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ON NON-HOMOGENEOUS CANONICAL THIRD-ORDER LINEAR DIFFERENTIAL EQUATIONS

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Abstract

In this paper sufficient conditions have been obtained for non-oscillation of non-homogeneous canonical linear differential equations of third order. Some of these results have been extended to non-linear equations.

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1. Introduction

In [1] Barrett considered homogeneous third-order linear differential equations of the form

(H) $$\left[r_2(t)\left\{(r_1(t)y')' + q_1(t)y\right\}\right]' + q_2(t)(r_1(t)y') = 0$$

where r_1, r_2, q_1 and $q_2 \in C([a, \infty), R)$, $a \in R$, $r_1(t) > 0$ and $r_2(t) > 0$. By a solution of (H) on $[a, \infty)$ we mean a function $y \in C^1([a, \infty), R)$ such that $r_1 y'$ and $r_2\{(r_1 y')' + q_1 y\} \in C^1([a, \infty), R)$ and (H) is satisfied identically. We call (H) the third-order canonical form. The adjoint of (H) is given by

(H*) $$\left[r_1(t)\left\{(r_2(t)y')' + q_2(t)y\right\}\right]' + q_1(t)(r_2(t)y') = 0.$$

We may note that (H*) is obtained from (H) by interchanging r_1 with r_2 and q_1 with q_2. The non-homogeneous equations associated with (H) and (H*) are given, respectively, by

(NH) $$\left[r_2(t)\left\{(r_1(t)y')' + q_1(t)y\right\}\right]' + q_2(t)(r_1(t)y') = f_1(t)$$

and

(NH*) $$\left[r_1(t)\left\{(r_2(t)y')' + q_2(t)y\right\}\right]' + q_1(t)(r_2(t)y') = g_1(t)$$

with f_1 and $g_1 \in C([a,\infty), R)$ such that $f_1(t) \geq 0$ and $g_1(t) \geq 0$.

Suppose that $\int_a^\infty dt/r_1(t) = \infty$. The Liouville transformation $s = R(t)$, $x(s) = y(t)$, where $R(t) = \int_a^t du/r_1(u)$, transforms (NH) into

$$\text{(1)} \qquad \frac{d}{ds}\left[\frac{r_2(t)}{r_1(t)}\frac{d^2x}{ds^2} + r_2(t)q_1(t)x\right] + r_1(t)q_2(t)\frac{dx}{ds} = r_1(t)f_1(t)$$

with $t = R^{-1}(s)$. If $\int_a^\infty dt/r_1(t) < \infty$, then the Kummer transformation $s = 1/\rho(t)$, $x(s) = sy(t)$, where $\rho(t) = \int_t^\infty du/r_1(u)$, transforms (NH) into

$$\text{(2)} \quad \frac{d}{ds}\left[\frac{r_2(t)}{r_1(t)}s^3\frac{d^2x}{ds^2} + \frac{r_2(t)}{s}q_1(t)x\right] + \frac{r_1(t)q_2(t)}{s}\frac{dx}{ds} - \frac{r_1(t)q_2(t)}{s^2}x = \frac{r_1(t)}{s^2}f_1(t)$$

with $t = \rho^{-1}(1/s)$. However, Equation (2) may be written as

$$\text{(3)} \quad \frac{d}{ds}\left[\sigma(s)\frac{d^2x}{ds^2} + \left(\lambda(s) - \int_a^s \nu(u)\,du\right)x\right] + \left[\mu(s) + \int_a^s \nu(u)\,du\right]\frac{dx}{ds} = \frac{r_1(t)}{s^2}f_1(t)$$

where $\sigma(s) = r_2(t)s^3/r_1(t)$, $\lambda(s) = r_2(t)q_1(t)/s$, $\mu(s) = r_1(t)q_2(t)/s$ and $\nu(s) = r_1(t)q_2(t)/s^2$.

We may note that $x(s)$ is non-oscillatory if and only if $y(t)$ is non-oscillatory. Furthermore, Equations (1) and (3) have the same general form. If $\int_a^\infty dt/r_2(t) = \infty$ or $\int_a^\infty dt/r_2(t) < \infty$, then (NH*) is transformed into an equation of the type (1) or (3) which is obtained by interchanging r_1 with r_2 and q_1 with q_2. Hence it is enough to study the equations of the form

$$\text{(E)} \qquad \big(r(t)y'' + p(t)y\big)' + q(t)y' = f(t)$$

where p, q, r and $f \in C([a,\infty), R)$, $r(t) > 0$ and $f(t) \geq 0$.

We recall that a function $y \in C([a,\infty), R)$ is said to be *oscillatory* if for every $t_1 \geq a$ there exist t_2 and t_3 ($t_1 < t_2 < t_3$) such that $y(t_2) > 0$ and $y(t_3) < 0$. It is said to be of *Z-type* if it has arbitrarily large zeros but is ultimately non-negative or non-positive. A function $y(t)$ is said to be *non-oscillatory* if it is neither oscillatory nor of Z-type. Equation (E) is said to be *non-oscillatory* if all of its solutions are non-oscillatory.

Linear non-homogeneous third order differential equations of the type

$$\text{(4)} \qquad \big(r(t)y''\big)' + q(t)y' + p(t)y = f(t)$$

occur in the study of the entry flow phenomenon in hydrodymics [3]. We note that Equation (4) is a particular case of (E). Indeed, we may write Equation (4) as

$$\left[r(t)y'' + \left(\int_a^t p(s)\,ds\right)y\right]' + \left(q(t) - \int_a^t p(s)\,ds\right)y' = f(t).$$

Unlike the second order case, equation (4) cannot be transformed to an equation of the type

$$x''' + c(t)x' + b(t)x = h(t)$$

when $\int_a^\infty dt/r(t) = \infty$ or $\int_a^\infty dt/r(t) < \infty$.

The purpose of this paper is to study non-oscillatory behaviour of solutions of (E). In the process, we obtain a result which generalizes a result in [5]. In Section 2 we obtain sufficient conditions for non-oscillation of (E). It is interesting to note that this study is applicable to a class of non-linear equations. Section 3 deals with the relation between three independent solutions of (E).

2. Non-oscillatory behaviour of solutions

In this section we obtain sufficient conditions for non-oscillation of (E). The same techniques are then used to obtain non-oscillation results for certain classes of non-linear equations (see Equations (7) - (11) below).

THEOREM 1. *If $p(t) \le 0$ and $q(t) \le 0$ for large t, then* (E) *is non-oscillatory.*

PROOF. Let $y(t)$ be a solution of (E) on $[a, \infty)$. Let $p(t) \le 0$ and $q(t) \le 0$ for $t \ge t_0 \ge a$. Let $y(t)$ be of non-negative Z-type with consecutive double zeros at t_1 and t_2 ($t_0 < t_1 < t_2$). So there exists a $b \in (t_l, t_2)$ such that $y'(b) = 0$, $y''(b) \le 0$ and $y'(t) > 0$ for $t \in (t_1, b)$. Integrating (E) from t_1 to b, we get

$$\begin{aligned} 0 &\ge r(b)y''(b) + p(b)y(b) - c(t_1)y''(t_1) \\ &= \int_{t_1}^{b} f(t)\,dt - \int_{t_1}^{b} q(t)y'(t)\,dt > 0 \end{aligned}$$

because $y''(t_1) \ge 0$. Suppose that y(t) is a non-positive Z-type solution with consecutive double zeros at t_1 and t_2 ($t_0 < t_1 < t_2$). Then there exists $b \in (t_1, t_2)$ such that $y'(b) = 0$ and $y'(t) > 0$ for $t \in (b, t_2)$. We note that $y''(b) \ge 0$ and $y''(t_2) \le 0$. Now integrating (E) from b to t_2 yields

$$\begin{aligned} 0 &\ge r(t_2)y''(t_2) + r(b)y''(b) - p(b)y(b) \\ &= \int_{b}^{t_2} f(t)\,dt - \int_{b}^{t_2} q(t)y'(t)\,dt > 0, \end{aligned}$$

a contradiction. Hence $y(t)$ cannot be of Z-type.

Suppose that $y(t)$ is an oscillatory solution with consecutive zeros at t_1, t_2 and t_3 ($t_0 < t_1 < t_2 < t_3$) such that $y(t) < 0$ for $t \in (t_1, t_2)$ and $y(t) > 0$ for $t \in (t_2, t_3)$. So there exist $b \in (t_1, t_2)$ and $c \in (t_2, t_3)$ such that $y'(b) = 0$, $y'(c) = 0$, $y'(t) > 0$ for

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$t \in (b, t_2)$ and $y'(t) > 0$ for $t \in (t_2, c)$. If $y''(t_2) \geq 0$, then integrating (E) from t_2 to c, we obtain

$$0 \geq r(c)y''(c) + p(c)y(c) - r(t_2)y''(t_2) = \int_{t_2}^{c} f(t)\,dt - \int_{t_2}^{c} q(t)y'(t)dt > 0,$$

a contradiction because $y''(c) \leq 0$. Furthermore, if $y''(t_2) < 0$ then integrating (E) from b to t_2 yields

$$0 > r(t_2)y''(t_2) - r(b)y''(b) - p(b)y(b) = \int_{b}^{t_2} f(t)\,dt - \int_{b}^{t_2} q(t)y'(t)\,dt > 0,$$

a contradiction, because $y''(b) \geq 0$. Hence $y(t)$ cannot be oscillatory. This completes the proof of the theorem.

THEOREM 1$'$. *If $\int_a^t p(\theta)\,d\theta \leq 0$ and $q(t) \leq \int_a^t p(\theta)\,d\theta$ for large t, then Equation* (4) *is non-oscillatory.*

PROOF. This result follows from Theorem 1.

REMARK. We note that $p(t) \leq 0$ implies $\int_a^t p(\theta)\,d\theta \leq 0$ but the converse is not necessarily true. Furthermore, $p(t) - q'(t) \geq 0$ implies $q(t) \leq \int_a^t p(\theta)\,d\theta$, if $q(a) \leq 0$. Hence Theorem 1$'$ improves Theorem 2.1 in [5].

THEOREM 2. *If $p(t) \geq 0$, $q(t) \leq 0$ and $p(s) + q(t) \leq 0$, for t and $s \in [a, \infty)$ and $p(s) + q(t) \not\equiv 0$ on any subinterval of $[a, \infty)$, then* (E) *is non-oscillatory.*

PROOF. Let $y(t)$ be a solution of (E) on $[a, \infty)$. If $y(t)$ is of non-negative Z-type with consecutive double zeros at t_1 and t_2 ($a < t_1 < t_2$), then there exists a point $b \in (t_1, t_2)$ such that $y'(b) = 0$ and $y'(t) > 0$ for $t \in (t_1, b)$. Since $y'' \geq 0$ and $y''(b) \leq 0$, then integrating (E) from t_1 to b, we obtain

$$\begin{aligned} 0 &\geq r(b)y''(b) - r(t_1)y''(t_1) \\ &\geq -p(b)y(b) - \int_{t_1}^{b} q(t)y'(t)\,dt \\ &\geq -\int_{t_1}^{b} [q(t) + p(b)]\, y'(t)\,dt > 0, \end{aligned}$$

a contradiction. If $y(t)$ is of non-positive Z-type with consecutive double zeros at t_1 and t_2 ($a < t_1 < t_2$), then there exists a point $b \in (t_1, t_2)$ such that $y'(b) = 0$ and

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$y'(t) > 0$ for $t \in (b, t_2)$. Clearly $y''(b) \geq 0$ and $y''(t_2) \leq 0$. So integrating (E) from b to t_2 yields

$$\begin{aligned} 0 &\geq r(t_2)y''(t_2) - r(b)y''(b) \\ &\geq p(b)y(b) - \int_b^{t_2} q(t)y'(t)\,dt \\ &\geq -\int_b^{t_2} [q(t) + p(b)]\,y'(t)\,dt > 0, \end{aligned}$$

a contradiction. Hence $y(t)$ cannot be of Z-type.

Suppose that $y(t)$ is oscillatory. Let t_1, t_2, t_3 $(a < t_1 < t_2 < t_3)$ be consecutive zeros of $y(t)$ such that $y'(t_1) \leq 0$ and $y'(t_2) \geq 0$ and $y'(t_3) \leq 0$. So there exist $b \in (t_1, t_2)$ and $c \in (t_2, t_3)$ such that $y'(t) > 0$ for $t \in (b, t_2)$ and $t \in (t_2, c)$. Clearly, $y''(b) \geq 0$ and $y''(c) \leq 0$. If $y''(t_2) \geq 0$, then integrating (E) from t_2 to c, we obtain

$$\begin{aligned} 0 &\geq r(c)y''(c) - r(t_2)y''(t_2) \\ &\geq -p(c)y(c) - \int_{t_2}^{c} q(t)y'(t)\,dt \\ &\geq -\int_{t_2}^{c} [q(t) + p(c)]\,y'(t)\,dt > 0, \end{aligned}$$

a contradiction. If $y''(t_2) < 0$, then integrating (E) from b to t_2, we get

$$\begin{aligned} 0 &> r(t_2)y''(t_2) - r(b)y''(b) \\ &\geq p(b)y(b) - \int_b^{t_2} q(t)y'(t)\,dt \\ &\geq \int_b^{t_2} [q(t) + p(b)]\,y'(t)\,dt > 0. \end{aligned}$$

This contradiction completes the proof of the theorem.

REMARK. The condition $p(s) + q(t) \leq 0$ for t and $s \in [a, \infty)$ is equivalent to $p(s) \leq |q(t)|$. Hence $0 \leq p(s) \leq K \leq |q(t)|$ for t and $s \in [a, \infty)$, where $K > 0$ is a constant, implies that $p(s) + q(t) \leq 0$.

THEOREM 2′. *If* $\int_a^t p(u)\,du \geq 0$, $q(t) \leq \int_a^t p(u)\,du$ *and* $\int_a^s p(u)\,du \leq \int_a^t p(u)\,du - q(t)$, *then Equation* (4) *is non-oscillatory.*

This follows from Theorem 2.

EXAMPLE. Consider

$$\left(2t^3 y'' + \frac{1}{t+2}y\right)' - 4ty' = 4t^2 + \frac{t(t+4)}{(t+2)^2}, \qquad t \geq 1. \tag{5}$$

Clearly $p(s) = 1/(s+2) \leq 1/3 < 4t = |q(t)|$ for $s, t \geq 1$. From Theorem 2 it follows that Equation (5) is non-oscillatory. In particular, $y(t) = t^2$ is a non-oscillatory solution of the equation. Note that Equation (5) may be written as

$$(2t^3 y'')' - \left(4t - \frac{1}{t+2}\right) y' - \frac{1}{(t+2)^2} y = 4t^2 + \frac{t(t+4)}{(t+2)^2}, \qquad t \geq 1.$$

Clearly, Theorem 2′ cannot be applied to (5). We note that

$$\int_1^t - \left[\frac{1}{(u+2)^2}\right] du = \frac{1}{t+2} - \frac{1}{3}.$$

However Theorems 2 and 2′ can be applied to the equation

$$(5t^4 y'' + 2y)' - 8y' = 40t^3 - 12t, \qquad t \geq 0,$$

which admits the non-oscillatory solution $y(t) = t^2$.

The proofs of the following two results are similar to the proofs of Theorem 2 and 2′ and hence will be omitted.

THEOREM 3. *If* $p(t) \leq 0, q(t) \geq 0$ *and* $p(t) + q(s) \leq 0$ *for* t *and* $s \in [a, \infty)$ *such that* $p(t) + q(s) \not\equiv 0$ *on any subinterval of* $[a, \infty)$, *then* (E) *is non-oscillatory.*

THEOREM 3′. *If* $\int_a^t p(u)\,du \leq 0$, $q(t) \geq \int_a^t p(u)\,du$ *and* $\int_a^t p(u)\,du \leq \int_a^s p(u)\,du - q(s)$, *then Equation* (4) *is non-oscillatory.*

Our last non-oscillation result for linear equations is contained in the following theorem

THEOREM 4. *Let* $p(t) \geq 0$ *and* $q(t) \geq 0$. *If* $\lim_{t\to\infty} f(t)/(p(s) + q(t)) = \infty$ *uniformly for* $s \geq a$, *then every solution of* (E) *whose first derivative is bounded is non-oscillatory.*

PROOF. Let $y(t)$ be a solution of (E) on $[a, \infty)$ such that $|y'(t)| \leq L$ for $t \geq a$. From the given hypothesis it follows that there exists a $T > a$, independent of s, such that $f(t) > L(p(s) + q(t))$ for $t \geq T$.

Suppose that $y(t)$ is of non-negative Z-type with consecutive double zeros at t_1 and t_2 $(T < t_1 < t_2)$. Then there exists $b \in (t_1, t_2)$ such that $y'(b) = 0$ and $y'(t) > 0$ for $t \in (t_1, b)$. Now integrating (E) from t_1 to b, we get

$$\begin{aligned} 0 &\geq r(b)y''(b) - r(t_1)y''(t_1) \\ &= -p(b)y(b) - \int_{t_1}^b q(t)y'(t)\,dt + \int_{t_1}^b f(t)\,dt \end{aligned}$$

$$= -\int_{t_1}^{b} [q(t) + p(b)]\, y'(t)\, dt + \int_{t_1}^{b} f(t)\, dt$$
$$\geq \int_{t_1}^{b} [f(t) - L(q(t) + p(b))]\, dt > 0,$$

a contradiction. Similar contradiction may be obtained in case $y(t)$ is non-positive Z-type or oscillatory. Hence the theorem is proved.

REMARK. The Liouville transformation transforms

(6) $$\left[r_2(t)\left((r_1(t)y')' + q_1(t)y^{\alpha}\right)\right]' + q_2(t)(r_1(t)y')^{\beta} = f_1(t),$$

where q_1, q_2, r_1, r_2 and f_1 are as in (NH) and each of $\alpha > 0$ and $\beta > 0$ is a quotient of odd integers, to an equation of the type

(7) $$\left(r(t)y'' + p(t)y^{\alpha}\right)' + q(t)(y')^{\beta} = f(t).$$

However, the Kummer transformation fails to do so.

THEOREM 5. *If* $p(t) \leq 0$ *and* $q(t) \leq 0$, *then* (7) *is non-oscillatory.*

The proof of this theorem is similar to that of Theorem 1 and hence is omitted.

REMARK. Theorems 1-5 all remain true if the condition, '$f(t) \geq 0$' is replaced by '$f(t) \leq 0$'.

Equations of the type

(8) $$y''' + yy'' + \lambda\left[1 - (y')^2\right] = 0$$

arise in boundary layer theory in fluid Mechanics cite[p. 520]2. The particular case of (8), $y''' + yy'' = 0$, is known as the Blasius equation. In the following we study the non-oscillatory behaviour of solutions of the non-homogeneous Blasius equation

(9) $$y''' + yy'' = f(t)$$

where $f \in C([a, \infty), R)$ is such that $f(t) \geq 0$.

THEOREM 6. *All solutions of Equation* (9) *are non-oscillatory.*

PROOF. Equation (9) may be written as

(10) $$[y'' + yy']' = (y')^2 + f(t).$$

Let $y(t)$ be a solution of (10) on $[a, \infty)$. Proceeding exactly as in Theorem 1, one may show that $y(t)$ cannot be of Z-type or oscillatory. Hence $y(t)$ is non-oscillatory.

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The following examples illustrate the theorem.

EXAMPLES.

(i) The equation $y''' + yy'' = 0$ admits both positive and negative solutions $y_1(t) = t$ and $y_2(t) = -t$,

(ii) The equation $y''' + yy'' = 8/t^4$, $t \geq 1$, admits the positive bounded solution $y(t) = 4/t$,

(iii) $y(t) = -e^{-t}$ is a bounded negative solution of

$$y''' + yy'' = e^{-t} + e^{-2t}, \qquad t \geq 0.$$

The asymptotic behaviour of solutions of Equation (8) has been studied by Hartman [2]. Equation (8) with $\lambda = 1/2$ is often called the Homann differential equation. In the following we obtain a theorem concerning non-oscillatory behaviour of solutions of non-homogeneous equation associated with Equation (8), that is,

$$y''' + yy'' + \lambda\left[1 - (y')^2\right] = f(t), \tag{11}$$

where $f \in C([a, \infty), R)$ is such that $f(t) \geq 0$.

THEOREM 7. *If* $-1 \leq \lambda < 0$ *then all solutions of Equation* (11) *are non-oscillatory. If* $\lambda > 0$ *and* $\lim_{t\to\infty} f(t) = \infty$, *then all solutions of Equation* (11) *are non-oscillatory. If* $\lambda < -1$ *and* $\lim_{t\to\infty} f(t) = \infty$ *then all solutions of Equation* (11) *whose first derivatives are bounded are non-oscillatory.*

PROOF. The equation (11) can be written as

$$\left(y'' + yy'\right)' = (1 + \lambda)(y')^2 + f(t) - \lambda.$$

In each case we see that the right-hand side of the above identity is positive for sufficiently large t. Then proceeding as in Theorem 1 we may show that all solutions of (11) are non-oscillatory. Hence the proof of the theorem is complete.

EXAMPLES.

(i) All solutions of

$$y''' + yy'' - \left[1 - (y')^2\right] = 6t^2 - 1, \qquad t \geq 1,$$

are non-oscillatory. In particular, $y(t) = t^2$ is a non-oscillatory solution of the equation.

(ii) The equation

$$y''' + yy'' + \left[1 - (y')^2\right] = 1 + \frac{7}{t^4}, \qquad t \geq 1,$$

is non-oscillatory with a particular non-oscillatory solution $y(t) = -1/t$.

(iii) The equation

$$y''' + yy'' + \left[1 - (y')^2\right] = 1 + e^t, \qquad t \geq 0,$$

is non-oscillatory. In particular, $y(t) = e^t$ is a non-oscillatory solution of the equation.

3. Relation between linearly independent solutions

In this section we study the relation between three linearly independent solutions of (E). Let $y_1(t)$, $y_2(t)$ and $y_3(t)$ be solutions of (E) with initial conditions

$$\begin{array}{lllllllll} y_1(a) & = & 0 & y_1'(a) & = & 1 & y_1''(a) & = & 0 \\ y_2(a) & = & 1 & y_2'(a) & = & 0 & y_2''(a) & = & -q(a)/r(a) \\ y_3(a) & = & 0 & y_3'(a) & = & 0 & y_3''(a) & = & 1/r(a) \end{array}$$

THEOREM 8. *If $p(t) \leq 0$, $q(t) \leq 0$ and $q'(t) \geq 0$, then $y_1(t)$ cannot meet $y_2(t)$ in the strip $[a, t_1)$, where t_1 is given by*

$$t_1 \geq 1 + a + p(a) \int_a^{t_1} \left(\int_a^s \frac{du}{r(u)} \right) ds.$$

PROOF. From Theorem 1 it follows that $y_1(t)$ and $y_2(t)$ are non-oscillatory. Successive integrations yield

$$\begin{aligned} y_1(t) = (t-a) &+ \int_a^t \left(\int_a^u \left(\frac{1}{r(s)} \int_a^s f(\theta)\, d\theta \right) ds \right) du \\ &- \int_a^t \left(\int_a^u \frac{1}{r(s)} (p(s) + q(s)) y_1(s)\, ds \right) du \\ &+ \int_a^t \left(\int_a^u \frac{1}{r(s)} \left(\int_a^s q'(\theta) y_1(\theta)\, d\theta \right) ds \right) du \end{aligned}$$

and

$$\begin{aligned} y_2(t) = 1 + p(a) \int_a^t \left(\int_a^s \frac{du}{r(u)} \right) ds &+ \int_a^t \left(\int_a^u \left(\frac{1}{r(s)} \int_a^s f(\theta)\, d\theta \right) ds \right) du \\ &- \int_a^t \left(\int_a^u \frac{p(s) + q(s)}{r(s)} y_2(s)\, ds \right) du \\ &+ \int_a^t \left(\int_a^u \left(\frac{1}{r(s)} \int_a^s q'(\theta) y_2(\theta)\, d\theta \right) ds \right) du. \end{aligned}$$

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If $t_1 > a$ is the first point where $y_1(t)$ meets $y_2(t)$, then $y_1(t_1) = y_2(t_1)$ and $y_1(t) < y_2(t)$ for $t \in [a, t_1)$. Thus $y_2(t_1) \geq 1 + p(a) \int_a^{t_1} (\int_a^s du/r(u))\, ds + y_1(t_1) - (t_1 - a)$, that is,

$$t_1 \geq 1 + a + p(a) \int_a^{t_1} \left(\int_a^s \frac{du}{r(u)} \right) ds.$$

Hence the theorem is proved.

REMARK. The conclusion of Theorem 8 holds if
(i) $p(t) \geq 0$, $q(t) \leq 0$, such that $p(t) + q(t) \leq 0$ and $q'(t) \geq 0$;
(ii) $p(t) \leq 0$, $q(t) \geq 0$ such that $p(t) + q(t) \leq 0$ and $q'(t) \geq 0$

However, if $p(t) \geq 0$, $q(t) \geq 0$ and $q'(t) \leq 0$, then $y_1(t)$ cannot meet $y_2(t)$ in the strip $[a, t_1)$, where t_1 is given by

$$t_1 \leq 1 + a + p(a) \int_a^{t_1} \left(\int_a^s \frac{du}{r(u)} \right) ds$$

THEOREM 9. *If $p(t) \leq 0$, $q(t) \leq 0$ and $q'(t) \geq 0$, then $y_3(t)$ cannot meet $y_1(t)$ in the strip (a, t_1), where t_1 is given by*

$$t_1 \leq a + \int_a^{t_1} \left(\int_a^s \frac{du}{r(u)} \right) ds$$

and $y_3(t)$ cannot meet $y_2(t)$ in the strip $[a, t_1)$, where t_1 is given by

$$1 \leq (1 - p(a)) \int_a^{t_1} \left(\int_a^s \frac{du}{r(u)} \right) ds$$

The proof of this theorem is similar to that of Theorem 8 and hence is omitted.

REMARK. The conclusion of the above theorem remains true if
(i) $p(t) \geq 0$, $q(t) \leq 0$, such that $p(t) + q(t) \leq 0$ and $q'(t) \geq 0$,
(ii) $p(t) \leq 0$, $q(t) \geq 0$ such that $p(t) + q(t) \leq 0$ and $q'(t) \geq 0$.

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