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COMPACTNESS AND CONVEXITY OF CORES OF TARGETS FOR NEUTRAL SYSTEMS

ANTHONY N. EKE

In this paper we prove the convexity and the compactness of the cores of targets for neutral control systems. We make use of a weak compactness argument; but in the crucial part where we establish the boundedness of the cores of the target we make use of the notion of asymptotic direction from Convex Set Theory. Let E^n be n-dimensional Euclidean space. We prove that the core of the target $H = L + E$ (where $L = \{x \in E^n \mid Mx = 0\}$, M is a constant $m \times n$ matrix and E is a compact, convex set containing 0) of the neutral system

$$\dot{x}(t) - A\dot{x}(t-h) = Bx(t) + Cx(t-h) + Du(t)$$

is convex, and is compact if, and only if, the system

$$\dot{x}(t) - A\dot{x}(t-h) = B^T x(t) + C^T x(t-h) + M^T u(t)$$

is Euclidean controllable.

1. INTRODUCTION

The study of controllability of systems to the core of targets was studied first in the case of linear control systems by Hajek [4].

In this paper, we consider the neutral control system

$$\text{(1.1)} \qquad \begin{cases} \dot{x}(t) - A\dot{x}(t-h) &= Bx(t) + Cx(t-h) + Du(t) \\ x(t) &= \phi(t), \qquad T \in [-h, 0], \qquad h > 0; \end{cases}$$

where A, B and C are $n \times n$ constant matrices, D is a constant $n \times m$ matrix and ϕ is continuous. The control u is an m-vector measurable function having values $u(t)$ constrained to lie in a compact, convex, non-empty set Ω, Ω being a subset of the Euclidean space E^m, and $u \in L_2([0,t], \Omega)$ for $0 < t < \infty$. This u is said to be *admissible*. The target set H is a closed, convex and non-empty subset of E^n.

Now suppose $W_2^{(1)}$ is the Sobolev space $W_2^{(1)}([-h,0], E^n)$ of functions $\phi\colon [-h,0] \to E^n$ which are absolutely continuous with square integrable derivatives. If $x\colon [-h, t_1] \to E^n$ then, whenever $t \in [0, t_1]$, we write x_t as the continuous function on $[-h, 0]$ defined by $x_t(s) = x(t+s)$, $s \in [-h, 0]$. Provided $\phi \in W_2^{(1)}$ and u is an

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admissible control, there always exists a unique solution for (1.1) such that $x(t) = \phi(t)$ for $t \in [-h, 0]$. This solution is given by the Variation-of-Constants formula

$$x(t,\phi,u) = x(t,\phi,0) + \int_0^t X(t-\tau)Du(\tau)d\tau, \tag{1.2}$$

where the fundamental matrix $X(t)$ satisfies the equation

$$\dot{x}(t) - A\dot{x}(t-h) = Bx(t) + Cx(t-h) \tag{1.3}$$

$$X(t) = \begin{cases} 0, & t < 0 \\ 1, & t = 0 \end{cases} \tag{1.4}$$

and for $t \neq kh$, $k = 0, 1, 2, \ldots$, $X(t)$ has a continuous first derivative so is of bounded variation on each compact interval $(kh, (k+1)h)$, $k = 0, 1, 2, \ldots$ (see Hale [5, p. 29]). In (1.2) above, we have

$$x(t,\phi,0) = X(t)[\phi(0) - A\phi(-h)] + C\int_{-h}^0 X(t-\tau-h)\phi(\tau)d\tau - A\int_{-h}^0 [dX(t-\tau-h)]\phi(\tau), \qquad k \geq h, \qquad h > 0. \tag{1.5}$$

In view of (1.5) above, we can write (1.2) as follows

$$x(t,\phi,u) = X(t)[\phi(0) - A0(-h)] + \int_0^t X(t-\tau)Du(\tau)d\tau + C\int_{-h}^0 X(t-\tau-h)\phi(\tau)d\tau - A\int_{-h}^0 [dX(t-\tau-h)]\phi(\tau), \qquad t > 0. \tag{1.6}$$

Definition 1.1.. The core of the target set H, core(H), is the set of all initial points $\phi(0) \in E^n$ for which $\phi \in W_2^{(1)}$ such that there exists a measurable control $u\colon [0,\infty] \to \Omega$ for which the solution $x(t) = x(t,\phi,u)$ of (1.1) satisfies $x(t) \in H$ for all $t \geq 0$.

Definition 1.2. The system (1.1) is said to be *Euclidean controllable* if for each $\phi \in W_2^{(1)}$ and each $x_1 \in E^n$ there exist a $t_1 \geqslant 0$ and an admissible control u such that the solution $x(t,\phi,u) = x(t)$, say, of (1.1) satisfies $x_0(0,\phi,u) = \phi$ and $x(t_1,\phi,u) = x_1$.

Definition 1.3. The system (1.1) is said to be *proper on* $[0, t_1]$ if and only if $q^T X(t_1 - s)D = 0$ a.e. where $s \in [0, t_1]$, and $q \in E^n$ implies $q = 0$.

The system (1.1) is controllable on $[0, t_1]$ if and only if it is proper on $[0, t_1]$.

Remark. The above was shown to be true in Chukwu and Silliman [1].

Hence, we have the following lemma

LEMMA 1.1. *The system (1.1) is Euclidean controllable on* $[0, t_1]$ *if and only if* $q^T X(t_1 - s)D = 0$, $q \in E^n$, $s \in [0, t_1]$ *implies* $q = 0$.

2. Preliminaries

We shall give some facts in convex set theory which are crucial to our work. In this section we shall also establish a very important lemma which will be needed in proving the main result of this paper.

Definition 2.1. A point $a \in E^n$ is an *asymptotic direction* of a convex set $S \subseteq E^n$ if for $x \in S$ and all $t \geq 0$, we have $x + ta \in S$; that is, the half-ray issuing from x in direction a is entirely contained within S.

Proposition 2.1. *A non-empty convex set of E^n is bounded if and only if 0 is its only asymptotic direction.*

Proposition 2.2. *Suppose $P \subseteq E^n$ is a non-empty convex set of the form $P = L + E$, where E is bounded and contains 0, and L is a linear subspace of P, then L is the largest linear subspace of P and coincides with the set of asymptotic directions of P.*

Lemma 2.1. *If $0 \in H$ and $0 \in \Omega$ then $0 \in \operatorname{core}(H)$ and so $\operatorname{core}(H) \neq \emptyset$.*

Proof: From (1.6), we have

$$x(t,\phi,u) = X(t)[\phi(0) - A0(-h)] + \int_0^t X(t-\tau)Du(\tau)d\tau + C\int_{-h}^0 X(t-\tau-h)\phi(\tau)d\tau - A\int_{-h}^0 [dX(t-\tau-h)]\phi(\tau), \qquad t \geq 0. \tag{2.1}$$

We choose $\phi(\cdot) = 0 \in H$, $u = 0 \in \Omega$ so that we get from (2.1) above

$$x(t,0,0) = X(t)0 + \int_0^t X(t-\tau)D0d\tau + 0 + 0 = 0, \qquad t \geq 0.$$

Thus for $0 \in H$ we get $x(t,0,0) = 0 \in H$, $t \geq 0$. This shows that $\phi(0) = 0 \in \operatorname{core}(H)$ and so $\operatorname{core}(H) \neq \emptyset$. ∎

Lemma 2.2. *$a \in E^n$ is an asymptotic direction of $\operatorname{core}(H)$ if and only if $X(t-s)a$ is an asymptotic direction of H.*

Proof: Now, for fixed t, s we can write (1.6) as

$$x(t-s,\phi,u) = X(t-s)[\phi(0) - A\phi(-h)] + \int_0^{t-s} X(t-s-\tau)Du(\tau)d\tau + C\int_{-h}^0 X(t-s-\tau-h)\phi(\tau)d\tau - A\int_{-h}^0 [dX(t-s-\tau-h)]\phi(\tau), \qquad t-s \geqslant 0.$$

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We can take an asymptotic direction $a \in \operatorname{core}(H)$ and choose $\phi(0) \in \operatorname{core}(H)$ so that for all $\theta \geq 0$ we have $\phi(0) + \theta a \in \operatorname{core}(H)$. We choose an appropriate admissible control $u_\theta \colon [0, \infty) \to \Omega$ such that the right hand side equals

$$X(t-s)[\phi(0) + \theta a - A\phi(-h)] + \int_0^{t-s} X(t-s-\tau) D u_\theta(\tau) d\tau$$
$$+ C \int_{-h}^{0} X(t-s-\tau-h)\phi(\tau) d\tau - A \int_{-h}^{0} [dX(t-s-\tau-h)]\phi(\tau) \in H,$$

for $t - s \geqslant 0$.

Dividing throughout by θ we obtain

$$X(t-s)[\frac{\phi(0)}{\theta} a - \frac{A}{\theta}\phi(-h)] + \frac{1}{\theta}\int_0^{t-s} X(t-s-\tau) D u_0(\tau) d\tau$$
$$+ \frac{C}{\theta}\int_{-h}^{0} X(t-s-\tau-h)\phi(\tau) d\tau - \frac{A}{\theta}\int_{-h}^{0} [dX(t-s-\tau-h)]\phi(\tau) \in \frac{H}{\theta}, \qquad t-s \geq 0.$$

Since C and A are constants we have

$$\lim_{\theta \to \infty} \frac{C}{\theta} \int_{-h}^{0} X(t-s-\tau-h)\phi(\tau) d\tau = 0,$$

and

$$\lim_{\theta \to \infty} \frac{A}{\theta} \int_{-h}^{0} [dX(t-s-\tau-h)]\phi(\tau) = 0.$$

Also, $\lim_{\theta \to \infty} (A/\theta)\phi(-h) = 0$ and as $\phi(0) \in E^n$, $\lim(\phi(0)/\theta) = 0$.

Finally, since the control u_θ is measurable and it is defined on a bounded set, we have

$$\lim_{\theta \to \infty} \frac{1}{\theta} \int_0^{t-s} X(t-s-\tau) D u_\theta(\tau) d\tau = \lim_{\theta \to \infty} \int_0^{t-s} X(t-s-\tau) D \frac{u_\theta}{\theta} d\tau = 0.$$

Taking limits, we obtain

$$X(t-s)a = \lim_{\theta \to \infty} \frac{1}{\theta} b_\theta \text{ for some } b_\theta \in H. \tag{2.2}$$

We claim that $X(t-s)a$ in (2.2) above is an asymptotic direction of H. Indeed, for $c \in H$, $\lambda \geq 0$, it is sufficient to show that $c + \lambda X(t-s)a \in H$ provided (2.2) above is satisfied. Assuming λ is fixed and $\theta \geq \lambda$, we have $\lambda \leq \theta$, that is, $0 \leq \lambda \leq \theta$ and so $0 \leq (\lambda/\theta) \leq 1$.

Since H is convex, $c \in H$, $b_\theta \in H$, then we have

$$\left(1-\frac{\lambda}{\theta}\right)C+\frac{\lambda}{\theta}b_\theta \in H. \tag{2.3}$$

In (2.3) above, we take limits as $\theta \to \infty$ and since H is closed, the limit points of H also belong to H.

Therefore $\lim_{\theta\to\infty}(1-(\lambda/\theta))c+\lambda\lim_{\theta\to\infty}(1/\theta)b_\theta \in H$, or $c+\lambda X(t-s)a \in H$, since from (2.2), we have $\lim_{\theta\to\infty}(1/\theta)b_\theta = X(t-s)a$. This concludes the proof of the claim. Conversely, let $X(t-s)a$ be an asymptotic direction of H. For $t-s\geq 0$, we have

$$H+\theta X(t-s)a \in H, \qquad \theta \geq 0. \tag{2.4}$$

Take $\phi(0)\in \text{core}(H)$. Now, choose an admissible control $u_0\colon [0,\infty)\to \Omega$ such that

$$\begin{aligned} &X(t-s)[\phi(0)-A\phi(-h)]+\int_0^{t-s}X(-s-\tau)Du_0(\tau)d\tau \\ &+C\int_{-h}^0 X(t-s-\tau-h)\phi(\tau)d\tau - A\int_{-h}^0[dX(t-s-\tau-h)\phi(\tau)\ \in H, \qquad t-s\geq 0. \end{aligned} \tag{2.5}$$

If $X(t-s)a$ is an asymptotic direction of H, then for all $\theta\geq 0$, in view of definition 2.1, we have

$$\begin{aligned} X(t-s)[\phi(0)-A\phi(-h)]+\int_0^{t-s}X(t-s-\tau)Du_0(\tau)d\tau + C\int_{-h}^0 X(t-s-\tau)\phi(\tau)d\tau \\ -A\int_{-h}^0[dX(t-s-\tau-h)]\phi(\tau)+\theta X(t-s)a \text{ belongs to } H \text{ for } t-s\geq 0; \end{aligned}$$

that is,

$$\begin{aligned} X(t-s)[\phi(0)+\theta a-A\phi(-h)]+\int_0^{t-s}X(t-s-\tau)Du_0(\tau)d\tau \\ +C\int_{-h}^0 X(t-s-\tau-h)\phi(\tau)d\tau - A\int_{-h}^0[dX(t-s-\tau-h)]\phi(\tau) \end{aligned}$$

belongs to H and from this we infer that $\phi(0)+\theta a \in \text{core}(H)$. Now since the same control u_0 holds this point within H, this implies that a is an asymptotic direction of $\text{core}(H)$. ∎

3. Main results

Theorem 3.1. *Consider the linear neutral control system (1.1) in which the control u is an m-vector measurable function having values $u(t)$ lying in a compact, convex, non-empty set Ω. Then the core of the target set H (H being a closed, convex, non-empty subset of E^n),* core(H), *is convex.*

Proof: Suppose $\phi_1(0)$, $\phi_2(0) \in \text{core}(H)$. Then to two admissible controls, u_1 and u_2, there correspond two solutions, $x(t,\phi_1,u_1)$ and $x(t,\phi_2,u_2)$ such that

$$(3.1)\quad x(t,\phi_i u_i) = X(t)[\phi_i(0) - A\phi_i(-h)] + \int_0^t X(t-\tau)u_i(\tau)d\tau \\ + C\int_{-h}^0 X(t-\tau-h)\phi_i(\tau)d\tau - A\int_{-h}^0 [dX(t-\tau-h)]\phi_i(\tau) \in H, \text{ for } i = 1,2.$$

Suppose α is a constant such that $0 \le \alpha \le 1$, α being a constant.

Since the target set H is convex, and since each of $x(t,\phi_i,u_i)$, for $i = 1,2$, belongs to H, then a convex combination of (3.1) belongs to H. Thus we have

$$\alpha x(t,\phi_1,u_1) + (1-\alpha)x(t,\phi_i,u_i) \in H;$$

that is,

$$(3.2)\quad \alpha X(t)[\phi_1(0) - A\phi_1(-h)] + \alpha\int_0^t X(t-\tau)Du_1(\tau)d\tau \\ + C\int_{-h}^0 X(t-\tau-h)\phi_1(\tau)d\tau - A\int_{-h}^0 [dX(t-\tau-h)]\phi_1(\tau) \\ + (1-\alpha)X(t)[\phi_2(0) - A\phi_2(-h)] + (1-\alpha)\int_0^t X(t-\tau)Du_2(\tau)d\tau \\ + (1-\alpha)C\int_{-h}^0 X(t-\tau-h)\phi_2(\tau)d\tau \\ - (1-\alpha)A\int_{-h}^0 [dX(t-\tau-h)]\phi_2(\tau) \in H.$$

Since α is a constant, we can re-arrange (3.2) to obtain

$$(3.3)\quad X(t)[\{\alpha\phi_1 + (1-\alpha)\phi_2\}(0) - A\{\alpha\phi_1(-h) + (1-\alpha)\phi_2(-h)\}] \\ + \int_0^t X(t-\tau)D[\alpha u_1 + (1-\alpha)u_2](\tau)d\tau + C\int_{-h}^0 X(t-\tau-h)[\alpha\phi_1 + (1-\alpha)\phi_2](\tau)d\tau \\ - A\int_{-h}^0 [dX(t-\tau-h)\{\alpha\phi_1 + (1-\alpha)\phi_2](\tau) \in H.$$

Since Ω is a convex set, there is an admissible control $\overline{u}$ such that $\overline{u}(\tau) = \alpha u_1(\tau) + (1-\alpha)u_2(\tau)$. Also, since $\phi_i(\tau) \in E^n$ and E^n is convex, it follows that there exists $\overline{\phi}$ as follows

$$\overline{\phi}(-h) = \alpha\phi_1(-h) + (1-\alpha)\phi_2 \quad \text{and} \quad \overline{\phi}(\tau) = \alpha\phi_1(\tau) + (1-\alpha)\phi_2(\tau).$$

When these facts are taken into account in (3.3) above, we see that

$$\alpha\phi_1(0) + (1-\alpha)\phi_2(0) \in \operatorname{core}(H).$$

This shows that $\operatorname{core}(H)$ is convex. ∎

THEOREM 3.2. *Consider the neutral control system (1.1). The control functions $u\colon [0,\infty) \to \Omega$ are square integrable on finite intervals. The target set H is a closed, convex and non-empty subset of E^n. Then, the core of the target H, $\operatorname{core}(H)$, is closed.*

PROOF: The admissible controls $|M$ given by the set

$$|M = \{u\colon u \in L_2([0,t],\Omega)\},$$

where u is square integrable, is a closed, convex and bounded subset of $L_2([0,t],E^m)$. The space $L_2([0,t],E^m)$ is reflexive and so from [**3**, p.425] we infer that $|M$ is weakly compact.

Now, let $\phi_k(0)$, for $k = 1,2,\ldots$ be a sequence of points belonging to $\operatorname{core}(H)$ with $\phi_k \in W_2^{(1)}$ the corresponding functions such that

$$\lim_{k\to\infty} \phi_k = \phi \text{ in } W_2^{(1)}. \tag{3.4}$$

Thus in E^n $\lim_{k\to\infty}\phi_k(0) = \phi(0)$ and $\lim_{k\to\infty}\phi_k(-h) = \phi(-h)$. Let u_k, for $k = 1,2,\ldots$ be the corresponding admissible controls such that for $k = 1,2,\ldots$ we have

$$\begin{aligned} x(t,\phi_k,u_k) &= X(t)[\phi_k(0) - A\phi_k(-h)] + \int_0^t X(t-\tau)Du_k(\tau)d\tau \\ &+ C\int_{-h}^0 X(t-\tau-h)\phi_k(\tau)d\tau - A\int_{-h}^0 [dX(t-\tau-h)]\phi_k(\tau) \in H, \qquad t \geqslant 0. \end{aligned} \tag{3.5}$$

Since $|M$ is weakly compact, there exists a subsequence u_{kj} of u_k, with $j = 1,2,\ldots$ which converges weakly to a control function $\overline{u}_0 \in |M$ on $[0,t_1]$.

In other words,

$$\lim_{j\to\infty}\int_0^t X(t-\tau)Du_{kj}(\tau)d\tau = \int_0^t X(t-\tau)D\overline{u}_0(\tau)d\tau. \tag{3.6}$$

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Suppose now that $\{\phi_{kj},$ for $j = 1,2,\ldots\}$, are the subsequences of $\{\phi_k,$ for $k = 1,2,\ldots\}$ corresponding to $\{u_{kj},$ for $j = 1,2,\ldots\}$. Then we have

$$(3.7)\quad x(t,\phi_{kj},u_{kj}) = X(t)[\phi_{kj}(0) - A\phi_{kj}(-h)] + \int_0^t X(t-\tau)Du_{kj}(\tau)d\tau$$
$$+ C\int_{-h}^0 X(t-\tau-h)\phi_{kj}(\tau)d\tau - A\int_{-h}^0 [dX(t-\tau-h)]\phi_{kj}(\tau) \in H, \qquad t \geqslant 0.$$

Since H is closed, if we take the limits of both sides of (3.7) these limits belong to H; that is

(3.8)
$$\lim_{j\to\infty} x(t,\phi_{kj},u_{kj}) = \lim_{j\to\infty} X(t)[\phi_{kj}(0) - A\phi_{kj}(-h)] + \lim_{j\to\infty}\int_0^t X(t-\tau)Du_{kj}(\tau)d\tau$$
$$+ \lim_{j\to\infty} C\int_{-h}^0 X(t-\tau)\phi_{kj}(\tau)d\tau - \lim_{j\to\infty} A\int_{-h}^0 [dX(t-\tau-h)]\phi_{kj}(\tau) \in H.$$

Thus from (3.4), (3.6) and (3.8) we have

$$\lim_{j\to\infty} x(t,\phi_{kj},u_{kj}) = X(t)[\phi(0) - A\phi(-h)] + \int_0^t X(t-\tau)D\overline{u}_0(\tau)d\tau$$
$$+ C\int_{-h}^0 X(t-\tau-h)\phi(\tau)d\tau - A\int_{-h}^0 [dX(t-\tau-h)]\phi(\tau) \in H,$$

which implies that $\phi(0) \in \operatorname{core}(H)$ and so $\operatorname{core}(H)$ is closed. ∎

THEOREM 3.3. *Let us consider the neutral control system*

$$(1.1)\qquad \begin{cases} \dot{x}(t) - A\dot{x}(t-h) & = Bx(t) + Cx(t-h) + Du(t) \\ x(t) & = \phi(t),\ t \in [-h,0],\ \text{and } h > 0. \end{cases}$$

Suppose the target set H is of the form $H = L + E$, with $L = \{x \in E^n : Mx = 0\}$ a linear subspace of H, and E a compact, convex set containing 0 of the control system (1.1) and M is an $m \times n$ constant matrix. Let $0 \in \Omega$ and also $0 \in H$. Under these conditions, $\operatorname{core}(H)$ *is compact if and only if the control system*

$$\dot{x}(t) - A\dot{x}(t-h) = B^T x(t)C^T x(t-h) + M^T u(t)$$

is Euclidean controllable.

PROOF: Suppose $\{\phi_n(0) \mid n = 1,2,\ldots\}$ is the set of asymptotic directions of $\operatorname{core}(H)$. Then Lemma 2.2 implies that $\{X(t-s)\phi_n \mid n = 1,2,\ldots\}$ is the set of

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asymptotic directions of H. From Proposition 2.2, which says that L coincides with the set of asymptotic directions of H, we conlude that $L = \{X(t-s)\phi_n(0) \mid n = 1,2,\ldots\}$.

The hypothesis on H in the above theorem implies that

$$MX(t-s)\phi_n(0) = 0.$$

Taking the transposes, we have

$$\phi_n^T(0)X^T(t-s)M^T = 0, \text{ for all } n, \qquad t-s \geqslant 0. \tag{3.9}$$

Let us suppose now that the system

$$\dot{x}(t) - A\dot{x}(t-h) = B^T x(t) + C^T x(t-h) + M^T u(t)$$

is Euclidean controllable on $[0,t_1]$ for each $t_1 > 0$. Then by Lemma 1.1 this means that $\phi_n^T(0)X^T(t-s)M^T = 0$, $\phi_n(0) \in E^n$ implies $\phi_n(0) = 0$, $\forall t-s \geqslant 0$, for each n. Hence by hypothesis, this shows that 0 is the only asymptotic direction of core(H). Lemma 2.1 gives that core(H) is non-empty. Also Theorem 3.1 shows that core(H) is convex. Thus, core(H) is a non-empty convex subset of E^n with 0 as its only asymptotic direction; then Proposition 2.1 implies that core(H) is bounded. But Theorem 3.2 shows that core(H) is also closed. Thus core(H) is compact.

Conversely, assume that core(H) is compact. This implies that core(H) is bounded. So Proposition 2.1 gives that 0 is the sole asymptotic direction. Reffer-ring now to (3.9) above, we have that $\phi_n^T(0)X^T(t-s)M^T = 0$ implies $\phi_n(0) = 0$ for all $t-s \geqslant 0$, and for all n. Hence Lemma 1.1 implies by this that the control system

$$\dot{x}(t) - A\dot{x}(t-h)B^T x(t) + C^T x(t-h) + M^T u(t)$$

is Euclidean controllable on $[0,t_1]$, for $t_1 \geqslant 0$. We have thus proved our main result. ∎

4. Example

Consider in E^2, the $x-y$ plane, say, the target set H defined by

$$H = \{x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \mid x_1 = 0, x_2 \neq 0\}, \tag{4.1}$$

where $x \in E^2$. Then systems of vectors of the form $\binom{0}{\eta}$ for all finite non-zero entries $\eta \in E^1$ belong to core(H). Thus any neutral control system in E^2 of the form (1.1) with initial function $\phi_0 \in W_2^{(1)}([-1,0], E^2)$ such that

$$\phi_0(t) = \{\begin{pmatrix} \phi_1(t) \\ \phi_2(t) \end{pmatrix} \mid \phi_1(t) = 0, \phi_2(t) \neq 0 \text{ for all } t \geq 0\} \tag{4.2}$$

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implies that $\phi_0(t) \in \text{core}(H)$.

Following Theorem 3.1, we infer that this $\text{core}(H)$ is convex and it is definitely bounded. In (4.1) above we define $M = (1 \quad 0)$, a 1×2 constant matrix.

Now, consider the neutral system in E^2 given as

$$\dot{x}(t) - A\dot{x}(t-1) = Bx(t) + Cx(t-1) + Du(t) \tag{4.3}$$

where

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & 0 \\ 1 & -2 \end{pmatrix}, \qquad C = \begin{pmatrix} -1 & -1 \\ 0 & -1 \end{pmatrix}, \qquad D = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

which satisifes the initial condition (4.2) and has the target set (4.1) above.

Following Hale [5, p.144] we need to find the fundamental matrix $X(t-s)$ of (4.3). With the data for the system (4.3) we obtain, after lengthy but straightforward calculations as in Driver [2],

$$X(t-s) = e^{2\tau} \begin{pmatrix} 1+\tau & -\tau \\ \tau & 1-\tau \end{pmatrix}$$

for some $\tau = s - T \geq 0$, where $T \geq 0$, for which the u in (4.3) is admissible. Choosing any $\xi = \binom{\xi_1}{\xi_2} \in E^2$ we see that

$$e^{-2\tau}(\xi_1, \xi_2) \begin{pmatrix} 1+\tau & \tau \\ -\tau & 1-\tau \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = e^{-2\tau}(\xi_1 + \tau\xi_1 - \tau\xi_2) = 0$$

is true if and only if $\xi_1 = 0$ and $\xi_2 = 0$, which implies

$$\xi = \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = 0.$$

That is,

$$\xi^T X^T(t-s) M^T = 0 \text{ implies } \xi = 0$$

which, in turn, implies by Lemma 1.1 that the system

$$\dot{x}(t) - A\dot{x}(t-1) = B^T x(t) + C^T x(t-1) + M^T u(t)$$

is Euclidean controllable.

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Department of Mathematics,
University of Nigeria,
Nsukka,
Nigeria.