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THE REDUCED GROUP C^*-ALGEBRA OF A TRIANGLE BUILDING

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Let Δ be an affine building of type $\widetilde{A}_2$ and let Γ be a discrete group of type-rotating automorphisms acting simply transitively on the vertices of Δ. We prove that the reduced group C^*-algebra $C_r^*(\Gamma)$ is simple. To prove this result we use the sufficient condition for the simplicity of $C_r^*(\Gamma)$ given in a recent paper by M. Bekka, M. Cowling and P. de la Harpe.

1. Introduction

Let Γ be a discrete group. The reduced C^*-algebra $C_r^*(\Gamma)$ of Γ is the norm closure in the C^*-algebra of all bounded linear operators on $\ell^2(\Gamma)$ of the linear span of $\lambda_\Gamma(\Gamma)$, where λ_Γ is the left regular representation of Γ on $\ell^2(\Gamma)$.

Powers proved in [7] that when Γ is a non-abelian free group, $C_r^*(\Gamma)$ is simple (that is, it has no non-trivial two-sided ideals) and the map $\tau : C_r^*(\Gamma) \to \mathbb{C}$ defined by $\tau(e) = 1$ and $\tau(\lambda_\Gamma(\gamma)) = 0$ for all γ in $\Gamma \setminus \{e\}$ is the unique normalised trace on the C^*-algebra. This result has been generalised by several authors. In [1] Bekka, Cowling and de la Harpe proved that, when Γ is a discrete group acting on a compact space Ω, then $C_r^*(\Gamma)$ is simple and it has a unique normalised trace if the action of Γ on Ω satisfies the following geometric condition:

Property P_{geo}. Let Γ be a discrete group acting on a compact space Ω. Then (Γ, Ω) is said to have Property P_{geo} if, for any finite subset F of $\Gamma \setminus \{e\}$, there exist γ_0 in Γ, a finite subset $\{\omega_s, s \in S\}$ of Ω, and open neighbourhoods V_s of ω_s in Ω for each s in S, such that

(i) $\{\omega_s, s \in S\}$ is the set of fixed points of the action of γ_0 on Ω and, for each ω in Ω, there exists s in S such that

$$\lim_{j \to \infty} \gamma_0^j \omega = \omega_s;$$

(ii) $\gamma V_s \cap V_{s'} = \emptyset$, for all s, s' in S and all γ in F;

(iii) for all s in S and j in $\mathbb{Z}^+$, if ω in V_s and $\gamma_0^j \omega \notin V_s$, then $\gamma_0^{j+1} \omega \notin V_s$.

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In this paper we prove that this geometric condition is satisfied when Γ is a discrete group of type-rotating automorphisms of an affine building Δ of type $\widetilde{A}_2$, acting simply transitively on the vertices of Δ and the compact space Ω is the maximal boundary of the building.

Recently Robertson and Steger have shown that, if Γ is a linear group acting simply transitively on the vertices of a triangle building, then the C^*-algebra $C(\Omega) \rtimes_r \Gamma$ is simple (and $C_r^*(\Gamma)$ is subnuclear). The minimality of the action of Γ on Ω ([**9**, Proposition 4.1.1]), [**1**, Theorem 5] and our result imply that the C^*- algebra $C(\Omega) \rtimes_r \Gamma$ is simple also for non-linear buildings. Another proof of this more general result appears in [**9**, Theorem 5.1].

2. Notation

Following Tits, a triangle building Δ (of order $q \geqslant 2$) is any thick affine building of type $\widetilde{A}_2$ and of order q. Thus Δ is a simplicial complex of rank 2, consisting of vertices, edges and triangles (called chambers), such that each edge lies on $(q+1)$ chambers. We refer the reader to [**10**] for more details on buildings.

We denote by $\mathcal{V}$ the set of vertices of Δ. There is a function $\tau : \mathcal{V} \to \mathbb{Z}/3\mathbb{Z}$, called the type, such that each chamber contains exactly one vertex of each type.

Let d be the usual graph-theoretic distance on $\mathcal{V}$. For each $x_0 \in \mathcal{V}$ we denote by $\mathcal{S}(x_0)$ the subcomplex consisting of all vertices x satisfying $d(x, x_0) = 1$, and of all edges connecting them. The complex $\mathcal{S}(x_0)$ has the structure of a finite projective plane (P, L) of order q, where P and L are the sets of the vertices of $\mathcal{S}(x_0)$ having type $\tau(x_0)+1$ and $\tau(x_0)-1$ respectively, and $x \in P$, $y \in L$ are incident if x, y and x_0 lie on a common triangle. Hence $\mathcal{S}(x_0)$ consists of (q^2+q+1) vertices of type $\tau(x_0)+1$, (q^2+q+1) vertices of type $\tau(x_0)-1$ and $(q+1)(q^2+q+1)$ edges, and each vertex lies on $(q+1)$ edges.

An apartment $\mathcal{A}$ of Δ is any thin subcomplex of Δ isomorphic to the Coxeter complex of type $\widetilde{A}_2$; it may be realised as an Euclidean plane tessellated by equilateral triangles. For each vertex $x \in \mathcal{A}$, the apartment $\mathcal{A}$ may be decomposed into six simplicial cones emanating from x, called sectors based at x and denoted Q_x.

Given a sector Q_x and an apartment $\mathcal{A}$ containing it, each vertex y in $\mathcal{A}$ may be identified by a pair of integer coordinates (with respect to Q_x), as shown in [**5**] and [**3**]. If $\mathcal{A}'$ is another apartment containing y and Q_x, then the coordinates of y in $\mathcal{A}'$ are the same as those in $\mathcal{A}$. Moreover, if $y \in Q_x \cap Q'_x$, then y has the same coordinates with respect to both Q_x and Q'_x.

Two sectors Q_x and Q_y are said to be equivalent, or parallel, if they contain a common sector. The set Ω of equivalence classes of sectors of Δ is called the maximal boundary of Δ. Ω is in fact the set of chambers of the spherical building at infinity Δ^∞ associated to Δ, as defined in [**10**]. For any fixed vertex x, there is a canonical bijection between the maximal boundary Ω and the collection of sectors based at x. For every $\omega \in \Omega$, we denote by $Q_x(\omega)$ the sector based at x associated with ω. An element ω is a boundary point of an apartment $\mathcal{A}$ if it is represented by a sector lying on $\mathcal{A}$. Hence each apartment has six boundary points.

Fix a vertex x_0. For $\omega \in \Omega$, and $N \geqslant 1$, we define

$$Q_N(\omega) = \Big\{x \in Q_{x_0}(\omega) : d(x, x_0) \leqslant N\Big\}$$

and

$$E_N(\omega) = \Big\{\omega' \in \Omega : Q_N(\omega) \subset Q_{x_0}(\omega')\Big\}.$$

Then [5] the family of sets

$$\mathcal{E} = \{E_N(\omega) : \ N \geqslant 1, \ \omega \in \Omega\}$$

generates a totally disconnected compact Hausdorff topology on Ω.

A particular class of triangle buildings was introduced in [2]. Let (P, L) be the projective plane of order q, where q is any power of a prime number, and let us fix a point-line correspondence $\lambda : P \to L$. A "triangle presentation" (compatible with λ) is defined to be a subset $\mathcal{T}$ of P^3 with the following properties:

(1) given $\xi, \eta \in P$ there exists $\zeta \in P$ such that $(\xi, \eta, \zeta) \in \mathcal{T}$ if and only if $\eta \in \lambda(\xi)$,

(2) $(\xi, \eta, \zeta) \in \mathcal{T}$ implies $(\zeta, \xi, \eta), (\eta, \zeta, \xi) \in \mathcal{T}$,

(3) given $\xi, \eta \in P$ there exists at most one $\zeta \in P$ such that $(\xi, \eta, \zeta) \in \mathcal{T}$.

Let $A = \{a_\xi : \xi \in P\}$ be a set of $(q^2 + q + 1)$ distinct letters and form the multiplicative group

$$\Gamma = \Big\langle \{a_\xi : \xi \in P\} : a_\xi a_\eta a_\zeta = e \ \text{ if } \ (\xi, \eta, \zeta) \in \mathcal{T} \Big\rangle,$$

where e denotes the identity of Γ. Then [2, Theorem 3.4] $\mathcal{T}$ gives rise to a triangle building Δ. The vertices and the edges of Δ form the Cayley graph of Γ constructed via right multiplication with respect to $A \cup A^{-1}$, and its chambers are the triples $\{\gamma, \gamma a_\xi, \gamma a_\eta^{-1}\}$, where $\gamma \in \Gamma$, and $\xi, \eta \in P$, with $(\zeta, \eta, \xi) \in \mathcal{T}$ for some $\zeta \in P$. The type is the homomorphism $\tau : \Gamma \to \mathbb{Z}/3\mathbb{Z}$ determined by $\tau(a_\xi) = 1$, for each $\xi \in P$. The triangle group Γ acts simply transitively by left multiplication on the vertices of Δ, as a group of "type-rotating" automorphisms [2]. It was also shown in [2] that any triangle building, on whose vertices a group acts simply transitively and in type-rotating way, is isomorphic to a building arising from a triangle presentation.

In the present paper we assume Δ is the triangle building arising from a triangle presentation and Γ is its type-rotating simply transitive automorphisms group. It is natural to take the identity element of Γ as the special vertex x_0, and each element γ as the vertex x of Δ if $x = \gamma \cdot x_0$; then, for every $\gamma' \in \Gamma$,

$$\gamma' \cdot (\gamma \cdot x_0) = (\gamma'\gamma) \cdot x_0.$$

For every $\gamma \in \Gamma$, any minimal word for γ in the generators and their inverses contains the same number m of generators and the same number n of inverses; in particular we may write γ in both the following forms

$$\gamma = a_{\xi_1} \dots a_{\xi_m} a_{\eta_1}^{-1} \dots a_{\eta_n}^{-1} = a_{\zeta_1}^{-1} \dots a_{\zeta_n}^{-1} a_{\theta_1} \dots a_{\theta_m}.$$

Moreover the positive integers m, n are the coordinates of the vertex $x = \gamma \cdot x_0$, with respect to a sector Q_{x_0} containing x. By abuse of notation, we simply write $(a, b, c) \in \mathcal{T}$ if $a = a_\xi, b = a_\eta$ and $c = a_\zeta$, with $(\xi, \eta, \zeta) \in \mathcal{T}$. Given $\gamma, \gamma' \in \Gamma$ corresponding to an edge of the building, there exists a generator $a \in A$ such that $\gamma' = \gamma a$ or $\gamma = \gamma' a$. We provide the edge with the orientation $\gamma \to \gamma'$, if $\gamma' = \gamma a$. So we assign to each of the edges of Δ a generator of Γ, and we represent the chamber $C = \{\gamma, \gamma a, \gamma b^{-1}\}$ by the triple $(c, b, a) \in \mathcal{T}$, where c denotes the unique generator satisfying $cba = e$, according to the following diagram.

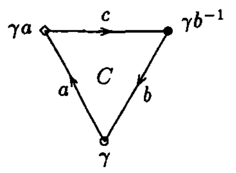

FIGURE 1

The action of Γ on the vertices of Δ induces a natural action, by left multiplication, of the group on the maximal boundary. We refer the reader to [**3**] for a proof that this action is in fact well-defined.

3. Periodic apartments

Let $\mathcal{A}$ be an apartment of Δ, not necessarily containing the fundamental vertex x_0. Fix a sector Q_x of the apartment and consider the coordinate system for the vertices of $\mathcal{A}$ determined by Q_x. For every $(j, k) \in \mathbb{Z}^2$, let $a_{j,k}$ be the element of Γ such that the vertex $a_{j,k} \cdot x_0$ of $\mathcal{A}$ has coordinates (j, k). Hence $\mathcal{A}$ may be represented via its vertices as

$$\mathcal{A} = \{a_{j,k}\}_{j,k \in \mathbb{Z}}.$$

In particular $a_{0,0}$ is the element of Γ corresponding to the origin x of the coordinate system. We note that if $a_{j,k}$ and $a_{l,m}$ correspond to an edge of $\mathcal{A}$, then $a_{l,m} = a_{j,k} a$ or $a_{l,m} = a_{j,k} b^{-1}$, for suitable $a, b \in A$, according as the edge is oriented from $a_{j,k}$ to $a_{l,m}$ or vice versa (see Figure 2).

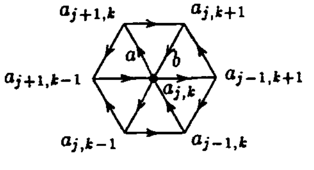

FIGURE 2

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DEFINITION 3.1: The label associated to the ordered pair $(a_{j,k}, a_{l,m})$ of elements of $\mathcal{A}$ is defined as the element of Γ

$$\varepsilon_{j,k}^{l,m} = a_{j,k}^{-1} a_{l,m},$$

and the labeling of any region $\mathcal{R}$ of the apartment $\mathcal{A}$ is defined as the collection

$$\left\{\varepsilon_{j,k}^{l,m}, \quad a_{j,k}, a_{l,m} \in \mathcal{R}\right\}.$$

REMARK 3.2. (i) If $a_{j,k}$, $a_{l,m}$ are adjacent, then $\varepsilon_{j,k}^{l,m} = a$ or $\varepsilon_{j,k}^{l,m} = b^{-1}$, according to the orientation of the corresponding edge, and in this case we refer to $\varepsilon_{j,k}^{l,m}$ as the label of the edge.

(ii) The labels of the edges determine the labeling of the apartment. In fact, if $\{a_{j_i,k_i}\}_{i=0}^{n}$ is a minimal path from $a_{j,k}$ to $a_{l,m}$, then

$$\varepsilon_{j,k}^{l,m} = \prod_{i=0}^{n} \varepsilon_{j_i,k_i}^{j_{i+1},k_{i+1}},$$

where $\varepsilon_{j_i,k_i}^{j_{i+1},k_{i+1}}$ belongs to $A \cup A^{-1}$ for every i.

(iii) The length of a minimal word for $\varepsilon_{j,k}^{l,m}$ is the distance between $a_{j,k}$ and $a_{l,m}$; moreover $\varepsilon_{l,m}^{j,k} = \left(\varepsilon_{j,k}^{l,m}\right)^{-1}$.

REMARK 3.3. Let

(1) $$R_1(j,k) = \{a_{j,k}, a_{j+1,k}, a_{j,k+1}, a_{j+1,k+1}\}$$

be the parallelogram of base vertices $a_{j,k}$, $a_{j+1,k}$, $a_{j,k+1}$ and $a_{j+1,k+1}$ (see Figure 3). The labels, say a and b^{-1}, of two adjacent edges of $R_1(j,k)$, connecting $a_{j,k}$ and $a_{j+1,k+1}$, determine the labeling of the parallelogram. Indeed, there exist unique c, d such that

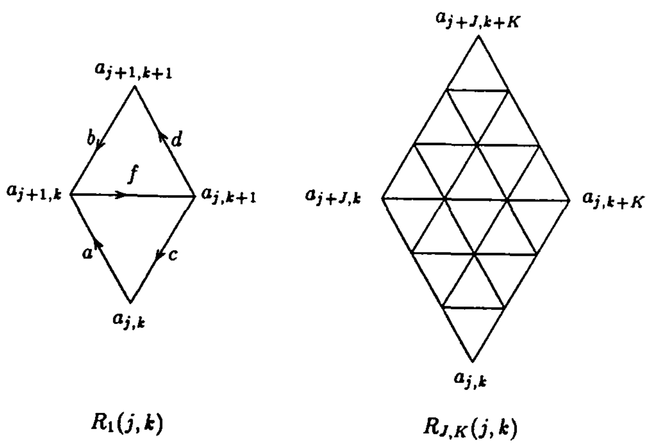

FIGURE 3

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$ab^{-1} = c^{-1}d$, and a unique f such that $(f,c,a),(f,d,b) \in \mathcal{T}$. More generally, let

(2) $$R_{J,K}(j,k) = \{a_{j,k}, a_{j+J,k}, a_{j,k+K}, a_{j+J,k+K}\}$$

be the parallelogram of base vertices $a_{j,k}, a_{j+J,k}$, $a_{j,k+K}$ and $a_{j+J,k+K}$, as in Figure 3. The label $\varepsilon_{j,k}^{j+J,k+K}$, consisting of the labels of the edges of a minimal path joining $a_{j,k}$ to $a_{j+J,k+K}$, determines the labeling of the parallelogram.

REMARK 3.4. If $\mathcal{A}' = \gamma\mathcal{A}$ for some $\gamma \in \Gamma$, then the apartments $\mathcal{A}$ and $\mathcal{A}'$ have the same labeling with respect to corresponding sectors Q_x and γQ_x. Indeed, for every $(j,k) \in \mathbb{Z}^2$, $a'_{j,k} = \gamma a_{j,k}$ and therefore

$$\varepsilon'^{l,m}_{j,k} = a'^{-1}_{j,k}a'_{l,m} = a^{-1}_{j,k}\gamma^{-1}\gamma a_{l,m} = \varepsilon^{l,m}_{j,k}.$$

Conversely, if two apartments $\mathcal{A}$ and $\mathcal{A}'$ have the same labeling with respect to sectors Q_x and $Q_{x'}$, then $\mathcal{A}' = \gamma\mathcal{A}$ for $\gamma = a'_{0,0}a^{-1}_{0,0}$. In fact, for every $(j,k) \in \mathbb{Z}^2$, $\varepsilon'^{j,k}_{0,0} = \varepsilon^{j,k}_{0,0}$, so $a'_{j,k} = a'_{0,0}a^{-1}_{0,0}a_{j,k}$.

DEFINITION 3.5: The pair $(J,K) \in \mathbb{Z}^2$ is a period for the apartment $\mathcal{A}$ (with respect to Q_x) if, for every $(j,k),(l,m) \in \mathbb{Z}^2$,

(3) $$\varepsilon^{l,m}_{j,k} = \varepsilon^{l+J,m+K}_{j+J,k+K}.$$

The periods of $\mathcal{A}$ (with respect to Q_x) form a subgroup of $\mathbb{Z}^2$, denoted by $\mathcal{L}$. If $(J,K) \in \mathcal{L}$, then $a_{j+J,k+K}a^{-1}_{j,k}$ does not depend on (j,k), and it has length greater than one.

DEFINITION 3.6: The apartment $\mathcal{A}$ is periodic (with respect to Q_x) if $\mathcal{L}$ is non-trivial; it is doubly periodic if $\mathcal{L}$ is a cofinite subgroup of $\mathbb{Z}^2$. We denote

(4) $$\mu = \min_{(J,K)\in\mathcal{L}\setminus\{(0,0)\}} \left|a_{j+J,k+K}a^{-1}_{j,k}\right|.$$

REMARK 3.7. (i) The property of periodicity does not depend on the choice of the sector Q_x on the apartment. Actually the apartment $\mathcal{A}$ is (J,K)-periodic if and only if the condition (3) holds for (j,k) and (l,m) corresponding to adjacent vertices. Indeed, if $\{a_{j_i,k_i}\}_{i=0}^n$ is a minimal path from $a_{j,k}$ to $a_{l,m}$, then $\{a_{j_i+J,k_i+K}\}_{i=0}^n$ is a minimal path from $a_{j+J,k+K}$ to $a_{l+J,m+K}$; so Remark 3.2 enables us to conclude. So in geometric terms, the (J,K)-periodicity of $\mathcal{A}$ means that corresponding edges of the sets $\mathcal{S}(a_{j,k}) \cap \mathcal{A}$ and $\mathcal{S}(a_{j+J,k+K}) \cap \mathcal{A}$ have the same labels, for every $(j,k) \in \mathbb{Z}^2$. This characterisation allows us to deduce that the property of periodicity does not depend on the choice of the coordinate system on the apartment.

(ii) If $\mathcal{A}$ is doubly periodic, then $\mathcal{L}$ contains a subgroup of the form $N\mathbb{Z}^2$, for some positive integer N; so it contains vectors of $\mathbb{Z}^2$ in every rational direction.

We refer the reader to **[6]** for more details about periodicity.

Lemma 3.8. *Let $\mathcal{A}$ be an apartment and let $(J,K) \in \mathbb{Z}^2$, with $J, K \neq 0$ and $J + K \neq 0$. The following facts are equivalent:*

(i) (J,K) *is a period;*

(ii) $\varepsilon_{rJ,rK}^{(r+1)J,(r+1)K} = \varepsilon_{(r+1)J,(r+1)K}^{(r+2)J,(r+2)K}, \quad \forall r \in \mathbb{Z}.$

Moreover, if $J = K$, then (i) and (ii) are equivalent to

(iii) $\varepsilon_{j,j}^{j+1,j+1} = \varepsilon_{j+J,j+J}^{j+1+J,j+1+J}, \quad \forall j \in \mathbb{Z}.$

Proof: It is obvious that (i) implies (ii), and (iii) for $J = K$. We prove that (iii) implies (i), assuming, without loss of generality, that $J = K > 0$. Consider

$$\mathcal{C}_i = \{R_1(j+i,j)\}_{j\in\mathbf{Z}}, \ \forall i \in \mathbb{Z}. \tag{5}$$

As shown in Remark 3.3, the labels $\left(\varepsilon_{j,j}^{j+1,j+1}\right)_{j\in\mathbf{Z}}$ determine the labeling of $\mathcal{C}_0$. Then condition (iii) implies that (J,J) is a period for this chain, that is (1) holds for every pair of vertices of this region. Consider now the chain $\mathcal{C}_1$ adjacent on the left to $\mathcal{C}_0$ (the same argument applies to the chain $\mathcal{C}_{-1}$ adjacent on the right). For every j, $R_1(j+1,j)$ contains one edge of $R_1(j,j)$ and one edge of $R_1(j+1,j+1)$, connecting $a_{j+1,j}$ to $a_{j+2,j+1}$. Therefore the labeling of $\mathcal{C}_1$ is determined by that of $\mathcal{C}_0$. Hence (1) holds for every pair of vertices of $\mathcal{C}_{-1} \cup \mathcal{C}_0 \cup \mathcal{C}_1$ (see Figure 4).

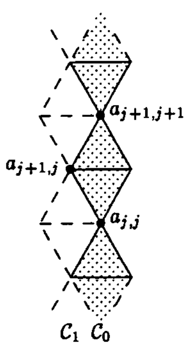

Figure 4

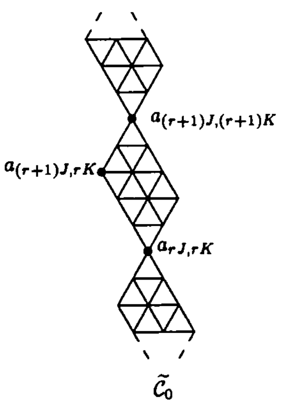

Figure 5

Iterating this procedure we argue that (J,J) is a period for $\bigcup_{i=-M}^{M} \mathcal{C}_i$, for every $M > 0$. Since, given $(j,k),(l,m) \in \mathbb{Z}^2$, there exists M such that $a_{j,k}$ and $a_{l,m}$ belong to $\bigcup_{i=-M}^{M} \mathcal{C}_i$, we conclude that (J,J) is a period for the whole apartment.

We consider now (J,K), with $J, K \neq 0$ and $J + K \neq 0$, and we prove that (ii) implies (i). We assume $J, K > 0$; the other cases are analogous. For every $r \in \mathbb{Z}$ we consider the parallelogram $R_{J,K}(rJ, rK)$. The argument used to prove that (iii) implies (i) for $J = K$, applied to the chain

$$\tilde{\mathcal{C}}_i = \left\{R_{J,K}((r+i)J, rK)\right\}_{r\in\mathbf{Z}} \tag{6}$$

instead of C_i, enables us to show that (J, K) is a period for the apartment, provided (ii) is true (see Figure 5). ∎

REMARK 3.9. The equivalence of (ii) and (iii) for $J = K$, in Lemma 3.8 is a straightforward consequence of the fact that, for every r, the labeling of $R_{J,J}(rJ, rJ)$ is determined by that of the finite chain $\left\{R_1(j,j)\right\}_{j=rJ}^{(r+1)J-1}$ herein contained (see Figure 6).

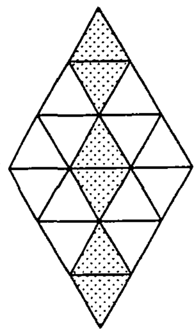

FIGURE 6

Conditions (ii) and (iii) in Lemma 3.8 may be replaced by the following slightly different conditions:

(ii)′ for some $r_0 \in \mathbb{Z}$, $\varepsilon_{(r+r_0)J,rK}^{(r+r_0+1)J,(r+1)K} = \varepsilon_{(r+r_0+1)J,(r+1)K}^{(r+r_0+2)J,(r+2)K}$, $\forall r \in \mathbb{Z}$;

(iii)′ for some $j_0 \in \mathbb{Z}$, $\varepsilon_{j+j_0,j}^{j+j_0+1,j+1} = \varepsilon_{j+j_0+J,j+J}^{j+j_0+1+J,j+1+J}$, $\forall j \in \mathbb{Z}$.

PROPOSITION 3.10. *Let $(J, K) \in \mathbb{Z}^2$, with $J, K \neq 0$ and $J + K \neq 0$. If the apartment $\mathcal{A}$ is (J, K)-periodic, then it is doubly periodic.*

PROOF: Consider for every $i \in \mathbb{Z}$ the chain $\widetilde{C}_i$ defined by (5). The (J, K)-periodicity of $\mathcal{A}$ implies that any two parallelograms of $\widetilde{C}_i$ have the same labeling. Since the set of generators is finite, there are only finitely many possible choices for the labeling of the infinite collection of chains $\{\widetilde{C}_i\}_{i\in\mathbb{Z}}$. Thus there must exist M and N, with $M \neq N$, such that $\widetilde{C_M}$ has the same labeling as $\widetilde{C_N}$. On the other hand, Lemma 3.8 and Remark 3.9 imply that the labeling of $\mathcal{A}$ is determined by the labeling of each parallelogram either $R_{J,K}((r + M)J, rK)$ or $R_{J,K}((r + N)J, rK)$. Hence, for every $(j, k) \in \mathbb{Z}^2$, corresponding edges of $\mathcal{S}(a_{j+MJ,k}) \cap \mathcal{A}$ and of $\mathcal{S}(a_{j+NJ,k}) \cap \mathcal{A}$ have the same labels, and $((N - M)J, 0)$ is a period for the apartment. Since (J, J) and $((N - M)J, 0)$ generate a cofinite subgroup of $\mathbb{Z}^2$, the apartment is doubly periodic. ∎

REMARK 3.11. (i) If $J = 0$ or $K = 0$ or $J + K = 0$, Lemma 3.8 and Proposition 3.10 are false. In fact, in these cases each parallelogram $R_{J,K}(rJ, rK)$ collapses to a segment and therefore the labeling of this region does not determine the labeling of the apartment.

(ii) Δ contains apartments which are non-periodic. In fact, the cardinality of the set of all apartments containing a fixed vertex x is not countable, while the set of (J, K)-

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periodic apartments containing x is finite.

In order to prove the existence of periodic apartments with arbitrarily large μ, we state the following lemmata.

LEMMA 3.12. *There exists an element $\gamma_0 \in \Gamma$ such that, for every $N \in \mathbb{N}$, the vertex $y_0^N = \gamma_0^N \cdot x_0$ has coordinates $(2N, 2N)$ with respect to any sector based at x_0 and containing y_0^N.*

PROOF: For every $(c, b, a) \in \mathcal{T}$, let C be a chamber associated to this triple, and let x be its vertex opposite to the edge labeled a. We define $X_{(c,b,a)}$ to be the set of triples $(g, f, d) \in \mathcal{T}$ associated to the chambers D opposite to C with respect to x, where d denotes the label of the edge opposite to x in D (see Figure 7).

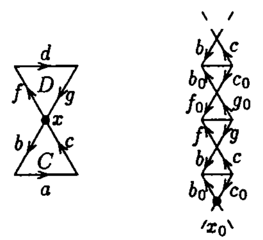

FIGURE 7

Furthermore for every generator d we define

$$X^d_{(c,b,a)} = \{(g, f, \delta) \in X_{(c,b,a)} : \delta = d\}.$$

We have $\left|X_{(c,b,a)}\right| = q^3$ and $0 \leqslant \left|X^d_{(c,b,a)}\right| \leqslant (q+1)$. Actually in a projective plane of order q, given a line L_0 and a point $x_0 \in L_0$, there are q^2 ways of choosing a point x outside L_0 and q ways of choosing a line through x but not through x_0. Fix a generator a and denote by (b_i, c_i), $i = 0, \dots, q$, all possible pairs such that $(c_i, b_i, a) \in \mathcal{T}$. For every i, consider the set $X_i = X_{(c_i,b_i,a)}$. We claim that, for some $j \neq k$, X_j and X_k contain triples (g_j, f_j, d_j) and (g_k, f_k, d_k) respectively, with $d_j = d_k$ and $(f_j, g_j) \neq (f_k, g_k)$. Otherwise, for every d, either $\left|X_i^d\right| = 1$, for all non-empty X_i^d, or only one X_i^d is non-empty. Hence $\sum_{i=0}^{q} \left|X_i^d\right| \leqslant q + 1$, and

$$\sum_{i=0}^{q} |X_i| = \sum_d \sum_{i=0}^{q} \left|X_i^d\right| \leqslant (q+1)\left(q^2+q+1\right).$$

On the other hand $\sum_{i=0}^{q} |X_i| = q^3(q+1)$, so the previous inequality is absurd, because $q^3(q+1) \geqslant (q+1)(q^2+q+1)$, for $q > 1$. Without loss of generality we may assume that the previous condition is satisfied for $j = 0, k = 1$. If we consider the pairs $(b_0, c_0), (f_0, g_0)$ and $(b_1, c_1), (f_1, g_1)$, it is obvious that, for every positive integer N, $\left(b_0 b^{-1} f f_0^{-1}\right)^N$ and

$\left(c_0^{-1}cg^{-1}g_0\right)^N$ are minimal words consisting of $2N$ generators and $2N$ inverses. Therefore, if

$$\gamma_0 = b_0 b^{-1} f f_0^{-1} = c_0^{-1} c g^{-1} g_0,$$

the vertex $y_0^N = \gamma_0^N \cdot x_0$ has coordinates $(2N, 2N)$, for every $N \geqslant 1$ (see Figure 7). ∎

Lemma 3.13. *For every positive integer $M > 2$, there exists an element $\gamma_0 \in \Gamma$, such that*

(i) for every $N \in \mathbb{N}$, the vertex $y_0^N = \gamma_0^N \cdot x_0$ has coordinates (MN, MN) with respect to any sector based at x_0 containing y_0^N;

(ii) γ_0 is not a true power in Γ.

Proof: Fix a non-periodic apartment $\mathcal{A}$ containing the fundamental vertex x_0, and write $\mathcal{A} = \{a_{j,k}\}$ with respect to a sector based at x_0. Consider the vertex $x_1 = a_{M-1,M-1} \cdot x_0$ and the convex hull $\mathcal{R}$ of the set $\{x_0, x_1\}$ (see Figure 8).

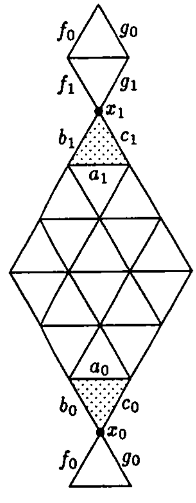

Figure 8

Let C_0 and C_1 be the chambers of $\mathcal{R}$ containing x_0 and x_1 respectively. Suppose $C_0 = (c_0, b_0, a_0)$ and $C_1 = (c_1, b_1, a_1)$, where a_0 and a_1 denote the labels of the edges opposite to x_0 and x_1 respectively. By altering the choice of the non-periodic apartment if necessary, we may assume $a_1 \neq a_0$, $(b_1, c_1) \neq (b_0, c_0)$. Following the notation of Lemma 3.12, we consider the sets $X_0 = X_{(c_0,b_0,a_0)}$ and $X_1 = X_{(c_1,b_1,a_1)}$. We claim that X_0 and X_1 contain triples (g_0, f_0, d_0) and (g_1, f_1, d_1) respectively, with $d_0 = d_1$ and $(f_0, g_0) \neq (f_1, g_1)$. Otherwise, if for some d the sets are both non-empty, then $\left|X_0^d\right| = \left|X_1^d\right| = 1$. Hence, for every d,

$$\left|X_0^d\right| + \left|X_1^d\right| \leqslant q + 1$$

and therefore

$$|X_0| + |X_1| \leqslant (q+1)\left(q^2 + q + 1\right).$$

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Since $|X_0| + |X_1| = 2q^3$, this inequality is absurd for $q \geqslant 3$.

For $q = 2$, we use a direct argument to prove the required property. For $i = 0, 1$, we consider, for every chamber based at x_i and opposite to C_i, the edge having the orientation away from x_i. We may think of these four edges as a system $\mathcal{M}_i$ of lines on the projective plane of order 2 and the three edges adjacent to each of them (but not containing x_i) as points of the corresponding line on this plane. Among these three points we neglect the one lying on a chamber adjacent to C_i. Since we suppose $(b_1, c_1) \neq (b_0, c_0)$, exactly two lines belong to both $\mathcal{M}_0$ and $\mathcal{M}_1$. We can prove by a direct computation that, in all possible configurations of $\mathcal{M}_0$ and $\mathcal{M}_1$, there exist two lines, say $L_0 \in \mathcal{M}_0$ and $L_1 \in \mathcal{M}_1$, with $L_0 \neq L_1$, which intersect each other in one point different from those discarded before. If f_i is the label corresponding to L_i, d is the label attached to their intersection point, and g_i is the unique generator such that $g_i f_i d = e$, then the triples (g_0, f_0, d) and (g_1, f_1, d) satisfy the required property. Thus $a_{M-1,M-1} f_1 f_0^{-1}$ and $a_{M-1,M-1} g_1^{-1} g_0$ are minimal words for an element of Γ, say γ_0, consisting of M generators and M inverses. Therefore $y_0^N = \gamma_0^N \cdot x_0$ has coordinates (MN, MN), for every $N \in \mathbb{Z}$. Finally, we note that, by construction, γ_0 cannot be a power of the form γ^m, for $\gamma \in \Gamma$ and $m \geqslant 2$. ∎

Proposition 3.14. *For every integer $M \geqslant 2$, there exists a (M, M)-periodic apartment, such that $\mu \geqslant M$.*

Proof: Let $M = 2$. Consider the sequence of vertices $\{y_0^N\}_{N \in \mathbb{Z}}$ constructed in Lemma 3.12. It is obvious that the convex hull of $\{y_0, y_0^{-1}\}$ also contains x_0. More generally, for every $N > 1$, the convex hull of $\{y_0^N, y_0^{-N}\}$ contains y_0^n, for all $n \in \{-N, \ldots, N\}$.

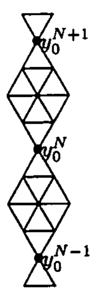

Figure 9

This implies that there exists an apartment $\mathcal{A}_0$ containing all vertices of the sequence. We write $\mathcal{A}_0 = \{a_{j,k}\}$ with respect to the sector containing the vertices y_0^N, $N > 0$. In particular for every integer N the vertex y_0^N corresponds to $a_{2N,2N}$. The apartment is uniquely determined by the chain consisting of the convex hulls of $\{a_{2N,2N}, a_{2(N+1),2(N+1)}\}$; moreover, for all $(j,k) \in \mathbb{Z}^2$,

$$\varepsilon_{j,k}^{j+1,j+1} = \varepsilon_{j+2,j+2}^{j+3,j+3}.$$

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Lemma 3.8 enables us to conclude that $(2,2)$ is a period for the apartment (see Figure 9). For $M > 2$, we consider the sequence of vertices $\{y_0^N\}_{N\in\mathbb{Z}}$ constructed in Lemma 3.13, and proceed as above to construct a (M,M)-periodic apartment. By altering the choice of the non-periodic apartment containing x_0, if necessary, we may assume that the convex hull $\mathcal{R}$ is not contained in any doubly periodic apartment having $\mu \leqslant M-1$. ∎

We conclude this section stating some interesting results about the action of the group Γ on a doubly periodic apartment and on its boundary points.

We note that if $\mathcal{A}$ is any (J,K)-periodic apartment, then $\gamma\mathcal{A}$ is, as follows from Remark 3.4; hence the property of periodicity is Γ-invariant.

Proposition 3.15. *Let $\mathcal{A}$ and $\mathcal{A}'$ be (J,K)-periodic apartments, with $J \neq 0$, $K \neq 0$ and $J+K \neq 0$. Let ω, ω' be boundary points of $\mathcal{A}$ and $\mathcal{A}'$ respectively. If $\omega' = \gamma\omega$ for some $\gamma \in \Gamma$, then $\mathcal{A}' = \gamma\mathcal{A}$.*

Proof: Assume $\omega' = \gamma\omega$. Then, for every sector $Q_x(\omega)$ associated to ω, $\gamma Q_x(\omega)$ is the sector $Q_{\gamma\cdot x}(\omega')$ representative of ω'. Since $Q_{y_1}(\omega) \supset Q_{y_2}(\omega)$ implies that $\gamma Q_{y_1}(\omega) \supset \gamma Q_{y_2}(\omega)$, we may deduce that there exists a vertex $x \in \mathcal{A}$ such that $Q_x(\omega) \subset \mathcal{A}$ and $Q_{\gamma\cdot x}(\omega) \subset \mathcal{A}'$. Set $\mathcal{A} = \{a_{j,k}\}$ and $\mathcal{A}' = \{a'_{j,k}\}$ with respect to $Q_x(\omega)$ and $Q_{\gamma\cdot x}(\omega')$. Thus, for all $j,k \geqslant 0$,

$$\gamma a_{j,k} = a'_{j,k}. \tag{7}$$

We prove that actually (3) holds for every $(j,k) \in \mathbb{Z}^2$. Assume $J,K > 0$ (the other cases are similar), and note that, for every $(j,k) \in \mathbb{Z}^2$, there exist $n, j_0, k_0 \in \mathbb{N}$, such that

$$j = j_0 - nJ, \quad k = k_0 - nK.$$

Since $j_0 + nJ, k_0 + nK$ are positive, the (J,K)-periodicity of $\mathcal{A}$ and $\mathcal{A}'$ implies

$$\begin{aligned} a'_{j,k} &= a'_{j_0,k_0}\left(a'_{j_0,k_0}\right)^{-1} a'_{j_0-nJ,k_0-nK} \\ &= a'_{j_0,k_0}\left(a'_{j_0+nJ,k_0+nK}\right)^{-1} a'_{j_0,k_0} \\ &= \gamma a_{j_0,k_0}\left(\gamma a_{j_0+nJ,k_0+nK}\right)^{-1} \gamma a_{j_0,k_0} \\ &= \gamma a_{j_0,k_0} a^{-1}_{j_0,k_0} a_{j_0-nJ,k_0-nK} \\ &= \gamma a_{j,k}. \end{aligned}$$

∎

Let $\mathcal{A}$ be a doubly periodic apartment containing x_0, and represent $\mathcal{A} = \{a_{j,k}\}$, with respect to a sector Q_{x_0}. We denote by Σ the finite group of symmetries of $\mathcal{A}$ generated by the reflections fixing $a_{0,0}$.

Proposition 3.16. *Let $\gamma \in \Gamma$ be such that $\gamma\mathcal{A} = \mathcal{A}$, and let T_γ be the operator on $\mathbb{Z}^2$ defined by $T_\gamma(j,k) = (l,m)$, if $a_{l,m} = \gamma a_{j,k}$. Then*

(i) there exist $(p,r) \in \mathbb{Z}^2$ and $\sigma \in \Sigma$ such that $T_\gamma(j,k) = \sigma(j,k) + (p,r)$; moreover $(p,r) \neq (0,0)$, if γ is non-trivial;

(ii) *if σ is the identity of Σ then $(p,r) \in \mathcal{L}$;*

(iii) *the length of γ is at least $1/3\mu$.*

PROOF: (i) Since $\gamma \cdot x_0$ is a vertex of $\mathcal{A}$, then $\gamma = a_{p,r}$ for some $(p,r) \in \mathbb{Z}^2$. If ω is the boundary point of $\mathcal{A}$ represented by the sector Q_{x_0}, and if $\omega' = \gamma\omega$, there exists $\sigma \in \Sigma$ such that $\sigma Q_{x_0}(\omega) = Q_{x_0}(\omega')$. Therefore $(l,m) = \sigma(j,k) + (p,r)$, if $a_{l,m} = \gamma a_{j,k}$. Moreover $(p,r) \neq (0,0)$ if γ is non-trivial, because γ cannot act on $\mathcal{A}$ according to a symmetry fixing $a_{0,0}$.

(ii) If σ is the identity of Σ, then γ fixes ω, and, for every $(j,k) \in \mathbb{Z}^2$,

$$a_{j,k}a_{j+p,k+r}^{-1} = a_{j,k}(\gamma a_{j,k})^{-1} = \gamma^{-1}.$$

This implies that (p,r) is a period for the apartment $\mathcal{A}$.

(iii) If T_γ is a translation, then σ is trivial and, by (ii), $|\gamma| = \left|a_{p,r}a_{0,0}^{-1}\right| \geqslant \mu$. If T_γ contains a non-trivial symmetry σ which is a reflection, then $T_{\gamma^2} = T_\gamma^2$ acts as a translation, so $2|\gamma| \geqslant |\gamma^2| \geqslant \mu$. Finally if T_γ contains a non-trivial symmetry σ which is a rotation, then $T_{\gamma^3} = T_\gamma^3$ acts as a translation, so $3|\gamma| \geqslant |\gamma^3| \geqslant \mu$. ∎

[8, Section 2] contains a parallel discussion of periodic apartments.

4. SIMPLICITY OF THE REDUCED GROUP C^*-ALGEBRA

In this section we prove (Γ, Ω) has property P_{geo}.

According to Lemma 3.13, for every $M > 2$, we fix an element $\gamma_0 \in \Gamma$ of length $2M$ such that $y_0^N = \gamma_0^N \cdot x_0$ has coordinates (MN, MN), for all $N \in \mathbb{Z}$, and consider the doubly periodic apartment $\mathcal{A}_0$ determined by $\{y_0^N\}_{N\in\mathbb{Z}}$. We denote by $Q_{x_0}^{\infty}$ and $Q_{x_0}^{-\infty}$ the sectors of $\mathcal{A}_0$ containing $\{y_0^N,\ N \geqslant 0\}$ and $\{y_0^{-N},\ N \geqslant 0\}$ respectively; moreover we denote by $\{\omega_1, \dots, \omega_6\}$ the boundary points of the apartment, assuming the following choice:

ω_1 and ω_6 are the points represented by the sectors $Q_{x_0}^{\infty}$ and $Q_{x_0}^{-\infty}$ respectively (see Proposition 3.14);

ω_2 and ω_3 are the points represented by the sectors based at x_0 adjacent to $Q_{x_0}^{\infty}$;

ω_4 and ω_5 are the points represented by the sectors based at x_0 adjacent to $Q_{x_0}^{-\infty}$.

PROPOSITION 4.1. *The following facts are true:*

(i) *the element γ_0 fixes ω_s, for every $s = 1, \dots, 6$;*

(ii) *for every $\omega \in \Omega$ there exists $s \in \{1, \dots, 6\}$ such that $\lim_{n\to\infty} \gamma_0^n\omega = \omega_s$;*

(iii) *if $\gamma_0\omega = \omega$, then $\omega = \omega_s$, for some $s \in \{1, \dots, 6\}$.*

PROOF: (i) The element γ_0 acts on $\mathcal{A}_0$ by translation. So for every s the sector $\gamma_0 Q_{x_0}(\omega_s)$ lies on $\mathcal{A}_0$ and is parallel to $Q_{x_0}(\omega_s)$; this means that γ_0 fixes all boundary points of the apartment.

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(ii) The property is obvious if $\omega = \omega_s$. For $\omega \neq \omega_s$, consider the boundary point ω_6 of $\mathcal{A}_0$ and an apartment $\mathcal{A}$ containing ω and ω_6. We sketch in Figure 10 one of the six different situations that may occur; all the others are similar.

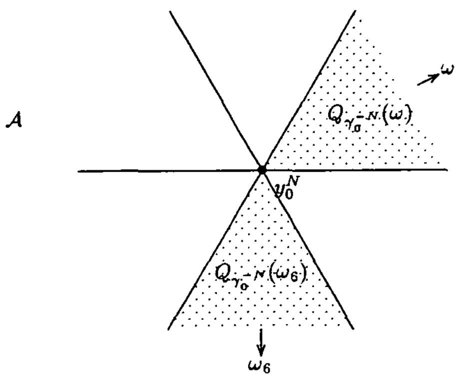

FIGURE 10

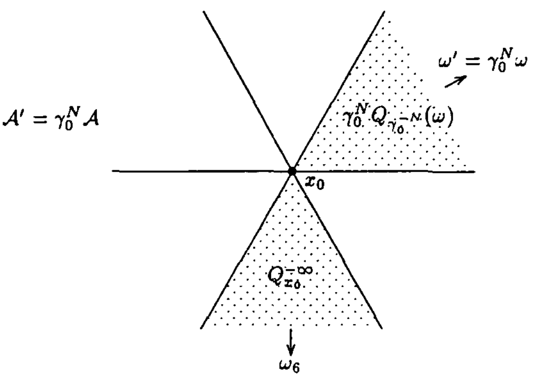

FIGURE 11

We may choose a positive integer N so big that $y_0^{-N} \in \mathcal{A}$; thus $Q_{y_0^{-N}}(\omega_6)$ and $Q_{y_0^{-N}}(\omega)$ belong to $\mathcal{A}$. On the translated apartment $\mathcal{A}' = \gamma_0^N \mathcal{A}$, consider the sector $\gamma_0^N Q_{y_0^{-N}}(\omega_6) = Q_{x_0}^{\infty}$ and the sector $\gamma_0^N Q_{y_0^{-N}}(\omega)$, based at x_0 and corresponding to the boundary point $\omega' = \gamma_0^N \omega$. (see Figure 11).

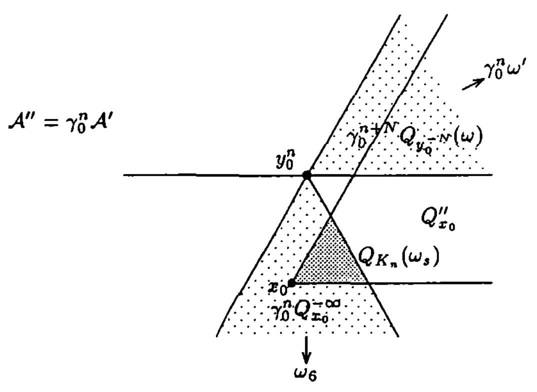

FIGURE 12

For every $n \geqslant 1$, consider, on the translated apartment $\mathcal{A}'' = \gamma_0^n \mathcal{A}'$, the sectors (based at y_0^n) $\gamma_0^n \gamma_0^N Q_{y_0^{-N}}(\omega)$ and $\gamma_0^n \gamma_0^N Q_{y_0^{-N}}(\omega_6) = \gamma_0^n Q_{x_0}^{-\infty}$, corresponding to $\gamma_0^n \omega'$ and ω_6 respectively. Since $\gamma_0^n Q_{x_0}^{-\infty}$ contains the fundamental vertex x_0 as an interior element, then $x_0 \in \mathcal{A}''$ and there exists a sector $Q''_{x_0} \subset \mathcal{A}''$ parallel to $\gamma_0^n \gamma_0^N Q_{y_0^{-N}}(\omega)$. This sector intersects $\gamma_0^n Q_{x_0}^{-\infty}$ in a set $Q_{K_n}(\gamma_0^n \omega')$, for some $K_n \geqslant 1$. (See Figure 12.)

Since $Q_{K_n}(\gamma_0^n\omega') \subset \mathcal{A}_0$, there exists s such that $Q_{K_n}(\gamma_0^n\omega') = Q_{K_n}(\omega_s)$. We claim that $\lim_{n\to\infty}\gamma_0^n\omega = \omega_s$. In fact, for every $m \geqslant n$, $K_m \geqslant K_n$, and for every $K \geqslant 1$ there exists $\nu \geqslant 1$ such that $K_\nu \geqslant K$, if $n \geqslant \nu$. It follows that

$$\lim_{n\to\infty}\gamma_0^n\omega = \lim_{n\to\infty}\gamma_0^{n-N}\omega' = \omega_s.$$

(iii) Assume $\gamma_0\omega = \omega$; then $\gamma_0^n\omega = \omega$, $\forall n \geqslant 1$, and (ii) implies $\omega = \omega_s$, for some $s \in \{1,\ldots,6\}$. ∎

PROPOSITION 4.2. *Let $V_{s,K} = E_K(\omega_s)$, for $K \geqslant 1$. If $\omega \in V_{s,K}$ and $\gamma_0^n\omega \notin V_{s,K}$, for some positive integer n, then also $\gamma_0^{n+1}\omega \notin V_{s,K}$.*

PROOF - CASE 1: $s = 1$: We prove that if $\omega \in V_{1,K}$, then $\gamma_0^n\omega \in V_{1,K}$, for all $n \geqslant 1$. Let $\mathcal{R}_K$ be the convex hull of the set $\{y_0^{-1}\} \cup Q_K(\omega_1)$, and let $\mathcal{A}$ be an apartment containing ω and y_0^{-1}. Thus $\mathcal{R}_K$ lies on both $\mathcal{A}$ and $\mathcal{A}_0$ (see Figure 13).

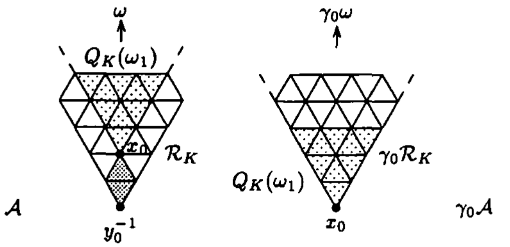

FIGURE 13

The translated apartment $\gamma_0\mathcal{A}$ intersects $\mathcal{A}_0$ in the region $\gamma_0\mathcal{R}_K$ containing $Q_K(\omega_1)$. This implies $\gamma_0\omega \in V_{1,K}$. By induction we can prove that $\gamma_0^n\omega \in V_{1,K}$, for all $n \geqslant 1$.

CASE 2: $s = 2, 3$. Denote by $\mathcal{R}_K$ and $\mathcal{R}'_K$ the convex hull of $\{y_0^{-n}\} \cup Q_K(\omega_s)$ and of $\{y_0\} \cup Q_K(\omega_s)$ respectively.

For every $\omega \in V_{s,K}$, the sector $Q_{x_0}(\omega)$ contains a wall, say S_K, of $Q_K(\omega_1)$. Therefore the region $\gamma_0^n\mathcal{R}_K$ intersects $Q_K(\omega_s)$ at least in S_K. So $Q_{x_0}(\gamma_0^n\omega)$ contains S_K (see Figure 14 for $s = 2$ and $n = 1$).

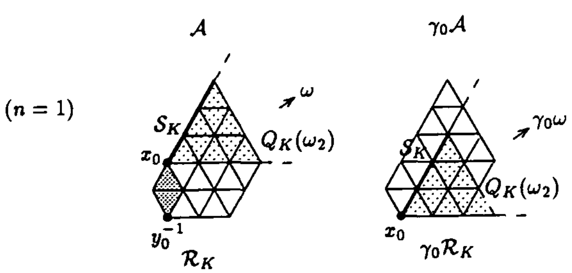

FIGURE 14

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We check that if $\omega \in V_{s,K}$ and $\omega' = \gamma_0^{n+1}\omega \in V_{s,K}$, then also $\gamma_0^{-1}\omega' \in V_{s,K}$. Let $\mathcal{A}$ be an apartment containing y_0 and ω'; then $\mathcal{R}'_K \subset \mathcal{A}$. Thus the translated region $\gamma_0^{-1}\mathcal{R}'_K$ lies on $\gamma_0^{-1}\mathcal{A}$ and on $\gamma_0^{-1}\mathcal{A}_0 = \mathcal{A}_0$. On the other hand the apartment $\gamma_0^{-1}\mathcal{A}$ contains the set S_K, since it contains the sector $Q_{x_0}\left(\gamma_0^{-1}\omega'\right) = Q_{x_0}(\gamma_0^n\omega)$. We conclude that the apartments $\mathcal{A}_0$ and $\gamma_0^{-1}\mathcal{A}$ share the convex hull of the set $S_K \cup \gamma_0^{-1}\mathcal{R}'_K$, which contains $Q_K(\omega_s)$. This proves that $\gamma_0^{-1}\omega' \in V_{s,K}$ (see Figure 15 for $s = 2$ and $n = 1$).

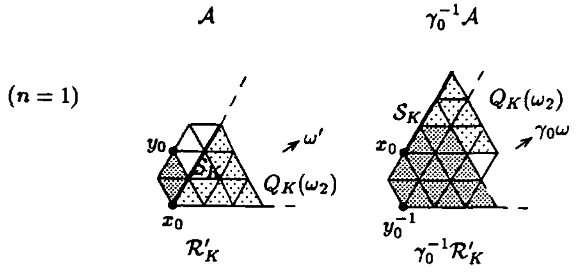

FIGURE 15

CASE 3: $s = 4, 5$. We prove that, if $\omega \in V_{s,K}$ and $\omega' = \gamma_0^{n+1}\omega \in V_{s,K}$, then also $\gamma_0^{-1}\omega' = \gamma_0^n\omega \in V_{s,K}$. Let $\mathcal{R}_K$ be the convex hull of the set $\{y_0^{-(n+1)}\} \cup Q_K(\omega_s)$ and let $\mathcal{A}$ be an apartment containing $\mathcal{R}_K$ and ω. Then the apartment $\mathcal{A}' = \gamma_0^{n+1}\mathcal{A}$ contains the region $\gamma_0^{n+1}\mathcal{R}_K$. Moreover the hypothesis $\gamma_0^{n+1}\omega \in V_{s,K}$ implies that $\mathcal{A}'$ contains also $Q_K(\omega_s)$. Then it contains the convex hull $\widetilde{\mathcal{R}_K}$ of the set $\gamma_0^{n+1}\mathcal{R}_K \cup Q_K(\omega_s)$. The region $\gamma_0^{-1}\widetilde{\mathcal{R}_K}$ lies in the translated apartment $\mathcal{A}'' = \gamma_0^{-1}\mathcal{A}' = \gamma_0^n\mathcal{A}$ and contains $Q_K(\omega_s)$. This proves that

$$\gamma_0^{-1}\omega' = \gamma_0^n\omega \in V_{s,K}$$

(see Figure 16 for $s = 4$ and $n = 1$).

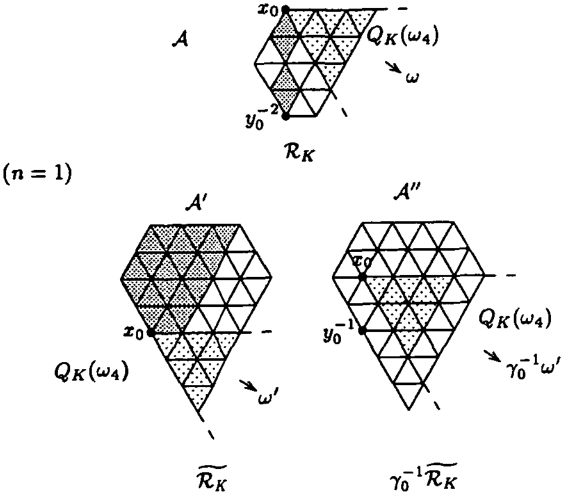

FIGURE 16

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CASE 4: $s = 6$. If we change y_0 with y_0^{-1} we may use the same argument as in case 1, to prove that $\omega \in V_{6,K}$ implies $\gamma_0^{-1}\omega \in V_{6,K}$. So if $\omega \in V_{6,K}$ and $\omega' = \gamma_0^n\omega \notin V_{6,K}$, then also $\gamma_0\omega' \notin V_{6,K}$. ∎

PROPOSITION 4.3. *Let $F \subset \Gamma \setminus \{e\}$ be a finite set and denote*

$$m(F) = \max\{|\gamma|, \gamma \in F\}.$$

Suppose $M > 3m(F)$. Then for each $s \in \{1, \dots, 6\}$ there exists an open neighbourhood $V_{s,K}$ of ω_s, such that

$$\gamma V_{s,K} \cap V_{s',K} = \emptyset, \qquad \forall \gamma \in F, \quad \forall s, s'.$$

PROOF: It suffices to prove that $\gamma\omega_s \neq \omega_{s'}$, for every $\gamma \in F$ and each pair s, s'. In fact, if $\gamma\omega_s = \omega_{s'}$ for some element γ, Proposition 3.16 implies $|\gamma| \geqslant \mu/3 \geqslant M/3$. Because of the choice of M, γ can not be an element of F. ∎

THEOREM 4.4. *Let Γ be a discrete group acting simply transitively on a triangle building. Then the reduced C^*-algebra $C_r^*(\Gamma)$ is simple.*

PROOF: Propositions 4.2, 4.3 and 4.4 prove that (Γ, Ω) has property P_{geo}. Then Lemmata 2.1, 2.3 and 2.4 of [**1**] imply that the reduced C^*-algebra of Γ is simple. ∎

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