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REGULAR GERMS FOR $\not{p}$-ADIC $Sp(4)$

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1. Introduction. Shalika [**6**] proved the existence of germs (associated with a connected semi-simple algebraic group and a maximal torus over a non-archimedean local field), established many of their properties, and conjectured that the germ associated to the trivial unipotent class in $GL(n)$ should be a constant. R. Howe and Harish-Chandra proved that it is a constant and J. Rogawski [**5**] proved that it had the value predicted previously by J. Shalika.

Recently, J. Repka published his papers about germs for $\not{p}$-adic $GL(n)$ and $SL(n)$ ([**2**], [**3**], [**4**]) which suggested to me that the same work for $Sp(2m)$ could possibly be done. Since $Sp(2) = SL(2)$, we investigate $Sp(4)$ prior to higher dimensions. Our result can in principle be obtained from the recent work of Langlands and Shelstad (cf. [**1**], [**7**]) and can be seen as an explicit version of their work in a special case.

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2. On symplectic groups and notations. Let

$$G = Sp(4) = \{A \in SL(4) | {}^tAJA = J\},$$

where

$$J = \begin{bmatrix} 0 & E \\ -E & 0 \end{bmatrix} \quad \text{with} \quad E = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Let τ be the involution on $SL(4)$ defined by

$$\tau(A) = J({}^tA)^{-1}J^{-1}, \quad A \in SL(4).$$

We know immediately that $SP(4) = SL(4)^{\tau}$. G can also be expressed as the subgroup of $SL(4)$ generated by all the symplectic transvections whose most general forms are of the type

(2.1) $\quad M_5(c, \alpha_1, \alpha_2, \beta_1, \beta_2)$

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$$= \begin{bmatrix} 1 + c\alpha_1\beta_1 & c\alpha_1\beta_2 & -\alpha_1^2 c & -\alpha_1\alpha_2 c \\ c\alpha_2\beta_1 & 1 + c\alpha_2\beta_2 & -\alpha_1\alpha_2 c & -\alpha_2^2 c \\ c\beta_1^2 & c\beta_1\beta_2 & 1 - c\alpha_1\beta_1 & -c\alpha_2\beta_1 \\ c\beta_1\beta_2 & c\beta_2^2 & -c\alpha_1\beta_2 & 1 - c\alpha_2\beta_2 \end{bmatrix}.$$

where $c \neq 0$ and α_i, β_i ($i = 1, 2$) are arbitrary variables in a ground field F. Thus any element is of the form

$$M = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix}, \quad M_{ij} \in M_2(F)$$

satisfying

$$(2.2) \quad \begin{cases} {}^tM_{11}M_{22} - {}^tM_{21}M_{12} = {}^tM_{22}M_{11} - {}^tM_{12}M_{21} = 1 \\ {}^tM_{11}M_{21} - {}^tM_{21}M_{11} = {}^tM_{22}M_{12} - {}^tM_{12}M_{22} = 0. \end{cases}$$

In what follows, we let F be a $\mathfrak{p}$-adic field with ring of integers $\mathcal{O}$; let $\mathfrak{p}$ be the maximal ideal of $\mathcal{O}$ and let $\#(\mathcal{O}/\mathfrak{p}) = q \neq 2^m$. Let $K = Sp(4, \mathcal{O})$, and let

$$K_r = \{k \in K: k \equiv \text{id} \bmod \mathfrak{p}^r\}.$$

For convenience write $\mathfrak{p}^r$ as $\tilde{\mathfrak{p}}$ (r is a fixed positive integer) and diag(a, b, a^{-1}, b^{-1}) as $d(a, b)$. When $a = b$, denote diag(a, b, a^{-1}, b^{-1}) as $d(a)$ for short. We write char(s) for the characteristic polynomial of a matrix s, and $c(s)$ for the pair consisting of the coefficients of the 2nd and the 3rd terms of the characteristic polynomial of s − id.

Henceforth conjugating a matrix s by a matrix r always means to produce $r^{-1}sr = s^r$. As for other symbols we shall follow standard conventions frequently used in the literature.

3. Unipotent conjugacy classes in G.

PROPOSITION (3.1). *Any unipotent element in $G = Sp(4)$ is G-conjugate to the form*

$$(3.2) \quad \begin{bmatrix} 1 & x & \alpha & \beta \\ 0 & 1 & \beta - \gamma x & \gamma \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -x & 1 \end{bmatrix}$$

where α, β and $\gamma \in F$.

Proof. Let v be any nonzero eigenvector of a unipotent element $u \in Sp(4)$; it is the first column of an element $g \in Sp(4)$. Then we get

$$g^{-1}ug = \begin{bmatrix} 1 & \\ 0 & \\ 0 & * \\ 0 & \end{bmatrix}.$$

Thus, considering (2.2), every unipotent symplectic element can be transformed into the form

$$\begin{bmatrix} 1 & u_{12} & u_{13} & \dfrac{u_{12} + u_{43}}{u_{42}} \\ 0 & 1 & \dfrac{u_{12} + u_{43}}{u_{42}} & 0 \\ 0 & 0 & 1 & 0 \\ 0 & u_{42} & u_{43} & 1 \end{bmatrix}$$

Lastly it can be transformed by $m_5(-1, 0, 1, 0, 1)$ (cf. (2.1)) into the form (3.2).

If $x = 0$ in (3.2), it is not a regular unipotent element, i.e., GL-conjugate to the element with all diagonal and super-diagonal entries equal to 1 and with all other entries equal to 0. If $x \neq 0$, we can compute directly to see that it is conjugate to either

$$(3.3) \quad \begin{bmatrix} 1 & 1 & 0 & \delta \\ 0 & 1 & 0 & \delta \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix}$$

with representatives

$$\delta \in F^{\times}/(F^{\times})^2$$

or a non-regular matrix. Hence we have

PROPOSITION (3.4). *The regular unipotent conjugacy classes of G are represented by* (3.3).

Now we set

$$u(\bar{a}) = \begin{bmatrix} 1 & 1 & 0 & \bar{a} \\ 0 & 1 & 0 & \bar{a} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix}.$$

Then $u(\bar{a})$ is G-conjugate to $u(1)$ if and only if $\bar{a} \in (F^{\times})^2$, and there are exactly four regular unipotent conjugacy classes.

Let

$$S(1) = \{g \in K: g \equiv u(1) \bmod \tilde{\mathfrak{p}}\},$$

and let

$$S(\bar{a}) = d(\sqrt{\bar{a}})S(1)d(\sqrt{\bar{a}})^{-1}.$$

Thus any element of $S(1)$ is of the form

$$(3.5)\quad \begin{bmatrix} 1+p_{11} & 1+X_{12} & p_{13} & 1+p_{14} \\ X_{21} & 1+p_{22} & p_{23} & 1+X_{24} \\ X_{31} & p_{32} & 1+p_{33} & X_{34} \\ p_{41} & X_{42} & -1+p_{43} & 1+p_{44} \end{bmatrix}$$

where the p_{ij}'s are arbitrary in $\tilde{\mathfrak{p}}$ and the X_{ij}'s $(\in \tilde{\mathfrak{p}})$ are rational functions of the p_{ij}'s with coefficients in $\mathcal{O}$ uniquely determined by (2.2). From this we see that $S(1) \approx (\tilde{\mathfrak{p}})^{10}$. We can also see that (3.5) is conjugate to the form

$$(3.6)\quad \begin{bmatrix} \dfrac{1}{1-p_{43}} & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ -p_{41} & 0 & 1-p_{43} & 0 \\ p_{41} & p_{44} & -1+p_{43} & 1+p_{44} \end{bmatrix}$$

by seven conjugation automorphisms, where p_{41}, p_{43} and p_{44} are arbitrary in $\tilde{\mathfrak{p}}$. For our convenience, let S_3 be the set of all such forms and let p be the composite map of these automorphisms.

PROPOSITION (3.7). *The G-conjugacy class of $u(\bar{b})$ meets $S(\bar{a})$ if and only if $\bar{b}/\bar{a} \in (F^{\times})^2$.*

Proof. If suffices to prove this is true for $\bar{a} = 1$. The "if part" is obvious from the proof of Proposition (3.1). Now for the "only if" part, suppose an element of $S(1)$ is G-conjugate to an element $u(\bar{b})$. Then it is also unipotent in $S(1)$ and is G-conjugate to an element u of the form (3.6). Since u is unipotent, we must have $(u - 1)^4 = 0$, and so it is not difficult to see that $p_{41} = p_{43} = p_{44} = 0$ in (3.6). Therefore u is transformed by conjugation into the simple form $u(1)$ which is G-conjugate to $u(\bar{b})$ by hypothesis (cf. [**4**], p. 259).

The next proposition also tells us the relation of $u(\bar{a})$ with $S(\bar{a})$.

PROPOSITION (3.8). *The only unipotent conjugacy class which intersects $S(\bar{a})$ is that of $u(\bar{a})$.*

Proof. We have only to show the result for the case $\bar{a} = 1$. Think of any other unipotent conjugacy class, and suppose that u is an element of its

$GL(4)$-conjugacy class which is in Jordan normal form; in other words u has 1's on the diagonal, 1's and 0's (at least one zero entry) on the super-diagonal, and zeros elsewhere.

Provided that some $GL(4)$-conjugate of u were in $S(1)$, then a $GL(4)$-conjugate of $u -$ id would be in $S(1) -$ id. We know that $u -$ id has rank less than 3 since there is at least one zero on the superdiagonal 1, while every element of $S(1) -$ id has rank at least 3 since even modulo $\tilde{\mathfrak{p}}$ its rank is 3, i.e., we are led to two incompatible facts. Hence our assertion is clear (cf. [**2**], p. 166 and [**4**], p. 174).

4. Elliptic tori and Shalika's germs. Suppose $\theta \in F^{\times}\backslash(F^{\times})^2$ and write $E^{\theta} = F(\sqrt{\theta})$. Let

$$E_1^{\theta} = \{x \in E^{\theta}: N_F^{E^{\theta}}(x) = 1\},$$

i.e.,

$$E_1^{\theta} = \{a + b\sqrt{\theta}: a, b \in F \text{ and } a^2 - b^2\theta = 1\}.$$

Now let T be the set of all matrices of the form

$$\begin{bmatrix} a & 0 & b & 0 \\ 0 & \alpha & 0 & \beta \\ b\theta_1 & 0 & a & 0 \\ 0 & \beta\theta_2 & 0 & \alpha \end{bmatrix}$$

where $\theta_1, \theta_2 \in F^{\times}\backslash(F^{\times})^2$ and $a^2 - b^2\theta_1 = 1, \alpha^2 - \beta^2\theta_2 = 1$. Then we have the following two results:

(i) T is isomorphic to $E_1^{\theta_1} \times E_1^{\theta_2}$,
(ii) T is an elliptic torus as a Cartan subgroup.

Shalika's theorem (cf. [**6**], p. 236) says that given $f \in C_c^{\infty}(G)$ (the space of locally constant, complex-valued functions on G having compact support), a maximal torus T and a regular element $t \in T'$ (the set of all regular elements in T) sufficiently close to the id (how close depends on f), we have

$$(4.1) \quad \int_{T\backslash G} f(g^{-1}tg)d\dot{g} = \sum_{i=1}^{n} \Gamma_i(t) \int_{Z(u_i)\backslash G} f(g^{-1}u_i g)d\dot{g}$$

where $\{u_i\}$ is a (finite) set of representatives of the unipotent conjugacy classes, and the functions Γ_i do not depend on f, though they do, of course, depend on T. We shall compute the function $\Gamma_{\bar{a}}(t)$ corresponding to the element $u(\bar{a})$ of Section 3. Hereafter we shall put $u(\bar{a}) = u_{\bar{a}}$, $S(\bar{a}) = S_{\bar{a}}$ for short.

We intend to figure out $\Gamma_{\bar{a}}(t)$ by letting $f = f_{S_{\bar{a}}}$ be the characteristic function of the set $S_{\bar{a}}$ defined in Section 3. By Propositions (3.9) and (3.10), the integrals on the right hand side of (4.1) all vanish, except for the one corresponding to $\bar{a}$. Therefore to calculate $\Gamma_{\bar{a}}(t)$ we have only to evaluate the orbital integrals of f over the conjugacy classes of t and $u_{\bar{a}}$ (cf. [**2**], p. 417).

5. Parametrization of variables and Jacobians. Let us assume that t is a regular element of T sufficiently close to the identity; write $t = \mathrm{id} + x$, and assume t is such that the nontrivial coefficients of the characteristic polynomial $t - \mathrm{id}$ are in $\tilde{\mathfrak{p}}$, i.e., $c(t) \in (\tilde{\mathfrak{p}})^2$ according to our convention. We can work out the characteristic polynomial of $g^{-1}tg$ easily since

$$\det(t - \lambda \cdot 1) = \lambda^4 - 2(a + \alpha)\lambda^3 + 2(1 + 2a\alpha)\lambda^2 - 2(a + \alpha)\lambda + 1.$$

On the other hand the characteristic polynomial of a matrix s in the form (3.6) turns out to be

$$\begin{aligned}\operatorname{char}(s) = \lambda^4 &- \lambda^3\left(4 - p_{43} + \frac{p_{43}}{1 - p_{43}} + p_{44}\right) \\ &+ \lambda^2\left\{2 - p_{41} + \left(1 - p_{43} + \frac{1}{1 - p_{43}}\right)(2 + p_{44})\right\} \\ &- \lambda\left(4 - p_{43} + \frac{p_{43}}{1 - p_{43}} + p_{44}\right) + 1.\end{aligned}$$

Thus we know that $c(s) \in (\tilde{\mathfrak{p}})^2$. Comparing these coefficients with those of $\operatorname{char}(t)$ yields a unique solution in terms of t and p_{43}, viz.

$$(5.1)\quad \begin{cases} p_{41} = \left(1 - p_{43} + \dfrac{1}{1 - p_{43}}\right)(2 + p_{44}) - 4a\alpha \\ \quad = 4(a + \alpha) - 3 - \dfrac{1}{(1 - p_{43})^2} - 2p_{43}(a + \alpha - 1) \\ \quad - p_{43}^2 + \dfrac{2(a + \alpha)}{1 - p_{43}}p_{43} - 4a\alpha \\ p_{44} = 2(a + \alpha) - 4 + p_{43} - \dfrac{p_{43}}{1 - p_{43}}. \end{cases}$$

Therefore supposing $t \in T'$ and p_{43} are given arbitrarily so that t and s are conjugate, then p_{41} and p_{44} are uniquely determined in $\tilde{\mathfrak{p}}$. We can return to (3.5) to work out a similar result. Suppose that we are given arbitrary p_{ij}'s except for p_{32}, p_{33} (set as X_{32}, X_{33} temporarily) and that this matrix is conjugate to t. Then, as above, we have eight independent variables p_{ij} and eight dependent variables which are uniquely determined by these variables and t.

Now we are going to determine whether we can find $g \in G$ so that $g^{-1}tg = s_3$, where s_3 is an arbitrary matrix of the form (3.8) determined by t and p_{43} uniquely. By the way, every element of T is G-conjugate to the form

$$t_1 = \begin{bmatrix} 1 & 0 & b & 0 \\ 0 & 1 & 0 & \beta \\ \dfrac{2a-2}{b} & 0 & 2a-1 & 0 \\ 0 & \dfrac{2\alpha-2}{\beta} & 0 & 2\alpha-1 \end{bmatrix}$$

by

$$g_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -\dfrac{a+1}{b} & 0 & 1 & 0 \\ 0 & -\dfrac{\alpha+1}{\beta} & 0 & 1 \end{bmatrix}.$$

By making use of this and by direct calculation, we can show that if

$$\frac{\bar{a}b}{2P(\alpha - a)} \in N_F^{E^{\theta_1}}[(E^{\theta_1})^\times] \quad \text{and}$$

$$\frac{\bar{a}\beta}{2Q(a - \alpha)} \in N_F^{E^{\theta_2}}[(E^{\theta_2})^\times]$$

with

$$P = (1 - p_{43})^2 - 2a(1 - p_{43}) + 1,$$

$$Q = (1 - p_{43})^2 - 2\alpha(1 - p_{43}) + 1,$$

there exists $g \in G$ such that $g^{-1}tg = s$ for every $s \in S(\bar{a})$ satisfying (5.1) and with $p_{41} \neq 0$ in the corresponding form (3.6). The converse also holds true.

Now for a fixed regular $t \in T$ we construct

$$c^t: T\backslash G \to G$$

given by $c^t(g) = t^g$. Let

$$\bar{G}_{\bar{a}}(t) = (c^t)^{-1}[S(\bar{a})].$$

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The measure of $\bar{G}_{\bar{a}}(t)$ is the orbital integral of $f_{S_{\bar{a}}}$ over the conjugacy class of t. Let $c^{\bar{a}}$ be the map $S(\bar{a}) \rightarrow S(1)$ obviously defined by

$$d(\sqrt{a})sd(\sqrt{a})^{-1} \mapsto s$$

for arbitrary $s \in S(1)$. Let

$$P': S_3 \times (\tilde{\mathfrak{p}})^7 \rightarrow \tilde{\mathfrak{p}} \times (\tilde{\mathfrak{p}})^7$$

be a map with

$$P'(p_{41}, p_{43}, p_{44}, \dots) = (p_{43}, \dots).$$

Hence we have the composite map

$$(5.2) \quad P' \circ P \circ c^{\bar{a}} \circ c^t : \bar{G}_{\bar{a}}(t) \rightarrowtail S_{\bar{a}} \rightarrowtail\!\!\twoheadrightarrow S_1 \rightarrowtail\!\!\twoheadrightarrow S_3 \times (\tilde{\mathfrak{p}})^7 \twoheadrightarrow \tilde{\mathfrak{p}} \times (\tilde{\mathfrak{p}})^7.$$

Due to the above description this map is bijective except for only a finite number of p_{43}, i.e., except for the case $p_{41} = 0$ in S_3. However, this does not affect the measure of $\bar{G}_{\bar{a}}(t)$. Thus the nonempty $\bar{G}_{\bar{a}}(t)$'s all have the same measure. Here we intend to find out the composite map's Jacobian so that we may calculate the measure of $\bar{G}_1(t)$.

For fixed $t \in T' \cap K_r$, let U be a neighborhood of t in $T' \cap K_r$ chosen so that no two elements of U are conjugate. Let $A \subset T' \times T\backslash G$ be the set

$$A = \{(t, g): t \in U, t^g \in S_1\}.$$

Consider the commuting diagram in Figure 1.

$$\begin{array}{ccccc}
T' \times T\backslash G \supset A & \xrightarrow{\quad c^T \quad} & S_1 \cong (\tilde{\mathfrak{p}})^{10} & \xrightarrow{\quad P \quad} & S_3 \times (\tilde{\mathfrak{p}})^7 \\
\downarrow c \times \mathrm{id} & & \downarrow c \times P' \circ P & & \downarrow P'' \\
(\tilde{\mathfrak{p}})^2 \times T\backslash G \supset B & \xrightarrow{\mathrm{id} \times P' \circ P \circ c^t} & (\tilde{\mathfrak{p}})^2 \times (\tilde{\mathfrak{p}} \times (\tilde{\mathfrak{p}})^7) & \xrightarrow{\quad \mathrm{id} \quad} & (\tilde{\mathfrak{p}})^2 \times (\tilde{\mathfrak{p}} \times (\tilde{\mathfrak{p}})^7)
\end{array}$$

Figure 1

The mapping labelled c^T in the diagram stands for the conjugation map taking $(t, g) \in T' \times T\backslash G$ to $t^g = g^{-1}tg$. The set B is just the image of A. For $s_1 \in S_1$,

$$c(s_1) = (c_1, c_2), \quad \text{where } c_1 = \operatorname{trace}(s_1 - 1)$$

and c_2 is the coefficient of λ^2 appearing in $\det(s_1 - 1 - \lambda \cdot 1)$. For

$$(s_3, p_1, \dots, p_7) \in S_3 \times (\tilde{\mathfrak{p}})^7,$$

the right vertical map P'' is defined as

$$P''(s_3, p_1, \dots, p_7) = (c(s_3), p_{43}, p_1, \dots, p_7);$$

explicitly,

$$c(s_3) = \left[p_{43} - p_{44} - \frac{p_{43}}{1 - p_{43}}, p_{43} - p_{41} - 2p_{44} - \frac{p_{43}}{1 - p_{43}} + \frac{p_{44}}{1 - p_{43}} - p_{43}p_{44}\right].$$

In the next paragraph we discuss the Jacobians of these maps.

The Jacobian of the map $T' \times T\backslash G \to G$ given by $(t, g) \to t^g$ is

$$D(t) = \det[\mathrm{id} - \mathrm{Ad}(t)]_{g/t},$$

where g and t are the associated Lie algebras of G and T respectively (cf. [**6**], p. 231). Now to find the Jacobian of the left vertical map c, we make a composite map:

$$(a, \alpha) \xrightarrow{c'} [a + \sqrt{a^2 - 1}, \alpha + \sqrt{\alpha^2 - 1}]$$

$$\xrightarrow{c} [2(a + \alpha) - 4, 8 - 6a - 6\alpha + 4a\alpha].$$

Here

$$|J(c \circ c')| = |8(a - \alpha)| = |a - \alpha| \quad \text{and}$$

$$|J(c')| = \left|\left[\frac{a + \sqrt{a^2 - 1}}{\sqrt{a^2 - 1}}\right]\left[\frac{\alpha + \sqrt{\alpha^2 - 1}}{\sqrt{\alpha^2 - 1}}\right]\right|$$

as is easily shown.

Hence

$$|J(c)| = |(a - \alpha)\sqrt{a^2 - 1}\sqrt{\alpha^2 - 1}|.$$

Furthermore we have

$$|J(c \times P' \circ P)| = 1$$

since $|J(P'')| = |J(P)| = 1$, and hence we have

$$(5.3) \quad |J(P' \circ P \circ c^t)| = \left|\frac{D(t)}{(a - \alpha)\sqrt{a^2 - 1}\sqrt{\alpha^2 - 1}}\right|.$$

Seeing, however, that for two roots

$$r_1 = a + b\sqrt{\theta_1}, \quad r_2 = \alpha + \beta\sqrt{\theta_2}$$

of the characteristic polynomial of t,

$$|D(t)| = |(1 - r_1^2)(1 - r_1^{-2})(1 - r_2^2)(1 - r_2^{-2})(1 - r_1r_2) \times (1 - r_1^{-1}r_2^{-1})(1 - r_1r_2^{-1})(1 - r_1^{-1}r_2)|$$

$$= |\sqrt{a^2 - 1}\sqrt{\alpha^2 - 1}(a - \alpha)|^2,$$

we have the following simple form

$$(5.3)' \quad |J(P' \circ P \circ c^t)| = |D(t)/(a - \alpha)\sqrt{a^2 - 1}\,\sqrt{\alpha^2 - 1}| = |D(t)|^{1/2}.$$

6. Normalization of measures and orbital integrals. The natural additive measure dx on F is the one for which $\mathcal{O}_F$ has measure 1. As was mentioned earlier in Section 4,

$$T \cong E_1^{\theta_1} \times E_1^{\theta_2};$$

moreover

$$[E^{\theta_1}]^\times / F^\times \supset E_1^{\theta_1}/\{\pm 1\}$$

and hence choices of measures on $[E^{\theta_1}]^\times$ and $F^\times$ determine a choice of measure on E_1^θ. We know that on $[E^{\theta_1}]^\times$ we can take the corresponding measure

$$d^\times x = \frac{dx}{|x|_{E^{\theta_1}}}$$

and on $F^\times$ the standard measure

$$d^\times s = \frac{ds}{|s|}.$$

We select the measure on G whose restriction to K is an extension of the standard measure of $S(1) \cong (\mathfrak{p})^{10}$. Haar measure of $S(1)$ must be the same as that of $(\mathfrak{p})^{10}$ since

$$|J(c \times P' \circ P)| = 1.$$

Choices of measures on G and T imply a choice of measure on $T\backslash G$, and so on K and $S_{\bar{a}}$, where the natural measure is just the restriction of Haar measure on G. Referring again to the diagram in Figure 1, we see that these measures are compatible with the other measures in the diagram.

PROPOSITION (6.1). *Let T be the set of all matrices of the form*

$$t = \begin{bmatrix} a & 0 & b & 0 \\ 0 & \alpha & 0 & \beta \\ b\theta_1 & 0 & a & 0 \\ 0 & \beta\theta_2 & 0 & \alpha \end{bmatrix}$$

where

$$\theta_1, \theta_2, \in F^\times \backslash (F^\times)^2 \quad \textit{and} \quad a^2 - b^2\theta_1 = \alpha^2 - \beta^2\theta_2 = 1.$$

Suppose $t \in T'$ is sufficiently close to id *that the coefficients of* char$(t - \text{id})$ *are in $(\mathfrak{p})^2$ ignoring the first and last term. Then we have*

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$$\int_{T\backslash G} f_{S_{\bar{a}}}(t^g)d\dot{g}$$

$$= \begin{cases} q^{-8r} \cdot |D(t)|^{-1/2}, & if \dfrac{\bar{a}b}{(1-a)(\alpha - a)} \in N_F^{E^{\theta_1}}[(E^{\theta_1})^\times] \\ & and \dfrac{\bar{a}\beta}{(1-\alpha)(a-\alpha)} \in N_F^{E^{\theta_2}}[(E^{\theta_2})^\times] \\ 0 & , otherwise. \end{cases}$$

Proof. We have seen already the conditions for the germ, i.e.,

$$\frac{\bar{a}b}{2P(\alpha - a)} \in N_F^{E^{\theta_1}}[(E^{\theta_1})^\times] \quad \text{and}$$

$$\frac{\bar{a}\beta}{2Q(a - \alpha)} \in N_F^{E^{\theta_2}}[(E^{\theta_2})^\times],$$

where

$$P = (1-p)^2 - 2a(1-p) + 1,$$
$$Q = (1-p)^2 - 2\alpha(1-p) + 1,$$

and p is any element in $\mathfrak{p}$. However,

$$P = 2 - 2p + p^2 - 2(1-p)\sqrt{1 + b^2\theta_1} \quad \text{and}$$
$$Q = 2 - 2p + p^2 - 2(1-p)\sqrt{1 + \beta^2\theta_2}.$$

Here

$$\sqrt{1 + b^2\theta_1} = 1 + p'\theta_1, \quad \sqrt{1 + \beta^2\theta_2} = 1 + p''\theta_2$$

for some p', $p'' \in \mathfrak{p}$ by Hensel's lemma, so we have

$$P = 2 - 2p + p^2 - 2(1-p)(1 + p'\theta_1) = p^2 - 2(1-p)p'\theta_1$$

and similarly

$$Q = p^2 - 2(1-p)p''\theta_2.$$

Hence whether or not P (resp. Q) belongs to

$$N_F^{E^{\theta_1}}[(E^{\theta_1})^\times]$$
$$[\text{resp. } N_F^{E^{\theta_2}}[(E^{\theta_2})^\times]]$$

does not depend upon $p \in \mathfrak{p}$. Therefore we can set $p = 0$ from the first.

Now, since the measure of $\mathfrak{p}$ is q^{-r} and so the measure of $(\mathfrak{p})^8$ is q^{-8r}, and since from (5.3) and (5.3)′ the Jacobian of $P' \circ P \circ c^t$ has modulus $|D(t)|^{1/2}$, we have the result immediately.

Next we have to look for the last ingredient, that is the orbital integral over the conjugacy class of $u(\bar{a}) = u_{\bar{a}}$. In order to compute it we need to specify the measure on the centralizer $Z(u_{\bar{a}})$. By the way, $Z(u_{\bar{a}})$ consists of all the matrices of the form:

$$\begin{bmatrix} \pm 1 & a_{12} & \bar{a}a_{13} & \frac{\bar{a}a_{12}}{2}(\pm a_{12} + 1) \\ 0 & \pm 1 & \frac{\bar{a}a_{12}}{2}(1 \mp a_{12}) & \bar{a}a_{12} \\ 0 & 0 & \pm 1 & 0 \\ 0 & 0 & -a_{12} & \pm 1 \end{bmatrix}$$

where $a_{12}, a_{13} \in F$ are arbitrary. We take the measure $da_{12}da_{13}$.

We write

$$G = BK = Z(u_{\bar{a}}) \cdot P \cdot K = Z(u_{\bar{a}}) \cdot B_1 \cdot B_0 \cdot K,$$

where B is the quasi-upper triangular symplectic subgroup, i.e.,

$$B = \left\{ \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ 0 & a_{22} & \frac{a_{14}a_{22} - a_{12}a_{24}}{a_{11}} & a_{24} \\ 0 & 0 & \frac{1}{a_{11}} & 0 \\ 0 & 0 & -\frac{a_{12}}{a_{11}a_{22}} & \frac{1}{a_{22}} \end{bmatrix} \right\}$$

where $a_{11}, a_{22} \in F^{\times}$ and $a_{12}, a_{13}, a_{14}, a_{24} \in F$,

$$B_0 = \left\{ \begin{bmatrix} 1 & 0 & 0 & a_{14} \\ 0 & 1 & a_{14} & a_{24} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \right\} \quad \text{where } a_{14}, a_{24} \in F,$$

and

$$B_1 = \{\mathrm{diag}(a_{11}, a_{22}, a_{11}^{-1}, a_{22}^{-1})\}$$

where $a_{11}, a_{22} \in F^{\times}$, so the integral over $Z(u_{\bar{a}})\backslash G$ can be replaced by an integral over $[Z(u_{\bar{a}})\backslash B] \cdot K$, and $Z(u_{\bar{a}})\backslash B$ can be represented by $P = B_1 \cdot B_0$. We write dp for the obvious measure on $Z(u_{\bar{a}})\backslash B$, that is, write

$$p = \mathrm{diag}(a_{11}, a_{22}, a_{11}^{-1}, a_{22}^{-1}) \cdot b_0$$

with $b_0 \in B$, and let dp be the product of the standard $F^{\times}$ measures $d^{\times}a_{11}$, $d^{\times}a_{22}$, and the standard F-measures $da_{14}da_{24}$.

The quotient measure on $Z(u_{\bar{a}})\backslash G$ is obtained by writing $\dot{g} = pk$ with

$$p \in B_1B_0 = P, \quad k \in K$$

and putting

$$d\dot{g} = \left[1 - \frac{1}{q}\right]^{-2} \Delta_p(p)\Delta_B(p)dpdk,$$

since B and P are not unimodular although G, K and $Z(u_{\bar{a}})$ are unimodular.

PROPOSITION (6.2). *Under the assumption of measures normalized as above, we have*

$$\int_{Z(u_{\bar{a}})\backslash G} f_{S_{\bar{a}}}[u_{\bar{a}}^g]d\dot{g} = q^{-8r}.$$

Proof. $G = BK$ implies that elements conjugate to u_1 are determined by $g = pk$ with $p \in P$ and $k \in K$. By the way

$$p^{-1}u_1p = \begin{bmatrix} a_{11} & 0 & 0 & a_{11}a_{14} \\ 0 & a_{22} & a_{22}a_{14} & a_{22}a_{24} \\ 0 & 0 & a_{11}^{-1} & 0 \\ 0 & 0 & 0 & a_{22}^{-1} \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} a_{11}^{-1} & 0 & 0 & -a_{22}a_{14} \\ 0 & a_{22}^{-1} & -a_{11} & -a_{22}a_{24} \\ 0 & 0 & a_{11} & 0 \\ 0 & 0 & 0 & a_{22} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & a_{11}a_{22}^{-1} & -2a_{11}^2a_{14} & a_{11}a_{22}(1 - a_{24}) \\ 0 & 1 & -a_{11}a_{22}a_{24} & a_{22}^2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -a_{11}a_{22}^{-1} & 1 \end{bmatrix},$$

and so

$$p^{-1}u_1p = ks_1k^{-1} \quad \text{for } k \in K, s_1 \in S_1$$

implies that $p \in P \cap K$, and so it is not difficult to show that

$$k = p^{-1}k'$$

with

$$k' \in [Z(u_1) \cap K] \cdot K_r \supset K_r.$$

Conversely,

$$p \in P_1 \cap K \quad \text{and} \quad k = p^{-1}k'$$

with

$$k' \in [Z(u_1) \cap K] \cdot K_r$$

obviously implies

$$u_1^{pk} \in S_1.$$

Since the modular functions are both 1 for $p \in K$, we have

$$\int_{Z(u_1)\backslash G} f_{S_1}(u_1^g)d\dot{g} = \int_{[Z(u_1)\cap K]\cdot K_r} \int_{P\cap K} \left[1 - \frac{1}{q}\right]^{-2} dpdk.$$

However, the measure of $P \cap K$ is $[1 - 1/q]^2$ and the measure of $[Z(u_1) \cap K] \cdot K_r$ is nothing but q^{-8r}. Thus we have

$$\int_{Z(u_1)\backslash G} f_{S_1}(u_1^g)d\dot{g} = q^{-8r}.$$

Since

$$u_{\bar{a}} = d(\sqrt{a})u_1 d(\sqrt{a})^{-1} \quad \text{and} \quad S_{\bar{a}} = d(\sqrt{a})S_1 d(\sqrt{a})^{-1},$$

we have

$$\{\dot{g} \in Z(u_{\bar{a}})\backslash G | u_{\bar{a}}^g \in S_{\bar{a}}\}$$
$$\approx \{\dot{h} = d(\sqrt{a})^{-1}\dot{g}d(\sqrt{a}) \in Z(u_1)\backslash G | u_1^h \in S_1\}.$$

Thus, if we change variables from $g \in G$ to $d(\sqrt{a})gd(\sqrt{a})^{-1}$, we have the same formula as above.

7. Main result. At last we are ready to establish our main result combining everything in the previous sections.

THEOREM. *Suppose that we are given T and t as above. Then the germ $\Gamma_{\bar{a}}(t)$ associated to the above regular element $t \in T'$ and $u_{\bar{a}}$ is obtained as follows: with the Haar measure normalization as in Section* 6,

$$\Gamma_{\bar{a}}(t) = |D(t)|^{-1/2},$$

if

$$\frac{\bar{a}b}{(1 - a)(\alpha - a)} \in N_F^{E^{\theta_1}}[(E^{\theta_1})^{\times}]$$

and

$$\frac{\bar{a}\beta}{(1 - \alpha)(a - \alpha)} \in N_F^{E^{\theta_2}}[(E^{\theta_2})^{\times}],$$

$$\Gamma_{\bar{a}}(t) = 0,$$

otherwise.

Proof. The result is immediate by Propositions (6.1) and (6.2).

Remarks. 1) We see

$$F^{\times}/N_F^{E^{\theta}}[(E^{\theta})^{\times}] = [(E^{\theta})^{\times}]^G/\mathrm{Tr}_G[(E^{\theta})^{\times}]$$
$$= \mathrm{Ext}^0[\mathbf{Z}, (E^{\theta})^{\times}] = \hat{H}^0[G, (E^{\theta})^{\times}],$$

where $G(E^\theta/F) = \{1, \sigma\}$ is the cyclic Galois group of order 2,

$$\mathrm{Tr}_G = 1 + \sigma,$$

and $[(E^\theta)^\times]^G$ is the G-submodule consisting of the elements fixed by G, which turns out to be identified with

$$\mathrm{Hom}_G[\mathbf{Z}, (E^\theta)^\times],$$

the group $\mathbf{Z}$ being considered as a G-module with trivial action, i.e., $\sigma n = n$ for all $n \in \mathbf{Z}$. So we see that

$$\#[\hat{H}^0[G, (E^\theta)^\times]] = [F^\times : N_F^{E^\theta}[(E^\theta)^\times]] = [E^\theta : F] = 2.$$

2) If $\theta_1(F^\times)^2 = \theta_2(F^\times)^2$, then there exist $t \in T'$ at which all the regular germs vanish. Indeed, suppose $\theta = \theta_1 = \theta_2 \in \mathcal{O}$, and choose b, $\beta \in \mathfrak{p}$ with

$$-b/\beta \notin N = N_F^{E^\theta}(E^\theta).$$

Solve for a, $\alpha \in 1 + \mathfrak{p}$ so that

$$a + b\sqrt{\theta}, \quad \alpha + \beta\sqrt{\theta} \in E_1^\theta,$$

and let $t \in T$ be the corresponding element (cf. Proposition (6.1)).

Now suppose

$$\frac{\bar{a}b}{(1 - a)(\alpha - a)} \in N.$$

Then

$$\frac{\bar{\alpha}\beta}{(1 - \alpha)(a - \alpha)} = \frac{\bar{a}b}{(1 - a)(\alpha - a)} \cdot \frac{(-\beta)(1 - a)}{b(1 - \alpha)},$$

which is in N if and only if

$$-\frac{1 - a}{b} \cdot \frac{\beta}{1 - \alpha} \in N.$$

But $a^2 - \theta b^2 = 1$ implies

$$(a - 1)/b = \theta b/(a + 1),$$

and similarly

$$(\alpha - 1)/\beta = \theta\beta/(\alpha + 1).$$

Now

$$(\alpha + 1)/(a + 1) \in 1 + \mathfrak{p} \subset N,$$

so

$$-\frac{1-a}{b}\frac{\beta}{1-\alpha} = \frac{\theta b}{a+1}\left[-\frac{\alpha+1}{\theta\beta}\right] = -\frac{b}{\beta}\frac{(\alpha+1)}{(a+1)} \notin N,$$

by the choice of b, β.

So the two conditions for the nonvanishing of $\Gamma_{\bar{a}}(t)$ are incompatible, and all regular germs vanish at t, a phenomenon not encountered for $GL(n)$ or $SL(n)$.

If $\theta_1/\theta_2 \notin (F^{\times})^2$, then it is easy to see that for any $t \in T'$ there is at least one $\bar{a}$ so that $\Gamma_{\bar{a}}(t) \neq 0$.

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