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ON ROSENBLOOM'S FIXED-POINT THEOREM AND RELATED RESULTS

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Abstract

In this paper, we improve the Rosenbloom's fixed-point theorem and prove a related normality criterion. We also consider the corresponding unicity theorem for transcendental entire functions.

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1. Introduction and the main results

Let $f(z)$ be a nonconstant meromorphic function in the whole complex plane. We use the following standard notations of value distribution theory,

$$T(r, f),\, m(r, f),\, N(r, f),\, \overline{N}(r, f), \dots$$

(see Hayman [5]). We denote by $S(r, f)$ any function satisfying

$$S(r, f) = o\{T(r, f)\},$$

as $r \to +\infty$, possibly outside a set of finite measure.

A meromorphic function $\alpha(z)$ is called a small function related to $f(z)$ if $T(r, \alpha) = S(r, f)$.

Let S be a set of complex numbers. Write

$$E(S, f) = \bigcup_{a \in S} \{z \mid f(z) - a = 0\},$$

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where a solution to $f(z) - a = 0$ with multiplicity m is counted m times in the above set.

In 1952, Rosenbloom [6] proved the following theorem.

THEOREM 1. *Let $P(z)$ be a polynomial with* $\deg P \geq 2$, *$f(z)$ a transcendental entire function. Then*

$$\varlimsup_{r\to\infty} \frac{N(r, 1/(P(f) - z))}{T(r, f)} \geq 1. \tag{1.1}$$

In 1995, Zheng and Yang [12] proved

THEOREM 2. *Let $P(z)$ be a polynomial with* $\deg P \geq 2$, *$f(z)$ a transcendental entire function, and $\alpha(z)$ a nonconstant meromorphic function satisfying $T(r, \alpha) = S(r, f)$. Then*

$$T(r, f) \leq k\overline{N}\left(r, \frac{1}{P(f) - \alpha}\right) + S(r, f). \tag{1.2}$$

Here $k = 2/(\deg P - 1)$ if $P'(z)$ has only one zero; otherwise $k = 2$.

Naturally, we ask what is the best possible k in (1.2). In this paper, we have obtained such a k by proving the following result.

THEOREM 3. *Let $P(z)$ be a polynomial with* $\deg P \geq 2$, *$f(z)$ a transcendental entire function, and $\alpha(z)$ a meromorphic function satisfying $T(r, \alpha) = S(r, f)$. If $\alpha(z)$ is a constant, we also require that there exists a constant $A \neq \alpha$ such that $P(z) - A$ has a zero of multiplicity at least 2. Then*

$$T(r, f) \leq k\overline{N}\left(r, \frac{1}{P(f) - \alpha}\right) + S(r, f). \tag{1.3}$$

Here $k = 1/(\deg P - 1)$ if $P'(z)$ has only one zero; otherwise $k = 1$.

Obviously, Theorem 3 improves Theorem 2 and implies the following corollary.

COROLLARY 1. *Let $P(z)$ be a polynomial with* $\deg P \geq 2$, *$f(z)$ a transcendental entire function, and $\alpha(z)$ a nonconstant meromorphic function satisfying $T(r, \alpha) = S(r, f)$. Then*

$$\varlimsup_{r\to\infty} \frac{\overline{N}(r, 1/(P(f) - \alpha))}{T(r, f)} \geq 1. \tag{1.4}$$

The following examples show that the condition in Theorem 3 when $\alpha(z)$ is a constant is necessary and the number k in Theorem 3 is sharp.

EXAMPLE 1. Let $f(z) = e^z - 1$, $P(z) = (z+1)^n + 1$, where $n \geq 2$ is a positive integer, and $\alpha = 1$. Thus $P(z) - \alpha = P(z) - 1 = (z+1)^n$, $P(f) - \alpha = e^{nz}$. Hence (1.3) does not hold. Obviously, α is the only constant A such that $P(z) - A$ has a zero with multiplicity ≥ 2.

EXAMPLE 2. Let $f(z) = e^z + z$, $P(z) = z$, $\alpha(z) = z$. Thus $P(f) - \alpha = e^z$ and (1.3) does not hold.

EXAMPLE 3. Let $f(z) = e^z$, $P(z) = (z+1)^n$, where $n \geq 2$ is a positive integer, and $\alpha = 1$. Thus $P(f) - 1 = (e^z + 1)^n - 1 = e^z \prod_{i=1}^{n-1}(e^z + 1 - e_i)$, where $e_i \neq 1$ is a distinct zero of $z^n - 1$ $(i = 1, 2, \dots, n-1)$. Thus we have

$$T(r, f) = \frac{1}{n-1}\overline{N}\left(r, \frac{1}{P(f) - 1}\right) + S(r, f).$$

Hence $k = 1/(\deg P - 1) = 1/(n-1)$ is sharp in Theorem 3.

EXAMPLE 4. Let $f(z) = e^z + 1$, $P(z) = z(z-1)^2$ and $\alpha = 0$. Thus $P(f) = (e^z + 1)e^{2z}$ and

$$T(r, f) = \overline{N}\left(r, \frac{1}{P(f)}\right) + S(r, f).$$

Thus $k = 1$ is sharp in Theorem 3.

We know that for the second Nevanlinna fundamental theorem there exists a corresponding Montel's normality criterion [5] and for Hayman's inequality there exists Gu's normality criterion (see [3]). Naturally, we ask whether there exists a corresponding normality criterion for inequality (1.3). The following theorem gives a positive answer to this question.

THEOREM 4. *Let $\mathscr{F}$ be a family of analytic functions in a domain D, $P(z)$ a polynomial with* $\deg P \geq 2$. *Suppose that $\alpha(z)$ is either a nonconstant analytic function or a constant function such that $P(z) - \alpha$ has at least two distinct roots. If $P(f(z)) \neq \alpha(z)$ for each $f(z) \in \mathscr{F}$, then $\mathscr{F}$ is normal in D.*

The following two examples illustrate that the conditions in Theorem 4 are necessary.

EXAMPLE 5. Take $P(z) = z$, $f_n(z) = z + e^{nz}$, $D = \{|z| < 1\}$. It is easy to see that $P(f_n(z)) \neq z$ in D and the analytic family $\{f_n(z)\}$ is not normal in D.

EXAMPLE 6. Let $P(z) = z^k + 1$, where $k \geq 2$ is a positive integer, $f_n(z) = e^{nz}$, $\alpha(z) = 1$, $D = \{|z| < 1\}$. It is easy to see that $P(f_n(z)) \neq 1$ and that $f_n(z)$ are analytic in D. But $\{f_n(z)\}$ is not normal in D.

Theorem 4 implies the following corollary.

COROLLARY 2. *Let $\mathcal{F}$ be a family of analytic functions in a domain D, $P(z)$ a polynomial with* $\deg P \geq 2$. *If $P(f(z)) \neq z$ for each $f(z) \in \mathcal{F}$, then $\mathcal{F}$ is normal in D.*

By the second fundamental theorem, Nevanlinna obtained the five-valueunicity theorem. Naturally, we ask whether there exists a corresponding unicity theorem for inequality (1.3). In this paper, we prove the following result.

THEOREM 5. *Let $f(z)$ and $g(z)$ be two transcendental entire functions, $\alpha(z) \not\equiv 0$ a common small function related to $f(z)$ and $g(z)$, and $P(z) = z^6(z-1)$. If $P(f(z)) - \alpha(z)$ and $P(g(z)) - \alpha(z)$ have the same zeros (counting multiplicity), then $f(z) \equiv g(z)$.*

REMARK 1. Let $f(z) = e^z$, $g(z) = e^{-z}$, $P(z) = z^6(z-1)$ and $\alpha(z) \equiv 0$. Obviously, $P(f(z)) - \alpha(z)$ and $P(g(z)) - \alpha(z)$ have the same zeros (counting multiplicity). But $f(z) \not\equiv g(z)$. Hence, $\alpha(z) \not\equiv 0$ is necessary in Theorem 5.

From Theorem 5, we can easily obtain the following corollaries.

COROLLARY 3. *Let $f(z)$ and $g(z)$ be two transcendental entire functions, and $P(z) = z^6(z-1)$. If $P(f(z)) - z$ and $P(g(z)) - z$ have the same zeros (counting multiplicity), then $f(z) \equiv g(z)$.*

Note that $P(z) = z^6(z-1)$, and that $P(f(z)) - 1$ and $P(g(z)) - 1$ have the same zeros (counting multiplicity) if and only if $E(S, f) = E(S, g)$, where $S = \{z \mid z^6(z-1) = 1\}$. Thus Theorem 5 implies.

COROLLARY 4. *Let $S = \{z \mid z^6(z-1) = 1\}$, $f(z)$ and $g(z)$ be two transcendental entire functions. If $E(S, f) = E(S, g)$, then $f(z) \equiv g(z)$.*

Note that Corollary 4 gives a positive answer to a question of Gross (see Gross [2], Yi [9]).

2. Proof of Theorem 3

In order to prove Theorem 3 we need the following lemmas.

LEMMA 2.1 (see [1, 5]). *Let $f(z)$ be a meromorphic function. If there exist two functions $a_i(z)$ such that $T(r, a_i) = S(r, f)$, $i = 1, 2$, then*

$$T(r, f) \le \overline{N}(r, f) + \overline{N}\left(r, \frac{1}{f - a_1}\right) + \overline{N}\left(r, \frac{1}{f - a_2}\right) + S(r, f).$$

LEMMA 2.2 ([10]). *Let*

$$P(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0,$$

where $a_n (\ne 0), a_{n-1}, \ldots, a_0$, are constants.

If $f(z)$ is a meromorphic function, then

$$T(r, P(f)) = nT(r, f) + S(r, f).$$

Next we prove Theorem 3.

PROOF. We consider two cases.

Case 1. $\alpha(z)$ is a constant function. Then by the assumpion in Theorem 3 we can choose a constant A such that $P(z) - A$ has a zero (say a) with multiplicity $m \ge 2$. Let $a_1, a_2, \ldots, a_{n-m}$ be the other zeros of $P(z) - A$, where $n = \deg P$. Then from Lemma 2.1, we have

$$\begin{aligned} T(r, P(f)) &\le \overline{N}\left(r, \frac{1}{P(f) - \alpha}\right) + \overline{N}\left(r, \frac{1}{P(f) - A}\right) + S(r, P(f)) \\ &\le \overline{N}\left(r, \frac{1}{P(f) - \alpha}\right) + \overline{N}\left(r, \frac{1}{f - a}\right) + \sum_{i=1}^{n-m} \overline{N}\left(r, \frac{1}{f - a_i}\right) + S(r, f) \\ &\le \overline{N}\left(r, \frac{1}{P(f) - \alpha}\right) + \overline{N}\left(r, \frac{1}{f - a}\right) + (n - m)T(r, f) + S(r, f). \qquad (2.1) \end{aligned}$$

On the other hand, by Lemma 2.2 we have

$$T(r, P(f)) = nT(r, f) + S(r, f). \qquad (2.2)$$

If $P'(z)$ has only one zero, then $m = n$. Thus we deduce from (2.1) and (2.2) that

$$T(r, f) \le \frac{1}{\deg P - 1} \overline{N}\left(r, \frac{1}{P(f) - \alpha}\right) + S(r, f).$$

Otherwise, $n - m \le n - 2$. Hence we deduce from (2.1) and (2.2) that

$$T(r, f) \le \overline{N}\left(r, \frac{1}{P(f) - \alpha}\right) + S(r, f).$$

Case 2. $\alpha(z)$ is a nonconstant meromorphic function satisfying $T(r, f) = S(r, f)$. In this case we can also choose A such that $P(z) - A$ has a zero (say a) with multiplicity $m \ge 2$. Using the same argument as in Case 1, we obtain (1.3). The proof of Theorem 3 is complete. □

3. Proof of Theorem 4

For the proof of Theorem 4, we need the Zalcman's Lemma [11].

LEMMA 3.1. *If a family $\mathscr{F}$ of functions analytic on the unit disc D is not normal at $z = 0$, then there exist a number $0 < r < 1$, a sequence of complex numbers $z_n \to 0$, a sequence of functions $f_n(z) \in \mathscr{F}$, a sequence of positive numbers $\rho_n \to 0$ such that*

$$g_n(\xi) = f_n(z_n + \rho_n \xi) \to g(\xi)$$

uniformly on any compact subset of C, where $g(\xi)$ is a non-constant entire function.

Now we prove Theorem 4.

PROOF. First, we prove the case when $\alpha(z)$ is a nonconstant analytic function in the domain D. We consider two cases.

Case I. $P(z) - \alpha(0)$ has at least two distinct zeros a and b.
Suppose that $\mathscr{F}$ is not normal in D. Without loss of generality, we assume that $\mathscr{F}$ is not normal at $z = 0$. By Lemma 3.1, there exist $0 < r < 1$, $z_n \to 0$, $f_n \in \mathscr{F}$, $\rho_n \to 0^+$ such that

$$g_n(\xi) = f_n(z_n + \rho_n \xi) \to g(\xi)$$

uniformly on compact subsets of C, where $g(\xi)$ is a non-constant entire function.
Hence

$$P(f_n(z_n + \rho_n \xi)) - \alpha(z_n + \rho_n \xi) \to P(g(\xi)) - \alpha(0) \tag{3.1}$$

uniformly on any compact subset of C. Since $P(f_n(z_n + \rho_n \xi)) - \alpha(z_n + \rho_n \xi) \neq 0$, using Hurwitz's theorem for (3.1), we get $P(g(\xi)) \neq \alpha(0)$. Thus $g(\xi) \neq a, b$. Noting that $g(\xi)$ is an entire function, we deduce that $g(\xi)$ is a constant (Picard's theorem), which is a contradiction.

Case II. $P(z) - \alpha(0)$ has only one zero.
We can write $P(z) - \alpha(0) = (az - b)^n$ ($a \neq 0, n \geq 2$). Obviously, there exists a neighbourhood (denoted by U) of point $z = 0$ such that $\alpha(z) \neq \alpha(0)$ for all $z \in U \backslash \{0\}$.

We claim that $\mathscr{F}$ is normal at $z_0 (\neq 0) \in U$. In fact, if $\mathscr{F}$ is not normal at z_0, then by using the similar argument as in Case I, we obtain $P(g(\xi)) \neq \alpha(z_0)$, that is, $(ag(\xi) - b)^n \neq \alpha(z_0) - \alpha(0)$. Therefore, $g(\xi)$ is not equal n distinct values $(1/a)(\{\alpha(z_0) - \alpha(0)\}^{1/n} + b)$. This means that $g(\xi)$ is a constant, which is a contradiction.

Next we prove $\mathscr{F}$ is normal at $z_0 = 0$. For any $f_n(z) \in \mathscr{F}$ and $C_r = \{z : |z| = r\} \subset U$, we know $\{f_n(z)\}$ is normal in C_r by the former conclusion. Thus there exists a subsequence f_{n_k} such that

$$f_{n_k}(z) \to g(z),$$

uniformly on C_r.

If $g(z) \not\equiv \infty$, then $g(z)$ is analytic on C_r. Hence there exist an integer N and a positive number M such that

$$|f_{n_k}(z)| \le M,$$

for all $k \ge N$, $z \in C_r$. By the maximum modulus theorem, we have

$$|f_{n_k}(z)| \le M,$$

for all $k \ge N$, $|z| \le r$. Hence $\{f_{n_k}(z)\}$ is normal in $\{z : |z| \le r\}$ by Montel's normality criterion (see [5]). Thus there exists a subsequence of $f_{n_k}(z)$ (which we continue to denote by $f_{n_k}(z)$) such that

$$f_{n_k}(z) \to g(z), \tag{3.2}$$

uniformly on $\{z : |z| \le r\}$.

If $g(z) \equiv \infty$, then there exist an integer N and a positive $M > M(r, \alpha(z))$ such that

$$|P(f_{n_k}(z))| \ge M,$$

for all $k \ge N$, $z \in C_r$, where $M(r, \alpha) = \max_{|z| \le r}\{|\alpha(z)|\}$. Thus

$$|P(f_{n_k}(z)) - \alpha(z)| \ge M - M(r, \alpha) > 0,$$

for all $k \ge N$, $z \in C_r$. Note that $P(f_{n_k}(z)) - \alpha(z)$ has no zeros in $\{z : |z| \le r\}$, and thus we have

$$|P(f_{n_k}(z)) - \alpha(z)| \ge M - M(r, \alpha),$$

for all $k \ge N$, $|z| \le r$ by the minimum modulus theorem. This means that

$$f_{n_k}(z) \to \infty \tag{3.3}$$

uniformly on $\{z : |z| \le r\}$. Thus we deduce from (3.2) and (3.3) that $\mathscr{F}$ is normal at $z = 0$. Therefore, $\mathscr{F}$ is normal in D in the case when $\alpha(z)$ is a nonconstant analytic function in D.

If $\alpha(z)$ is a constant, then by using the same argument as in Case I, we can prove $\mathscr{F}$ is normal in D. Thus the proof of Theorem 4 is complete. □

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4. Proof of Theorem 5

In order to prove our result, we need the following lemma.

LEMMA 4.1. *Let $f(z)$ be a meromorphic function. Then*

$$T\left(r, \frac{1}{f-a}\right) = T(r,f) + O(1); \qquad T\left(r, \frac{f^{(k)}}{f^{(l)}}\right) = S(r,f),$$

where k, l are two integer satisfying $k > l \geq 0$; and

$$(q-1)T(r,f) \leq N(r,f) + \sum_{i=1}^{q} N\left(r, \frac{1}{f-a_i}\right) - N_1(r,f) + S(r,f),$$

where a_i $(i = 1, \ldots, q)$ are distinct constants and

$$N_1(r,f) = N\left(r, \frac{1}{f'}\right) + 2N(r,f) - N(r,f').$$

PROOF (of Theorem 5). In the proof we use the following notation.

$N_{(2}(r, 1/(f-a))$ is the counting function which includes only multiple zeros of $f(z) - a$, $\overline{N}_{(2}(r, 1/(f-a))$ the corresponding reduced counting function, and $N_2(r, 1/(f-a)) = \overline{N}(r, 1/(f-a)) + \overline{N}_{(2}(r, 1/(f-a))$, $N_{1)}(r, 1/(f-a)) = N(r, 1/(f-a)) - N_{(2}(r, 1/(f-a))$.

Set

$$F(z) = \frac{P(f(z))}{\alpha(z)}, \quad \text{and} \quad G(z) = \frac{P(g(z))}{\alpha(z)}.$$

It follows from assumptions of Theorem 5 that

$$N\left(r, \frac{1}{F-1}\right) = N\left(r, \frac{1}{G-1}\right) + S(r,f), \tag{4.1}$$

$$N_2(r,F) = N_2(r,G) = S(r,f). \tag{4.2}$$

If z_0 is a zero of $F(z)$ and not a pole of $\alpha(z)$, then z_0 is either a zero of $f(z)$ or $f(z) - 1$. Thus

$$\overline{N}\left(r, \frac{1}{F}\right) \leq \overline{N}\left(r, \frac{1}{f}\right) + \overline{N}\left(r, \frac{1}{f-1}\right) + S(r,f). \tag{4.3}$$

If z_1 is a multiple zero of $F(z)$ and not a pole of $\alpha(z)$, then z_1 is a zero of $f(z)$ or a multiple zero of $f(z) - 1$. Hence

$$\overline{N}_{(2}\left(r, \frac{1}{F}\right) \leq \overline{N}\left(r, \frac{1}{f}\right) + \overline{N}_{(2}\left(r, \frac{1}{f-1}\right) + S(r,f). \tag{4.4}$$

Thus we deduce from (4.2), (4.3), (4.4), Lemma 4.1, and Lemma 2.2 that

$$
\begin{aligned}
N_2\left(r, \frac{1}{F}\right) &= \overline{N}\left(r, \frac{1}{F}\right) + \overline{N}_{(2}\left(r, \frac{1}{F}\right) \\
&\leq 2\overline{N}\left(r, \frac{1}{f}\right) + \overline{N}\left(r, \frac{1}{f-1}\right) + \overline{N}_{(2}\left(r, \frac{1}{f-1}\right) + S(r, F) \\
&\leq 2\overline{N}\left(r, \frac{1}{f}\right) + N\left(r, \frac{1}{f-1}\right) + S(r, F) \\
&\leq 3T(r, f) + S(r, F) \\
&\leq \left(\frac{3}{7} + o(1)\right) T(r, F). \qquad (4.5)
\end{aligned}
$$

In the same manner we obtain that

$$
N_2\left(r, \frac{1}{G}\right) \leq \left(\frac{3}{7} + o(1)\right) T(r, G). \qquad (4.6)
$$

Therefore, we deduce from (4.5) and (4.6) that

$$
N_2\left(r, \frac{1}{F}\right) + N_2\left(r, \frac{1}{G}\right) \leq \left(\frac{6}{7} + o(1)\right) T(r), \qquad (4.7)
$$

where $T(r) = \max\{T(r, F), T(r, G)\}$.

We claim that either $F(z) \equiv G(z)$ or $F(z)G(z) \equiv 1$. Set

$$
\Phi(z) = \frac{F''(z)}{F'(z)} - 2\frac{F'(z)}{F(z)-1} - \frac{G''(z)}{G'(z)} + 2\frac{G'(z)}{G(z)-1} \qquad (4.8)
$$

and suppose that $\Phi(z) \not\equiv 0$. Obviously, $m(r, \Phi) = S(r, F) + S(r, G)$.

If z_2 is a common simple 1-point of $F(z)$ and $G(z)$, substituting their Taylor series at z_2 into (4.8), we see that z_2 is a zero of $\Phi(z)$. Thus by Lemma 4.1 we have

$$
\begin{aligned}
N_{1)}\left(r, \frac{1}{F-1}\right) &= N_{1)}\left(r, \frac{1}{G-1}\right) \leq \overline{N}\left(r, \frac{1}{\Phi}\right) \\
&\leq T(r, \Phi) + O(1) \leq N(r, \Phi) + S(r, F) + S(r, G). \qquad (4.9)
\end{aligned}
$$

It is easy to show that $\Phi(z)$ is analytic at a simple pole or a multiple 1-point of $F(z)$ or $G(z)$. Hence if z_3 is a pole of $\Phi(z)$ and not a multiple pole of $F(z)$ or $G(z)$, then z_3 is a zero of $F'(z)$ or $G'(z)$. Note that z_3 is not a simple 1-point of $F(z)$ or $G(z)$, so if z_3 is also not a multiple zero of $F(z)$ or $G(z)$ then $F'(z_3) = 0$, $F(z_3) \neq 0, 1$ or $G'(z_3) = 0$, $G(z_3) \neq 0, 1$. Thus we have

$$
\begin{aligned}
\overline{N}(r, \Phi) \leq \overline{N}_{(2}(r, F) + \overline{N}_{(2}(r, G) + \overline{N}_{(2}\left(r, \frac{1}{F}\right) \\
+ \overline{N}_{(2}\left(r, \frac{1}{G}\right) + \overline{N}_0\left(r, \frac{1}{F'}\right) + \overline{N}_0\left(r, \frac{1}{G'}\right),
\end{aligned}
$$

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where $N_0(r, 1/F')$ is the counting function which only counts those zeros of F' but not those of $F(F-1)$.

Substituting the above inequality into (4.9) and noting (4.2), we have

$$\overline{N}_{1)}\left(r, \frac{1}{F-1}\right) \leq \overline{N}_{(2}\left(r, \frac{1}{F}\right) + \overline{N}_{(2}\left(r, \frac{1}{G}\right) + \overline{N}_0\left(r, \frac{1}{F'}\right) + \overline{N}_0\left(r, \frac{1}{G'}\right) + S(r, F) + S(r, G). \tag{4.10}$$

By the second fundamental theorem and (4.2), we have

$$T(r, F) \leq \overline{N}\left(r, \frac{1}{F}\right) + \overline{N}\left(r, \frac{1}{F-1}\right) - N_0\left(r, \frac{1}{F'}\right) + S(r, F), \tag{4.11}$$

$$T(r, G) \leq \overline{N}\left(r, \frac{1}{G}\right) + \overline{N}\left(r, \frac{1}{G-1}\right) - N_0\left(r, \frac{1}{G'}\right) + S(r, G). \tag{4.12}$$

Therefore, we deduce from (4.10), (4.11) and (4.12) that

$$\begin{aligned} T(r, F) + T(r, G) \leq{}& \overline{N}\left(r, \frac{1}{F}\right) + \overline{N}_{1)}\left(r, \frac{1}{F-1}\right) - N_0\left(r, \frac{1}{F'}\right) + S(r, F) \\ &+ \overline{N}\left(r, \frac{1}{G}\right) + \overline{N}\left(r, \frac{1}{G-1}\right) + \overline{N}_{(2}\left(r, \frac{1}{G-1}\right) \\ &- N_0\left(r, \frac{1}{G'}\right) + S(r, G) \\ \leq{}& N_2\left(r, \frac{1}{F}\right) + N_2\left(r, \frac{1}{G}\right) + N\left(r, \frac{1}{G-1}\right) \\ &+ S(r, F) + S(r, G). \end{aligned} \tag{4.13}$$

Without loss of generality, we assume that $T(r, G) \leq T(r, F)$ for $r \in I$ which is a set of infinite measure. Thus, (4.13) implies

$$T(r) \leq N_2\left(r, \frac{1}{F}\right) + N_2\left(r, \frac{1}{G}\right) + S(r, F) + S(r, G),$$

for $r \in I$, contradicting (4.7). Hence $\Phi(z) \equiv 0$, that is,

$$\frac{F''(z)}{F'(z)} - 2\frac{F'(z)}{F(z)-1} = \frac{G''(z)}{G'(z)} - 2\frac{G'(z)}{G(z)-1}. \tag{4.14}$$

Solving (4.14), we have

$$F(z) = \frac{(b+1)G(z) + (a-b-1)}{bG(z) + (a-b)}, \tag{4.15}$$

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where $a(\neq 0)$ and b are two constants.

If $b+1\neq 0$, $a-b-1\neq 0$, then

$$\overline{N}\left(r,\frac{1}{F}\right)=\overline{N}\left(r,\frac{1}{G+(a-b-1)/(b+1)}\right). \tag{4.16}$$

By Lemma 2.2 and Lemma 4.1, and (4.15) we deduce that

$$T(r,F)=T(r,G)+O(1). \tag{4.17}$$

Thus by the second fundamental theorem, we get from (4.2), (4.16) and (4.17) that

$$\begin{aligned} T(r) &= T(r,G)+O(1) \\ &\leq \overline{N}(r,G)+\overline{N}\left(r,\frac{1}{G}\right)+\overline{N}\left(r,\frac{1}{G+(a-b-1)/(b+1)}\right)+S(r,G) \\ &\leq \overline{N}\left(r,\frac{1}{G}\right)+\overline{N}\left(r,\frac{1}{F}\right)+S(r,G), \end{aligned}$$

which contradicts (4.7). Hence either $b+1=0$ or $a-b-1=0$.

If $b+1=0$, then (4.15) becomes

$$F(z)=\frac{a}{-G(z)+a+1}.$$

Clearly,

$$\overline{N}(r,F)=\overline{N}\left(r,\frac{1}{G-a-1}\right).$$

Using the same argument as in the former case, we can deduce that $a=-1$, which implies $F(z)G(z)\equiv 1$.

If $a-b-1=0$, then (4.15) becomes

$$F(z)=\frac{aG(z)}{bG(z)+1}.$$

If $b\neq 0$, then we have

$$\overline{N}(r,F)=\overline{N}\left(r,\frac{1}{G+1/b}\right).$$

Using the former method once more, we can obtain a contradiction. Hence $b=0$ and then $a=1$ which implies $F(z)\equiv G(z)$. Hence we deduce that either $F(z)G(z)\equiv 1$ or $F(z)\equiv G(z)$.

Now we prove $f(z)\equiv g(z)$.

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If $G(z)F(z) \equiv 1$, that is

$$f^6(z)(f(z)-1)g^6(z)(g(z)-1) \equiv \alpha^2(z), \tag{4.18}$$

then from (4.18) and the conditions of Theorem 5 we know that any zero or 1-point of $f(z)$ must be a zero of $\alpha(z)$. By the second fundamental theorem, we have

$$\begin{aligned} T(r,f) &\le \overline{N}(r,f) + \overline{N}\left(r,\frac{1}{f}\right) + \overline{N}\left(r,\frac{1}{f-1}\right) + S(r,f) \\ &\le N\left(r,\frac{1}{\alpha}\right) + S(r,f) = S(r,f), \end{aligned}$$

which is a contradiction. It shows that $F(z)G(z) \not\equiv 1$. Hence $F(z) \equiv G(z)$, that is,

$$f^6(z)(f(z)-1) \equiv g^6(z)(g(z)-1).$$

If $f(z) \not\equiv g(z)$, then $h(z) = f(z)/g(z) \not\equiv 1$. Substituting $h(z)$ into the above equation, we have

$$g(z) = \frac{1+h+\cdots+h^5}{1+h+\cdots+h^6}.$$

If $h(z)$ is not a constant function, then by Picard's theorem we deduce that $1+h+\cdots+h^6$ has zeros. Hence $g(z)$ has poles. Thus we obtain that $g(z)$ is either a constant or has poles but this is impossible. Hence $f(z) \equiv g(z)$. The proof of Theorem 5 is complete. □

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