, and such that* $L^1(\omega)$ *contains nonstandard ideals.*

Proof. By the results in [**18**], there is a weight sequence w such that (7) holds and $\ell^1(w)$ contains nonstandard ideals. For convenience, we suppose

$$w_{n+1}/w_n \leqq 1/2 \quad \text{for all } n \text{ in } \mathbf{Z}^+.$$

Let x be a generator of a nonstandard ideal in $\ell^1(w)$, and let y in $\ell^\infty(w^{-1})$ and z in $I_{n(x)}$ be chosen as before so that (2) and (3) are satisfied.

We first claim that there exists a sequence $\widetilde{w} = (\widetilde{w}_n)$ such that, if $\alpha_n = \widetilde{w}_n/w_n$, then, for $m, n \in \mathbf{Z}^+$:

(8) $\widetilde{w}_{m+n} \leqq \widetilde{w}_m \widetilde{w}_n$;

(9) $\alpha_{n+1} \geqq \alpha_n \geqq \alpha_0 = 1$;

(10) $\lim_{n\to\infty} \alpha_n = \infty$;

(11) $\widetilde{w}_{n+1} \leqq w_n$;

(12) $x, z \in \ell^1(\widetilde{w})$.

To show this, set $u_n = |x_n| + |z_n|$ and let $M = \sum u_n w_n$. Inductively define a sequence $(n_k) \subseteq \mathbf{Z}^+$ so that $n_0 = 0$, $n_{k+1} > n_k$, and so that, for $k \in \mathbf{N}$:

(13) $n_k \geqq \max\{n_i + n_j : i + j = k\}$;

(14) $\displaystyle\sum_{n=n_k}^{\infty} u_n w_n \leqq 4^{-k} M$;

(15) $w_n/w_{n+1} > 2^k \quad (n \geqq n_{k-1})$.

This is clearly possible by taking each n_k to be sufficiently large: we use (7) to see that we can obtain (15). Now set $B_0 = \{0\}$ and

$$B_k = \{n_{k-1} + 1, \dots, n_k\} \quad \text{for } k \in \mathbf{N},$$

and define $\alpha_n = 2^k \ (n \in B_k)$. Then, if $\widetilde{w}_n = \alpha_n w_n$, the sequence $\widetilde{w} = (\widetilde{w}_n)$ satisfies conditions (8)-(12). First, observe that (9) and (10) are immediate from the definition of (α_n) and that (11) follows from (15): if $n + 1 \in B_k$, then $n \geqq n_{k-1}$, and so

$$w_n/w_{n+1} > 2^k = \alpha_{n+1}.$$

Secondly, (12) follows from (14): for example,

$$\sum |x_n|\widetilde{w}_n \leqq \sum u_n \widetilde{w}_n = \sum_{k=0}^{\infty} 2^k \sum_{n\in B_k} u_n w_n \leqq 4M.$$

Finally, we verify that (8) holds. Given $m, n \in \mathbf{Z}^+$, suppose that $m \in B_p$, $n \in B_q$, and $m + n \in B_r$. Then, by (13), $r \leqq p + q$, and so

$$\widetilde{w}_{m+n} = 2^r w_{m+n} \leqq 2^{p+q} w_m w_n = \widetilde{w}_m \widetilde{w}_n.$$

This shows that the sequence $\widetilde{w}$ satisfies conditions (8)-(12), as required.

We now define a sequence $(\widetilde{\eta}_n)$ by setting

$$\widetilde{w}_n = \exp(-\widetilde{\eta}_n).$$

Note that $\tilde{\eta}_{n+m} \geqq \tilde{\eta}_n + \tilde{\eta}_m$, that $\tilde{\eta}_{n+1} \geqq \eta_n \geqq \tilde{\eta}_n$, and that $\eta_n - \tilde{\eta}_n = \log \alpha_n$, so that $(\eta_n - \tilde{\eta}_n)$ is a sequence which, by (9) and (10), is monotonically increasing to infinity.

Next, define a piecewise linear function $\tilde{\eta}$ on $[0, \infty)$ by setting

$$\tilde{\eta}(t) = \tilde{\eta}_n + (t - n)(\eta_n - \tilde{\eta}_n) \quad (t \in [n, n + 1))$$

for each $n \in \mathbf{Z}^+$. Note that $\tilde{\eta} = 0$ on $[0, 1]$. We claim that

$$(16) \quad \tilde{\eta}(s + t) \geqq \tilde{\eta}(s) + \tilde{\eta}(t) \quad (s, t \in [0, \infty)).$$

For suppose that $s \in [m, m + 1)$, $t \in [n, n + 1)$. If $s + t \in [m + n, m + n + 1)$, then

$$\begin{aligned}\tilde{\eta}(s + t) &= (1 - (s - m + t - n))\tilde{\eta}_{m+n} \\ &\quad + (s - m + t - n)\eta_{m+n} \\ &\geqq (1 - (s - m))\tilde{\eta}_m + (s - m)\eta_m + (1 - (t - n))\tilde{\eta}_n \\ &\quad + (t - n)\eta_n + (t - n)(\eta_m - \tilde{\eta}_m) + (s - m)(\eta_n - \tilde{\eta}_n) \\ &\geqq \tilde{\eta}(s) + \tilde{\eta}(t).\end{aligned}$$

If $s + t \in [m + n + 1, m + n + 2)$, then

$$\begin{aligned}\tilde{\eta}(s + t) &= \tilde{\eta}_{m+n+1} + (s + t - m - n - 1) \\ &\quad \times (\eta_{m+n+1} - \tilde{\eta}_{m+n+1}) \\ &\geqq \eta_{m+n} + (s + t - m - n - 1)(\eta_{n+m} - \tilde{\eta}_{m+n}) \\ &= \tilde{\eta}_{m+n} + (s + t - m - n)(\eta_{n+m} - \tilde{\eta}_{m+n}),\end{aligned}$$

and so $\tilde{\eta}(s + t) \geqq \tilde{\eta}(s) + \tilde{\eta}(t)$, as before. This establishes (16).

Since the function $\tilde{\eta}$ has gradient $\log \alpha_n$ on the interval $[n, n + 1)$, it follows from (10) that

$$\lim_{s\to\infty} (\tilde{\eta}(s + t) - \tilde{\eta}(s)) = \infty \quad \text{for each } t > 0.$$

In general, the function $\tilde{\eta}$ will not be continuous because it may have jumps at integral points of $(0, \infty)$, and we now modify $\tilde{\eta}$ to achieve continuity. We do this by replacing $\tilde{\eta}$ on the intervals $[n, n + \delta_n)$ for $n \in \mathbf{N}$, where $\delta_n \in (0, 1)$, by the obvious straight line which will make the new function, $\hat{\eta}$, continuous. If we do this, then certainly $\hat{\eta}(n) = \eta_{n-1}$, $\hat{\eta}(t) \leqq \tilde{\eta}(t)$ for $t \in [0, \infty)$, and

$$(17) \quad \lim_{s\to\infty} (\hat{\eta}(s + t) - \hat{\eta}(s)) = \infty \quad \text{for each } t > 0.$$

We claim that the numbers δ_n can be chosen so that

$$(18) \quad \hat{\eta}(s + t) \geqq \hat{\eta}(s) + \hat{\eta}(t) \quad (s, t \in [0, \infty)).$$

Clearly, any choice of δ_1 in (0, 1) will achieve this for $s + t \leqq 2$. Suppose that $\hat{\eta}$ has been defined to satisfy (18) on $[0, n]$. For $T \in [n, n + 1]$, let

$$\varphi(T) = \sup\{\hat{\eta}(T - t) + \hat{\eta}(t): T - n \leqq t \leqq n\}.$$

Then

$$\varphi(n) = \eta_{n-1} = \hat{\eta}(n), \; \varphi(n + 1) \leqq \eta_n, \quad \text{and}$$

$$\varphi(T) - \varphi(n) \leqq K(T - n),$$

where K is the maximum slope of any line segment occurring in $\hat{\eta}$ on $[0, n]$. Also, if $T - n \leqq t \leqq n$,

$$\hat{\eta}(T - t) + \hat{\eta}(t) \leqq \tilde{\eta}(T - t) + \tilde{\eta}(t) \leqq \tilde{\eta}(T),$$

using (16), and so $\varphi(T) \leqq \tilde{\eta}(T)$. Thus, we can choose $\delta_n \in (0, 1)$ so that, with the construction of $\hat{\eta}$ as described,

$$\varphi(T) \leqq \hat{\eta}(T) \leqq \tilde{\eta}(T) \quad \text{for } T \in [n, n + 1],$$

and condition (18) follows for $s + t \in [n, n + 1]$, as required.

By choosing each δ_n sufficiently small, as we may, we can also ensure that

$$(19) \quad \int_n^{n+1} \exp(-\hat{\eta}(t))dt \leqq \tilde{w}_n \quad (n \in \mathbf{N}).$$

We now set

$$\omega(t) = \exp(-\hat{\eta}(t)) \quad (t \in [0, \infty)).$$

By (18), ω is a weight function, and it is clear that $L^1(\omega)$ is a radical Banach algebra. Equation (17) shows that ω satisfies (6), and so, by Lemma 2.2, each element of $L^1(\omega)$ is compact.

As we explained above, to show that $L^1(\omega)$ contains nonstandard ideals, it suffices to prove that f and h belong to $L^1(\omega)$ and that g belongs to $L^\infty(\omega^{-1})$, where

$$f = \sum x_n\chi_{[n,n+1)}, \quad g = \sum y_n\chi_{[n,n+1)}, \quad \text{and} \quad h = \sum z_n\chi_{[n,n+1)}.$$

In fact

$$\int_0^\infty |f(t)|\omega(t)dt = \sum |x_n| \int_n^{n+1} \exp(-\hat{\eta}(t))dt \leqq \sum |x_n|\tilde{w}_n,$$

using (19), and so $f \in L^1(\omega)$ by (12). Similarly, $h \in L^1(\omega)$. The fact that $g \in L^\infty(\omega^{-1})$ follows because $\hat{\eta}(t) \leqq \eta_n \quad (t \in [n, n + 1))$.

This concludes the proof of the theorem.

To make a final observation about the nonstandard ideals produced by our construction, we introduce the following notation. If I is an ideal in $L^1(\omega)$, we write $\alpha(I)$ for $\inf\{\alpha(f): f \in I\}$. Then $I \subseteq M_{\alpha(I)}$, and

$I \not\subset M_\beta$ if $\beta > \alpha(I)$. In the case of a principal closed ideal I_f, we have $\alpha(I_f) = \alpha(f)$. Thus, if I_f is a nonstandard ideal constructed by the methods of this section, $\alpha(I_f)$ is a positive integer. By carrying out the construction as above, but consistently replacing intervals $[n, n + 1)$ by $[na, (n + 1)a)$, where a is a positive number, we can produce weight functions ω and nonstandard ideals I in $L^1(\omega)$ with $\alpha(I)$ any positive number. However, we cannot produce a nonstandard ideal I with $\alpha(I) = 0$. It would be interesting to find such an example.

3. Invariant subspaces. To motivate this section, we recall that one of the major reasons for studying the ideal structure of (radical) algebras of the form $\ell^1(w)$ was to determine the lattice of closed, invariant subspaces of a (quasinilpotent) weighted shift operator on ℓ^1; see [**9**], [**10**], [**12**], [**18**]. The weighted shift on ℓ^1 is unitarily equivalent to the right shift on a suitable weighted ℓ^1-space. If the latter space is an algebra with respect to convolution, then its closed ideals correspond to the invariant subspaces of the weighted shift, so that one has an interplay between the ideal theory of certain algebras and the invariant subspaces of the weighted shift operators. In the case of algebras $L^1(\omega)$, the corresponding object is not a single operator on L^1, but rather a semigroup $\{T_a : a \geqq 0\}$ of "weighted translation operators" on L^1. If one defines functions λ_a on $[0, \infty)$ by

$$\lambda_a(t) = \omega(t + a)/\omega(t),$$

and writes Λ_a for the operation of pointwise multiplication by λ_a, then for each $a \geqq 0$, the right translation S_a through a units on $L^1(\omega)$ is unitarily equivalent to the operator $T_a = S_a\Lambda_a$ on L^1. So, in this case, the closed ideals of $L^1(\omega)$ correspond to closed subspaces of L^1 which are invariant under the semigroup T_a. For a semigroup of class C_0 ([**3**], Lemma 1.6), the invariant subspaces of the semigroup are those of the generator. Thus, an algebra $L^1(\omega)$ with only standard ideals corresponds to what might be called a "unicellular semigroup", and a nonstandard ideal in $L^1(\omega)$ corresponds to a weighted translation semigroup on L^1 with invariant subspaces other than those of the form $\{f : f = 0 \text{ a.e. on } [0, \delta)\}$.

Now, questions about invariant subspaces of weighted shift operators are of interest not only for ℓ^1, but when the underlying Banach space is any ℓ^p, with $1 \leqq p < \infty$. The case $p = 2$ is of particular interest; see [**13**], Section 10, for a survey of work on invariant subspaces of weighted shifts on ℓ^2. The results in [**12**] are generally given in context of ℓ^p, with ℓ^1 being treated as a special case. In [**18**], it is shown that problems of constructing weighted shifts with nonstandard invariant subspaces on ℓ^p can be reduced to constructions of special weights w such that $\ell^1(w)$ has nonstandard ideals. In [**10**], such results are extended to more general

Banach spaces with basis. We wish to point out that, as in the case $p = 1$, such constructions can be transferred from ℓ^p to L^p. Thus, from Theorem 4 of [**12**] or Theorem 3.32 of [**18**] (see also Example 2, page 106 of [**13**]), we have the existence of a sequence $w = (w_n)$ such that the weighted shift operator $T = S\Lambda$ on ℓ^p ($\Lambda(x_n) = (\lambda_n x_n)$, where $\lambda_n = w_{n+1}/w_n$) is quasinilpotent and has nonstandard invariant subspaces. If we define the weight function

$$\omega(t) = \sum w_n \chi_{[n,n+1)}(t),$$

then it follows that the semigroup $T_a = S_a\Lambda_a$ of quasinilpotent operators on $L^p[0, \infty)$ has nonstandard invariant subspaces. In fact, the more difficult constructions of Section 2 can also be carried out, leading to the following result.

3.1 THEOREM. *For each p such that* $1 \leqq p < \infty$, *there exist continuous functions* $\omega(t)$ *on* $[0, \infty)$ *such that for each* $a > 0$, *the weighted translation* $T_a = S_a\Lambda_a$ *is quasinilpotent on* $L^p[0, \infty)$ *and the semigroup* $\{T_a:a \geqq 0\}$ *has nonstandard invariant subspaces in* $L^p[0, \infty)$.

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