A series of "cut" Bessel functions

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§ 1. *Introduction.*

1·1. In Kottler's theoretical discussion[1] of the diffraction of a plane wave of monochromatic light of wave-length $2\pi/k$ by a black half-plane, the function

$$f(r, \theta) = \frac{1}{2\pi}\int_0^\infty e^{ikr\cosh t}\frac{\sin\theta}{\cosh t + \cos\theta}\,dt, \tag{1·11}$$

where (r, θ, z) are cylindrical coordinates, plays an important part. In particular it is necessary to have asymptotic formulae for $f(r, \theta)$, valid when r is either very large or very small compared with the wave-length.

Kottler obtains these asymptotic formulae by showing that

$$\frac{1}{k}\frac{\partial f}{\partial r} + i\cos\theta f = -\tfrac{1}{4}\sin\theta\, H_0^{(1)}(kr), \tag{1·12}$$

where, in the usual notation[2], $H_n^{(1)}(\lambda)$ denotes the Bessel function $J_n(\lambda) + i\,Y_n(\lambda)$ of integral order n, and then making use of the known properties of $H_0^{(1)}(\lambda)$. Whilst this method gives correct results, it is difficult to place it on a sound basis.

1·2. The asymptotic formula valid as $r \to +\infty$ may be obtained by a direct method, well-known in the theory of Bessel functions. For if we make the substitution $\cosh t = 1 + iv$ and then rotate the path of integration through a right-angle, we obtain

$$f(r, \theta) = \frac{1}{2\pi}\exp\{i(kr + \tfrac{1}{4}\pi)\}\int_0^\infty e^{-krv}\frac{\sin\theta\,dv}{(1 + \cos\theta + iv)\sqrt{(2v + iv^2)}},$$

and an application of Watson's Lemma (**W.**, 236) gives the required expansion. There does not appear to be any corresponding very simple way of obtaining the expansion valid when r is small.

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[1] *Ann. der Phys.* 71 (1923), 457-508 (in particular, pages 496 and 499). We have found it convenient to change the sign of i throughout.

[2] G. N. Watson, *A treatise on the theory of Bessel functions* (Cambridge, 1922), 73. In later references, this work is cited as **W.**

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1·3. The form of the differential equation (1·12) suggests another line of approach. If we substitute in this equation the formal series

$$f(r, \theta) = \sum_{1}^{\infty} a_n H_n^{(1)}(kr),$$

where a_n is independent of r, and use the recurrence formula (W., 74 (4))

$$2\frac{dH_n^{(1)}(kr)}{d(kr)} = H_{n-1}^{(1)}(kr) - H_{n+1}^{(1)}(kr),$$

we find that

$$f(r, \theta) = \frac{1}{2i}\sum_{1}^{\infty} e^{-\frac{1}{2}n\pi i} H_n^{(1)}(kr) \sin n\theta \qquad (1·31)$$

is a particular solution of (1·12).

The series (1·31) is not easy to handle near $r = 0$ because $H_n^{(1)}(kr)$ is of the order of r^{-n} as r tends to zero; some other representation is needed. In the present note we use a series (viz. (2·12) *infra*) which is derived from (1·31) by omitting from each of the functions $H_n^{(1)}(kr)$ all negative powers of r. The sum of this series is discontinuous at $\theta = \pm\frac{1}{2}\pi$, and we are obliged to consider separately the two ranges $|\theta| < \frac{1}{2}\pi$ and $\frac{1}{2}\pi < |\theta| < \pi$.

The proofs of our results are less easy than one might expect, though the difficulty is perhaps due to the line of proof we have adopted.

Our methods can also be applied to obtain the expansion of a second integral considered by Kottler, viz.

$$g(r, \theta) = \frac{1}{\pi}\int_0^{\infty} e^{ikr\cosh t}\frac{\cos\frac{1}{2}\theta \cosh\frac{1}{2}t}{\cosh t + \cos\theta}\,dt, \qquad (1·32)$$

as a series of "cut" Bessel functions, in this case of order $n + \frac{1}{2}$. The discussion of this is omitted as no essentially new ideas are involved.

§ 2. *Enunciation of the main theorem: preliminary lemmas.*

2·1. Let $h_n^{(1)}(\lambda)$ denote the function obtained from the expansion of $H_n^{(1)}(\lambda)$ in the neighbourhood of $\lambda = 0$ by omitting all the terms that involve negative powers of λ. It is usual to call a function of this sort a *cut Bessel function.*

We shall prove

THEOREM 1. *Let*

$$F(\lambda) = \frac{1}{2\pi}\int_0^{\infty} e^{i\lambda\cosh t}\frac{\sin\theta}{\cosh t + \cos\theta}\,dt. \qquad (2·11)$$

Then if $\lambda > 0$ *and* $-\frac{1}{2}\pi < \theta < \frac{1}{2}\pi$,

$$F(\lambda) = \frac{1}{2i}\sum_{n=1}^{\infty} e^{-\frac{1}{2}n\pi i} h_n^{(1)}(\lambda) \sin n\theta. \qquad (2·12)$$

The proof rests on certain subsidiary lemmas which are given in the following sub-sections. Apart from Lemma 1, these lemmas are of some analytical interest; the reader will need only the results of Lemmas 2 and 3 in order to follow the proof of Theorem 1.

We suppose throughout that $\lambda > 0$ *and, in* §§ 2 *and* 3, *that* $-\frac{1}{2}\pi < \theta < \frac{1}{2}\pi$.

2·2. LEMMA 1. *If* Σa_n *is a bounded series, then* $\Sigma a_n x^n$ *is uniformly bounded in* $0 \leqq x \leqq 1$; *moreover,* $\Sigma a_n x^n$ *is convergent when* $0 < x < 1$, *and its sum* $s(x)$ *is bounded in this open interval.*

Let $$\sigma_n = a_0 + a_1 + \dots + a_n.$$

Then $$a_0 + a_1 x + a_2 x^2 + \dots + a_n x^n$$
$$= (1 - x)(\sigma_0 + \sigma_1 x + \sigma_2 x^2 + \dots + \sigma_{n-1} x^{n-1}) + \sigma_n x^n.$$

But $|\sigma_n| < K$ for all n; hence, if $0 < x < 1$,

$$|a_0 + a_1 x + \dots + a_n x^n| < K(1 - x^n) + Kx^n = K.$$

This proves the first part of the lemma.

By Dirichlet's test, the series $\Sigma a_n x^n$ is convergent when $0 < x < 1$; it follows at once that $|s(x)| \leqq K$ in this open interval.

LEMMA 2. *Let* Σa_n *be a bounded series,* $s(x)$ *the sum of* $\Sigma a_n x^n$ *when* $0 \leqq x < 1$; *let*

$$E(\lambda, x) = \exp\{\tfrac{1}{2} i\lambda (x + x^{-1})\} \qquad (x > 0).$$

Then $$\int_0^1 s(x) E(\lambda, x)\, dx = \Sigma a_n \int_0^1 x^n E(\lambda, x)\, dx.$$

The proof of this lemma might almost be taken for granted. But the fact that $s(1)$ and $E(\lambda, 0)$ are not defined raises minor difficulties, and we have thought it desirable to set out the proof.

Let $s_n(x) = a_0 + a_1 x + \dots + a_n x^n$. Take an arbitrary positive number ϵ. Then, by Lemma 1 and the fact that $|E(\lambda, x)| = 1$ when $0 < x < 1$, there exists a positive number δ such that, whenever $0 < \delta' < \delta$,

$$\left| \int_{1-\delta}^{1-\delta'} \{s(x) - s_n(x)\} E(\lambda, x)\, dx \right| < \tfrac{1}{3}\epsilon, \qquad (2·21)$$

$$\left| \int_{\delta'}^{\delta} \{s(x) - s_n(x)\} E(\lambda, x)\, dx \right| < \tfrac{1}{3}\epsilon, \qquad (2·22)$$

independently of the value of n. Hence

$$\int_0^1 \{s(x) - s_n(x)\} E(\lambda, x)\, dx \equiv \int_0^1 F_n(\lambda, x)\, dx,$$

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say, is defined as an improper integral. Moreover

$$\left|\int_0^\delta F_n(\lambda, x)\, dx\right| + \left|\int_{1-\delta}^1 F_n(\lambda, x)\, dx\right| \leqq \tfrac{2}{3}\epsilon,$$

independently of the value of n.

But, by the uniform convergence of $\Sigma a_n x^n$ in $(\delta, 1-\delta)$ and the boundedness of $E(\lambda, x)$, there exists an integer N such that, whenever $n \geqq N$,

$$\left|\int_\delta^{1-\delta} F_n(\lambda, x)\, dx\right| < \tfrac{1}{3}\epsilon.$$

It now follows that

$$\lim_{n\to\infty} \int_0^1 \{s(x) - s_n(x)\} E(\lambda, x)\, dx = 0. \qquad (2\cdot23)$$

Again, a modification of the argument used above shows that each of the integrals

$$\int_0^1 s(x)\, E(\lambda, x)\, dx, \quad \int_0^1 x^n E(\lambda, x)\, dx$$

is defined as an improper integral. Hence, by (2·23),

$$\int_0^1 s(x)\, E(\lambda, x)\, dx = \lim_{n\to\infty} \int_0^1 s_n(x)\, E(\lambda, x)\, dx,$$

and so the lemma is proved.

2·3. As a preliminary to our next lemma, we note that, by **W.**, 78 (8), with $z = -i\lambda$, and **W.**, 80 (15), the terms in the expansion of $H_n^{(1)}(\lambda)$ near $\lambda = 0$ which involve negative powers of λ are given by

$$-\frac{i}{\pi} \sum_{m=0}^{[\frac{1}{2}(n-1)]} \frac{(n-m-1)!}{m!\,(\frac{1}{2}\lambda)^{n-2m}} = -\frac{i}{\pi} S_n(\lambda),$$

where $S_n(\lambda)$ is Schläfli's polynomial, as defined, for example, in **W.**, 285 (1). Hence the "cut" Bessel function $h_n^{(1)}(\lambda)$ is given by

$$h_n^{(1)}(\lambda) = H_n^{(1)}(\lambda) + \frac{i}{\pi} S_n(\lambda). \qquad (2\cdot31)$$

Now, by **W.**, 288 (2) with $a = 0$, we have

$$S_n(\lambda) = \int_0^\infty \{e^{n\theta} - e^{-n(\pi i+\theta)}\} e^{-\lambda \sinh\theta}\, d\theta$$

$$= 2e^{-\frac{1}{2}n\pi i} \int_{\frac{1}{2}\pi i}^{\frac{1}{2}\pi i+\infty} \sinh n\phi\; e^{i\lambda\cosh\phi}\, d\phi,$$

on writing $\theta = \phi - \frac{1}{2}\pi i$. Moreover, by **W.**, 180, (8),

$$H_n^{(1)}(\lambda) = \frac{2}{\pi i} e^{-\frac{1}{2}n\pi i} \int_0^{\frac{1}{2}\pi i+\infty} \cosh n\phi\; e^{i\lambda\cosh\phi}\, d\phi. \qquad (2\cdot32)$$

From (2·31) and (2·32) follows

LEMMA 3. *The cut Bessel function* $h_n^{(1)}(\lambda)$ *is given by*

$$\tfrac{1}{2}\pi i\, e^{\frac{1}{2}n\pi i}\, h_n^{(1)}(\lambda) = \int_0^{\frac{1}{2}\pi i} \cosh n\phi\, e^{i\lambda \cosh\phi}\, d\phi + \int_{\frac{1}{2}\pi i}^{\frac{1}{2}\pi i+\infty} e^{-n\phi}\, e^{i\lambda\cosh\phi}\, d\phi.$$

§3. *Proof of Theorem* 1.

3·1. If in the definition (2·11) we put $e^{-t} = x$ and use the notation $E(\lambda, x)$ of Lemma 2, we obtain

$$F(\lambda) = \frac{1}{\pi}\int_0^1 E(\lambda, x)\, \frac{\sin\theta}{x^2 + 2x\cos\theta + 1}\, dx.$$

But when $0 \leqq x < 1$,

$$\frac{\sin\theta}{x^2 + 2x\cos\theta + 1} = -\sum_{n=1}^{\infty} x^{n-1}\sin(n\theta + n\pi);$$

moreover, for any fixed value of θ,

$$\Sigma \sin(n\theta + n\pi)$$

is a bounded series. Hence, by Lemma 2,

$$F(\lambda) = -\frac{1}{\pi}\sum_{n=1}^{\infty}(-1)^n \sin n\theta \int_0^1 E(\lambda, x)\, x^{n-1}\, dx,$$

or, on restoring the original variable of integration,

$$F(\lambda) = -\frac{1}{\pi}\sum_{n=1}^{\infty}(-1)^n \sin n\theta \int_0^\infty e^{i\lambda\cosh t}\, e^{-nt}\, dt. \qquad (3·11)$$

3·2. Now if $n \geqq 0$, we have, by Cauchy's theorem,

$$\int_0^\infty e^{i\lambda\cosh t}\, e^{-nt}\, dt = \int_0^{\frac{1}{2}\pi i} e^{i\lambda\cosh t}\, e^{-nt}\, dt + \int_{\frac{1}{2}\pi i}^{\frac{1}{2}\pi i+\infty} e^{i\lambda\cosh t}\, e^{-nt}\, dt. \qquad (3·21)$$

The integral over $(\frac{1}{2}\pi i, \frac{1}{2}\pi i + \infty)$ in (3·21) tallies with that in Lemma 3, whereas that over $(0, \frac{1}{2}\pi i)$ does not. We discuss the integral over $(0, \frac{1}{2}\pi i)$ in the next sub-section.

3·3. Let us consider the integral[1]

$$I = \int_0^{\frac{1}{2}\pi} e^{i\lambda\cos y}\, \frac{i\sin\theta}{\cos y + \cos\theta}\, dy \qquad (3·31)$$

$$= \int_0^{\frac{1}{2}\pi} e^{i\lambda\cos y}\left\{\frac{e^{i\theta}}{e^{iy} + e^{i\theta}} - \frac{e^{-i\theta}}{e^{iy} + e^{-i\theta}}\right\} dy$$

$$= \lim_{r\to 1-0}\int_0^{\frac{1}{2}\pi} e^{i\lambda\cos y}\left\{\frac{e^{i\theta}}{re^{iy} + e^{i\theta}} - \frac{e^{-i\theta}}{re^{iy} + e^{-i\theta}}\right\} dy. \qquad (3·32)$$

[1] The device used here was suggested by a more lengthy proof of the theorem on different lines.

The last step is justified by the fact that the integrand in (3·32) is a continuous function of both variables r and y on account of the restriction $-\frac{1}{2}\pi < \theta < \frac{1}{2}\pi$. Expanding in powers of r, we have

$$I = \lim_{r \to 1-0} \int_0^{\frac{1}{2}\pi} e^{i\lambda \cos y} \sum_{n=1}^{\infty} (-2i \sin n\theta)(-r)^n e^{niy}\, dy$$

$$= \lim_{r \to 1-0} (-2i) \sum_{n=1}^{\infty} (-r)^n \sin n\theta \int_0^{\frac{1}{2}\pi} e^{i\lambda \cos y} e^{niy}\, dy.$$

The series obtained by putting $r = 1$ is convergent, as an integration by parts beginning with

$$\frac{1}{ni} \int_0^{\frac{1}{2}\pi} e^{i\lambda \cos y}\, d(e^{niy})$$

will show. Hence, by Abel's theorem on the continuity of power series,

$$I = -2i \sum_{n=1}^{\infty} (-1)^n \sin n\theta \int_0^{\frac{1}{2}\pi} e^{i\lambda \cos y} e^{niy}\, dy$$

$$= -2 \sum_{n=1}^{\infty} (-1)^n \sin n\theta \int_0^{\frac{1}{2}\pi i} e^{i\lambda \cosh t} e^{nt}\, dt. \tag{3·33}$$

But we also have

$$\frac{i \sin \theta}{\cos y + \cos \theta} = \lim_{r \to 1} \left\{ \frac{e^{i\theta}}{re^{-iy} + e^{i\theta}} - \frac{e^{-i\theta}}{re^{-iy} + e^{-i\theta}} \right\},$$

and so we can carry out the previous transformations with e^{-iy} in place of e^{iy}. It then follows that

$$I = -2 \sum_{n=1}^{\infty} (-1)^n \sin n\theta \int_0^{\frac{1}{2}\pi i} e^{i\lambda \cosh t} e^{-nt}\, dt. \tag{3·34}$$

Combining (3·33) and (3·34), we have

$$\sum_{n=1}^{\infty} (-1)^n \sin n\theta \int_0^{\frac{1}{2}\pi i} e^{i\lambda \cosh t} e^{-nt}\, dt$$

$$= \sum_{n=1}^{\infty} (-1)^n \sin n\theta \int_0^{\frac{1}{2}\pi i} e^{i\lambda \cosh t} \cosh nt\, dt. \tag{3·35}$$

3·4. By (3·11), (3·21) and (3·35), we now have

$$F(\lambda) = -\frac{1}{\pi} \sum_{n=1}^{\infty} (-1)^n \sin n\theta\, G(n, \lambda),$$

where $G(n, \lambda) = \int_0^{\frac{1}{2}\pi i} e^{i\lambda \cosh t} \cosh nt\, dt + \int_{\frac{1}{2}\pi i}^{\frac{1}{2}\pi i + \infty} e^{i\lambda \cosh t} e^{-nt}\, dt.$

But, by Lemma 3,

$$G(n, \lambda) = \tfrac{1}{2}\pi i\, e^{\frac{1}{2}n\pi i}\, h_n^{(1)}(\lambda).$$

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Hence

$$F(\lambda) = \frac{1}{2i} \sum_{n=1}^{\infty} e^{-\frac{1}{2}n\pi i}\, h_n^{(1)}(\lambda) \sin n\theta.$$

But this is the required equation (2·12), and so Theorem 1 is proved.

§ 4. *Extensions of Theorem* 1.

4·1. The argument of § 3·3 fails when $\theta = \pm(\pi - \alpha)$ and $0 < \alpha < \frac{1}{2}\pi$, since the integral (3·31) exists only as a Cauchy principal value on account of the singularity at $y = \alpha$; moreover this principal value is not equal to (3·32) or to the corresponding limit with re^{-iy} replacing re^{iy}. It turns out that equation (3·35) no longer holds and has to be replaced by the identity of

LEMMA 4. *If* $\frac{1}{2}\pi < |\theta| < \pi$, *then*

$$-\sum_{n=1}^{\infty} (-1)^n \sin n\theta \int_0^{\frac{1}{2}\pi i} e^{i\lambda \cosh t}\, e^{-nt}\, dt$$
$$= \pm \tfrac{1}{2}\pi e^{-i\lambda\cos\theta} - \sum_{n=1}^{\infty} (-1)^n \sin n\theta \int_0^{\frac{1}{2}\pi i} e^{i\lambda\cosh t} \cosh nt\, dt, \quad (4·11)$$

where the upper or lower sign is taken according as θ is positive or negative.

Let us suppose first that $\theta = \pi - \alpha$ where $0 < \alpha < \frac{1}{2}\pi$, and consider

$$\int e^{i\lambda\cos z} \frac{i \sin\alpha}{\cos z - \cos\alpha}\, dz$$

along two paths both beginning at $z = 0$ and ending at $z = \frac{1}{2}\pi$. The first of these paths, Γ_1 say, lies in the lower half-plane, the other, Γ_2 say, in the upper half-plane. Then, by Cauchy's theorem of residues,

$$\int_{\Gamma_1} - \int_{\Gamma_2} = 2\pi i \,(\text{residue at } z = \alpha) = 2\pi e^{i\lambda\cos\alpha}. \quad (4·12)$$

Now the integral along Γ_2 is equal to

$$\lim_{r \to 1-0} \int_{\Gamma_2} e^{i\lambda\cos z} \left\{ \frac{e^{i\alpha}}{re^{iz} - e^{i\alpha}} - \frac{e^{-i\alpha}}{re^{iz} - e^{-i\alpha}} \right\} dz$$
$$= \lim_{r \to 1-0} \int_{\Gamma_2} 2ie^{i\lambda\cos z} \sum_{n=1}^{\infty} r^n e^{niz} \sin n\alpha\, dz$$
$$= \lim_{r \to 1-0} \sum_{n=1}^{\infty} 2i\, r^n \sin n\alpha \int_{\Gamma_2} e^{i\lambda\cos z}\, e^{niz}\, dz,$$

term-by-term integration being valid since $|re^{iz}| \leqq r$ on Γ_2. An application of Cauchy's theorem enables us to replace Γ_2 by a segment of the real axis, and so

$$\int_{\Gamma_2} = \lim_{r \to 1-0} \sum_{n=1}^{\infty} 2i\, r^n \sin na \int_0^{\frac{1}{2}\pi} e^{i\lambda \cos x}\, e^{nix}\, dx$$

$$= \lim_{r \to 1-0} \sum_{n=1}^{\infty} 2r^n \sin na \int_0^{\frac{1}{2}\pi i} e^{i\lambda \cosh t}\, e^{nt}\, dt.$$

Hence, by Abel's theorem on the continuity of power series,[1]

$$\int_{\Gamma_2} = \sum_{n=1}^{\infty} 2 \sin na \int_0^{\frac{1}{2}\pi i} e^{i\lambda \cosh t}\, e^{nt}\, dt. \tag{4·13}$$

By a similar argument, it follows from the identity

$$\int_{\Gamma_1} = \lim_{r \to 1-0} \int_{\Gamma_1} e^{i\lambda \cos z} \left\{ \frac{e^{ia}}{re^{-iz} - e^{ia}} - \frac{e^{-ia}}{re^{-iz} - e^{-ia}} \right\} dz$$

that

$$\int_{\Gamma_1} = \sum_{n=1}^{\infty} 2 \sin na \int_0^{\frac{1}{2}\pi i} e^{i\lambda \cosh t}\, e^{-nt}\, dt. \tag{4·14}$$

From equations (4·12), (4·13), (4·14), we easily obtain (4·11) for the case $\frac{1}{2}\pi < \theta < \pi$. The result when $-\pi < \theta < -\frac{1}{2}\pi$ follows by changing the sign of θ.

4.2. If we repeat the argument of § 3, using (4·11) instead of (3·35), we obtain

Theorem 2. *Let*

$$F(\lambda) = \frac{1}{2\pi} \int_0^{\infty} e^{i\lambda \cosh t}\, \frac{\sin \theta}{\cosh t + \cos \theta}\, dt. \tag{2·11}$$

Then if $\lambda > 0$ *and* $\frac{1}{2}\pi < |\theta| < \pi$,

$$F(\lambda) = \pm \tfrac{1}{2} e^{-i\lambda \cos \theta} + \frac{1}{2i} \sum_{n=1}^{\infty} e^{-\frac{1}{2}n\pi i}\, h_n^{(1)}(\lambda) \sin n\theta, \tag{4·21}$$

where the upper or lower sign is taken according as θ *is positive or negative.*

4·3. By Lemma 3, we have

$$h_n^{(1)}(\lambda) = \frac{2}{n\pi i} \cos \frac{n\pi}{2}\, e^{-\frac{1}{2}n\pi i} + O\left(\frac{1}{n^2}\right) \tag{4·31}$$

for any fixed value of λ. Hence the series

$$\sum_{n=1}^{\infty} e^{-\frac{1}{2}n\pi i}\, h_n^{(1)}(\lambda) \sin n\theta \tag{4·32}$$

[1] *Cf.* the argument which gave equation (3.33).

converges for all values of θ. But the integral (2·11) is a continuous function of θ when $|\theta| < \pi$; a comparison of (2·12) and (4·21) then shows that the sum of the series (4·32) is discontinuous at $\theta = \pm \frac{1}{2}\pi$.

Now by (4·31) and the Riesz-Fischer Theorem,[1] (4·32) is the Fourier series of an odd function $\Phi(\theta)$ of the class L^2, and evidently

$$\begin{aligned} \Phi(\theta) &= F & (0 < \theta < \tfrac{1}{2}\pi), \\ &= F - \tfrac{1}{2} e^{-i\lambda \cos\theta} & (\tfrac{1}{2}\pi < \theta < \pi). \end{aligned}$$

But since (4·32) is convergent,

$$\begin{aligned} \sum_{n=1}^{\infty} e^{-\frac{1}{2}n\pi i} h_n^{(1)}(\lambda) \sin n\theta &= \lim_{r \to 1-0} \sum_{n=1}^{\infty} r^n e^{-\frac{1}{2}n\pi i} h_n^{(1)}(\lambda) \sin n\theta \\ &= \lim_{r \to 1-0} \frac{1}{2\pi} \int_0^{2\pi} \frac{1 - r^2}{1 - 2r\cos(\theta - \phi) + r^2} \Phi(\phi)\, d\phi \\ &= \tfrac{1}{2}\{\Phi(\theta + 0) + \Phi(\theta - 0)\} \end{aligned}$$

whenever the latter limit exists.[2]

In particular, putting $\theta = \pm \frac{1}{2}\pi$, we obtain

THEOREM 3. *If* $\lambda > 0$ *and* $\theta = \pm \frac{1}{2}\pi$,

$$F(\lambda) = \pm \tfrac{1}{4} e^{-i\lambda\cos\theta} + \frac{1}{2i} \sum_{n=1}^{\infty} e^{-\frac{1}{2}n\pi i} h_n^{(1)}(\lambda) \sin n\theta,$$

where the upper or lower sign is taken according as θ *is positive or negative.*

§ 5. *The behaviour of* $F(\lambda)$ *when* $\lambda \to 0$ *or* ∞.

5·1. From Lemma 3, we have

$$\lim_{\lambda \to 0} h_n^{(1)}(\lambda) = \frac{2}{n\pi i} \cos \frac{n\pi}{2} e^{-\frac{1}{2}n\pi i}.$$

Hence it follows from Theorem 1 that, if $|\theta| < \frac{1}{2}\pi$,

$$\lim_{\lambda \to 0} F(\lambda) = \frac{1}{2\pi} \{\sin 2\theta - \tfrac{1}{2}\sin 4\theta + \tfrac{1}{3}\sin 6\theta - \dots\},$$

and so

$$\lim_{\lambda \to 0} F(\lambda) = \frac{\theta}{2\pi}. \tag{5·11}$$

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[1] See, for example, Titchmarsh, *Theory of Functions* (Oxford, 1932), 423-4.

[2] We have used here two well-known results in the theory of Fourier series. See, for example, Titchmarsh, *loc. cit.*, 440, Exx. 6, 7.

That the equation (5·11) holds in the wider range $|\theta| < \pi$ may be proved by means of Theorems 2 and 3, or, more directly, from the integral definition of $F(\lambda)$ by means of the calculus of residues.

Having obtained (5·11), we can show from the expansion of $h_n^{(1)}(\lambda)$ near $\lambda = 0$ that, if $|\theta| < \pi$,

$$F(\lambda) = \frac{\theta}{2\pi} - \frac{i \sin \theta}{2\pi} \lambda \log \lambda + O(\lambda)$$

as $\lambda \to 0$.

5·2. For completeness we observe that, if $|\theta| < \pi$,

$$2\pi F(\lambda) \sim \sqrt{(\tfrac{1}{2}\pi/\lambda)} \tan \tfrac{1}{2}\theta \, e^{i(\lambda + \pi/4)}$$

as $\lambda \to \infty$, a result best proved by the method of §1·2.

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