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THE ANALYTIC CHARACTER OF THE BIRKHOFF INTERPOLATION POLYNOMIALS

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1. Introduction. Let E be an $m \times (n + 1)$ regular interpolation matrix with elements $e_{i,k} = (E)_{i,k}$ which are zero or one, with $n + 1$ ones. Then for each $f \in C^n[a, b]$ and each set of knots X: $a \leqq x_1 < \ldots < x_m \leqq b$, there is a unique interpolation polynomial $P(f, E, X; t)$ of degree $\leqq n$ which satisfies

$$(1) \quad P^{(k)}(x_i) = f^{(k)}(x_i), \quad e_{i,k} = 1.$$

A recent paper [**1**] discussed the continuity of P, as a function of $x_1, \ldots, x_m$ (with coalescences allowed). We would like to study in this note the analytic character of P as a function of real or complex knots X: $x_1, \ldots, x_m$. This is easy for the Lagrange or the Hermite interpolation. In this case P is a polynomial in $x_1, \ldots, x_m$ if f is a polynomial, and an entire function in $x_1, \ldots, x_m$ if f is entire. This follows, for example, from the Hermite formula, which represents P by means of a contour integral. No formula of this type is known to exist in the general case of Birkhoff, non-Hermite interpolation.

We shall assume that the reader is acquainted with the terminology and the fundamental results of Birkhoff interpolation (see [**5**], [**3**], [**4**]).

For a set of functions $G = \{g_0, \ldots, g_n\}$ we have the determinant

$$D(E, X; G) = \det [g_0^{(k)}(x_i), \ldots, g_n^{(k)}(x_i); e_{i,k} = 1];$$

its rows are labeled by $n + 1$ pairs i, k with $e_{i,k} = 1$, and ordered lexicographically. In particular, for the system

$$G_s = \left\{1, \frac{x}{1!}, \ldots, \frac{x^{s-1}}{(s-1)!}, f, \ldots, \frac{x^n}{n!}\right\}$$

we have the determinant $D_s(E, X) = D(E, X; G_s)$

$$D_s(E, X) = \det \frac{x_i^{-k}}{(-k)!}, \ldots, \frac{x_i^{s-1-k}}{(s-1-k)!}, f^{(k)}(x_i), \ldots, \frac{x_i^{n-k}}{(n-k)!}, \quad e_{i,k} = 1,$$

(terms containing $r!$ with $r < 0$ are to be replaced by zero). We write $D(E, X)$ for the determinant with $G = \{1, x/1!, \ldots, x^n/n!\}$. The poly-

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nomial P given by (1) has the representation

$$(2)\quad P(f, E, X; t) = \frac{1}{D(E, X)} \sum_{s=0}^{n} \pm \frac{t^s}{s!} D(E, X; G_s).$$

It follows from this that if f is a fixed polynomial (of an arbitrary degree), then P is a rational function of X, and if f is an entire function, then P is meromorphic. We would like to improve these statements. It is essential to assume here that the function f remains fixed. For example, if f is a linear function with values c_1, c_2 at x_1, x_2, then $P = c_1l_1 + c_2l_2$, where l_1, l_2 are the fundamental Lagrange functions. Here P is only rational, and f depends on x_1, x_2. We prove:

THEOREM 1. *In order that P should be a polynomial (or an entire function) in X whenever f is a polynomial (or an entire function), it is necessary and sufficient that the canonical decomposition of E should consist only of Hermite and of two-row matrices.*

In other words, P has this property if and only if the matrix E is complex regular. This follows from a theorem of Lorentz and Riemenschneider [**6**], which is a natural generalization of D. Ferguson's theorem [**2**].

2. Proof of the theorem. The sufficiency of the conditions is easy to establish. From [**7**] it follows that the determinants $D = D(E, X)$ and $D_s = D(E, X; G_s)$ are divisible by $(x_i - x_j)^{\alpha_{ij}}$, $i, j = 1, \ldots, m$, $i \neq j$, if α_{ij} is the collision number of rows i and j. In the case when f is entire, the latter statement means that D_s is a product of $(x_i - x_j)^{\alpha_{ij}}$ with an entire function of X. The polynomial $D(E, X)$ is divisible by the product

$$\Pi = \prod_{1 \leqq i < j \leqq m} (x_i - x_j)^{\alpha_{ij}}.$$

On the other hand, it is known ([**3**], [**5**]) that the degree of $D(E, X)$ in one of the variables x_i is at most δ_i, which is the collision number of row i in E with the rest of the matrix E. Under the assumptions of Theorem 1 (see [**6**]), $\delta_i = \sum_{j \neq i} \alpha_{ij}$. This shows that $D = \text{Const}\ \Pi$. Therefore, the denominator in (2) cancels out.

The necessity requires a careful treatment of determinants $D(E, X; G)$ and of their derivatives. For the minors of $D = D(E, X; G)$ we use the notation

$$D(E, X; G)_{(i,k),s}.$$

This is the signed subdeterminant of D corresponding to its row, labelled (i, k) (with $e_{i,k} = 1$), and the column s, $s = 0, \ldots, n$.

For the derivatives of D we have the following (see [**5**], [**7**]). The simplest formula is

$$(3a)\quad \frac{d}{dx_i} D(E, X; G) = \sum_{e_{i,k}=1} U_{i,k} D(E, X; G),$$

where $U_{i,k}$ is the operation of differentiation of the row of D which corresponds to $e_{i,k} = 1$. A *shift* Λ of row i in E moves a one, $e_{i,k} = 1$ of this row to the next position $(i, k + 1)$. This shift is permissible if $e_{i,k+1} = 0$. As a variation of (3a) we have

$$(3b)\quad \frac{d}{dx_i} D = \sum_{\Lambda} D(\Lambda E, X; G).$$

For higher derivatives we shall use

$$(4)\quad \frac{d^r}{dx_i^{\,r}} D = \sum_{\Lambda^*} D(\Lambda^* E, X; G),$$

$$(5)\quad \frac{d^{r+1} D}{dx_i^{\,r+1}} = \sum_{\Lambda^*} \sum_{(\Lambda^* E)_{i,k}=1} U_{i,k} D(\Lambda^* E, X; G)$$

where Λ^* are *multiple shifts* of row i of G of order r, that is, products of r permissible simple shifts. After these preparations we can state and prove

LEMMA 2. *Let $x_1, \ldots, x_n$ be fixed. If for each polynomial f, all determinants $D_s = D(E, X; G_s)$ satisfy*

$$(6)\quad \frac{d^r D_s}{dx_1^{\,r}} = 0, \quad s = 0, \ldots, n,$$

then

$$(7)\quad \frac{d^{r+1}}{dx_1^{\,r+1}} D(E, X) = 0.$$

Proof of lemma. From (6) and (3), expanding the determinants with respect to their column s,

$$\begin{aligned} 0 = \frac{d^r D_s}{dx_1^{\,r}} &= \sum_{\Lambda^*} D(\Lambda^* E, X; G_s) \\ &= \sum_{\Lambda^*} \sum_{(\Lambda^* E)_{i,k}=1} D(\Lambda^* E, X; G_s)_{(i,k),s} f^{(k)}(x_i) \\ &= \sum_{(i,k)} f^{(k)}(x_i) \sum_{(\Lambda^* E)_{i,k}=1} D(\Lambda^* E, X; G_s)_{(i,k),s}. \end{aligned}$$

The minor in the last line does not contain f and is identical with $D(\Lambda^* E, X)_{(i,k),s}$. For a polynomial f of sufficiently high degree, the values $f^{(k)}(x_i)$ can be prescribed arbitrarily, hence we get from this

$$(8)\quad \sum_{(\Lambda^* E)_{i,k}=1} D(\Lambda^* E, X)_{(i,k),s} = 0, \quad i = 1, \ldots, m,\ k, s = 0, \ldots, n.$$

For the derivative (7), we use (5):

$$\frac{d^{r+1}D}{dx_1^{\,r+1}} = \sum_{\Lambda^*} \sum_{(\Lambda^*E)_{1,s}=1} D(\Lambda_s' \Lambda^* E, X). \tag{9}$$

For a fixed s with $(\Lambda^*E)_{1,s} = 1$ and fixed Λ^*, we expand the last determinant with respect to the row which contained the old one, $(\Lambda^*E)_{1,s}$. This gives, with $\beta_k = x_1^{k-s-1}/(k-s-1)!$,

$$D(\Lambda_s' \Lambda^* E, X) = \sum_{k=0}^{n} \beta_k D(\Lambda^* E, X)_{(1,s),k}.$$

Rearranging the sum (9) we have

$$\frac{d^{r+1}D}{dx_1^{\,r+1}} = \sum_{s=0}^{n} \sum_{k=0}^{n} \beta_k \sum_{\Lambda^*} D(\Lambda^* E, X)_{(1,s),k} = 0$$

by (8). This proves the lemma.

To prove the necessity of the condition of Theorem 1, we assume that it is not satisfied. By the theorem mentioned above, E is complex singular. There exist then distinct complex $x_1, \ldots, x_m$ for which $D(E, X) = 0$. Let $X^* = (x, x_2, \ldots, x_m)$ with variable x, and let r be the multiplicity of the zero $x = x_1$ of the polynomial $D(x) =: D(E, X^*)$. If one of the determinants $D_s(x) =: D(E, X^*; G_s)$ has a zero $x = x_1$ of order $< r$ for some polynomial f, then it follows from (2) that P is not an entire function. If these zeros are always of order $\geqq r$, then by the lemma, $D^{(r+1)}(x_1) = 0$, a contradiction. This completes the proof.

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