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SOME PROPERTIES OF VECTOR MEASURES TAKING VALUES IN A TOPOLOGICAL VECTOR SPACE

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Abstract

In this paper we study some properties of vector measures with values in various topological vector spaces. As a matter of fact, we give a necessary condition implying the Pettis integrability of a function $f\colon S \to E$, where S is a set and E a locally convex space.

Furthermore, we prove an iff condition under which (Q, E) has the Pettis property, for an algebra Q and a sequentially complete topological vector space E.

An approximating theorem concerning vector measures taking values in a Fréchet space is also given.

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Notations and terminology

We denote by S a non void set, Q (resp. Σ) an algebra (resp. σ-algebra) of subsets of S and E a real Hausdorff locally convex space.

A function μ from the algebra Q to E is said to be a finitely additive vector measure (or simply a vector measure) if $\mu(A_1 \cup A_2) = \mu(A_1) + \mu(A_2)$, whenever A_1, A_2 are disjoint members of Q.

If in addition $\mu(\bigcup_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} \mu(A_n)$ for all sequences (A_n) of pairwise disjoint members of Q with $\bigcup_{n=1}^{\infty} A_n \in Q$ in the topology of E, then μ is called a σ-additive vector measure. We say that μ is strongly bounded (s-bounded) iff $\lim_n \mu(A_n) = 0$ for every sequence (A_n) of mutually disjoint sets from Q.

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If μ is an E-valued vector measure on Q and P a seminorm on E, we shall define the P-semivariation $P(\mu)$ by $P(\mu)(A) = \sup\{P(\sum_{j=1}^{n} a_j \mu(A_j))\}$, $A \in Q$, where the supremum is taken over all disjoint sets $A_1, \dots, A_n$ from Q with $A = A_1 \cup \cdots \cup A_n$ and all scalars $a_1, \dots, a_n$ with $|a_i| \leqslant 1$ $(i = 1, 2, \dots, n)$. We say that the function $f\colon S \to E$ is weakly λ-summable with respect to measure $\lambda\colon Q \to [0, \infty)$ if $\int_A |x'f|\, d\lambda < \infty$ for all $x' \in E'$, A in Q. f is called λ-summable or Pettis integrable if it is weakly λ-summable for every A in Q and there exist an element $\int_A f\, d\lambda$, of E, such that

$$x' \int_A f\, d\lambda = \int_A x' f\, d\lambda, \qquad (x' \in E').$$

A locally convex space E has the *Bessaga-Pełczyński property* (shortly $(B$-$P)$-property), if for every sequence (x_n) from E with $\sum_{n=1}^{\infty} |x'(x_n)| < \infty$ for all $x' \in E'$, there exists $x \in E$ such that $x = \sum_{n=1}^{\infty} x_n$, where the series converges unconditionally.

Finally, a sequence $\{x_n\}$ in E is a Schauder basis if every $x \in E$ has a unique representation in the form $x = \sum_{n=1}^{\infty} a_n x_n$, where $\{a_n\}$ is a sequence of scalars. For each $n \in \mathbf{N}$ the nth coefficient functional f_n on E is defined by $f_n(x) = a_n$, for all $x \in E$ and so $\mu(A) = \sum_{n \in \mathbf{N}} f_n(\mu(A)) x_n = \sum_{n \in \mathbf{N}} \mu_n(A) x_n$, A in Q.

I. On Pettis integral

The purpose of this section is to extend a result of ([13], Theorem 1) to the case of vector measures which take values in a locally convex space E. This is given in 4. Theorem below.

1. LEMMA ([9], Proposition 1). *Let $\lambda\colon \Sigma \to [0, +\infty)$ be a measure and let $\mu\colon \Sigma \to E$ be a s-bounded vector measure with $x'\mu \ll \lambda$, for every $x' \in E'$. Then $\mu \ll \lambda$.*

2. LEMMA. *Let $f\colon S \to E$ be a vector function, $\nu\colon \Sigma \to E$ a vector measure and (s, Σ, λ) a finite non negative measure space. We denote by H the set $H = \{x' \in E'$: (i) $x'f \in L_1(\lambda)$ and (ii) $x' \circ \nu(A) = \int_A x' f\, d\lambda$ A in $\Sigma\}$. Then, for every $x' \in H$, there exist a continuous seminorm $P_{x'}$ on E such that*

$$\int_A |x'f|\, d\lambda \leqslant P_{x'}(\nu)(A), \qquad (A \text{ in } \Sigma).$$

PROOF. If $x' \circ \nu = \mu$, then $\mu(A) = \int_A x'f\,d\lambda$ and $\mathcal{U}(\mu, A) = \int_A |x'f|\,d\lambda$ (where $\mathcal{U}(\mu, A) \leqslant \|\mu\|(A)$ (where $\|\mu\|$ denotes the semivariation of μ), for if $A_1, \ldots, A_n$ are pairwise disjoint sets of Σ, then there exist complex numbers $a_1, \ldots, a_n$ with $|a_i| = 1$ $(i = 1, \ldots, n)$ such that

$$\sum_{i=1}^{n} |\mu(A_i)| = \left|\sum_{i=1}^{n} a_i\mu(A_i)\right| \leqslant \|\mu\|(A).$$

On the other hand,

$$\left|\sum_{i=1}^{n} a_i\mu(A_i)\right| = \left|x'\left(\sum_{i=1}^{n} a_i\nu(A_i)\right)\right| \leqslant \left|P_{x'}\left(\sum_{i=1}^{n} a_i\nu(A_i)\right)\right| \leqslant P_{x'}(\nu)(A)$$

for some continuous seminorm $P_{x'}$ on E, thus $\|\mu\|(A) \leqslant P_{x'}(\nu)(A)$. The results now follows.

3. LEMMA. *Let $f\colon S \to E$, $\lambda\colon \Sigma \to [0, +\infty)$ a σ-additive measure and $\nu\colon \Sigma \to E$ a λ-continuous s-bounded vector measure. Then the set*

$$H = \left\{x' \in E'\colon \text{(i) } x'f \in L_1(\lambda) \textit{ and } \text{(ii) } x' \circ \nu(A) = \int_A x'f\,d\lambda\right\}$$

is weak sequentially closed.*

PROOF. 2. Lemma implies that, for every $x' \in H$, there exists a continuous seminorm $P_{x'}$ on E such that

$$(1) \qquad \int_A |x'f|\,d\lambda \leqslant P_{x'}(\nu)(A), \qquad (A \text{ in } \Sigma).$$

Suppose $\{x'_n\}_{n=1}^{\infty}$ in H and $x'_n(x) \to x'(x)$ (for all $x \in H$). Since $\nu \ll \lambda$ we have that $P_{x'_n}(\nu) \ll \lambda$, $n = 1, 2, \ldots$.

In virtue of equality (1), we have $\lim_{\lambda(A)\to 0} \int_A |x'_n f|\,d\lambda = 0$ uniformly in $n \in \mathbf{N}$. Vitali's convergence theorem now says that $x'f \in L_1(\lambda)$, hence

$$\int_A x'f\,d\lambda = \int_A \lim_n (x'_n f)\,d\lambda = \lim_n \int_A x'_n f\,d\lambda = \lim_n x'_n\nu(A) = x'\nu(A)$$

and so $x' \in H$.

4. THEOREM. *Let $f\colon S \to E$, $\lambda\colon E \to [0, +\infty)$ a σ-additive measure and $\nu\colon \Sigma \to E$ a finite additive vector measure. Assume that:*

(i) *H is a weak* sequentially dense subset of E,*

(ii) *$x'f \in L_1(\lambda)$ (for all $x' \in H$),*

(iii) *$x'\nu(A) = \int_A x'f\,d\lambda$ (for all $A \in \Sigma$ and for all $x' \in H$).*

Then f is Pettis λ-integrable and

$$\nu(A) = (P)\int_A f\,d\lambda \qquad (A \in \Sigma)$$

PROOF. Assumption (iii) implies $x'\nu \ll \lambda$, for every $x' \in H$. Since H is a weak* sequentially dense subset of E', we have that $x'\nu \ll \lambda$, for every $x' \in E'$. Hence, $x'\nu$ is σ-additive for every $x' \in E'$ and thus ν is σ-additive by the Orlicz-Pettis theorem. Since Σ is a σ-algebra ν is also a s-bounded vector measure and from 1. Lemma we have that $\nu \ll \lambda$. 3. Lemma now implies that H is weak* sequentially closed and so $H = E'$. Hence we have that

$$x'\nu(A) = \int_A x'f\,d\lambda, \quad \text{for every } x' \in E',$$

which proves the assertion.

II. The Pettis property

If Q is a Boolean algebra and X is a Banach space, we shall say that the pair (Q, X) has the *Pettis property* if every weakly countably additive set function μ: $Q \to X$ is σ-additive. It is proved by [7] that a pair (Q, X) has the Pettis property, for every algebra Q, if and only if $X \not\supset c_0$. A generalization of this is 5. Theorem below for the case of a sequentially complete topological vector space.

5. THEOREM. *Let Q be an algebra of sets and let E be a q sequentially complete topological vector space. Then the following propositions are equivalent*:

(i) (Q, E) *has the Pettis property*,

(ii) E *has the* $(B\text{-}P)$*-property*.

PROOF. (i) ⇒ (i). We suppose that E does not have the (B-P)-property. Then, there exists a sequence (x_n) on E such that $\sum_{n=1}^{\infty}|x'(x_n)| < \infty$, for every $x' \in E'$ and the series $\sum_{n=1}^{\infty} x_n$ does not converge. From ([14], Theorem 4) now we have that c_0 is isomorphic to a subspace of E. But there exists a vector set function μ: $Q \to c_0$ which is weakly σ-additive but not σ-additive ([11], example 7).

(ii) ⇒ (i). Let μ: $Q \to E$ be weakly σ-additive and (A_n) a disjoint sequence of sets in Q with $\bigcup_{n=1}^{\infty} A_n \in Q$. Then $x'\mu(\bigcup_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} x'\mu(A_n)$ (the series converges unconditionally) for all $x' \in E'$. Hence $\sum_{n=1}^{\infty}|x'\mu(A_n)| < \infty$. Since E has the (B-P)-property, the series $\sum_{n=1}^{\infty}\mu(A_n)$ converges unconditionally and so, for $x' \in E'$, we have $x'(\sum_{n=1}^{\infty}\mu(A_n)) = \sum_{n=1}^{\infty} x'\mu(A_n) = x'\mu(\bigcup_{n=1}^{\infty} A_n)$ and $\sum_{n=1}^{\infty}\mu(A_n) = \mu(\bigcup_{n=1}^{\infty} A_n)$.

However, in the case of locally convex space with a Schauder basis, the σ-additivity of the measure, with respect to the topology, is equivalent to the σ-additivity of the real measures $\mu_n = f_n \circ \mu$, where the f_n are the functionals associated to the basis. As a matter of fact, one obtains

6. PROPOSITION ([8], PROPOSITION 2). *Let E be a locally convex space with a Schauder basis (x_n, f_n) and μ: $Q \to E$ a vector measure. Then the following are equivalent*:

(i) *μ is a σ-additive*,

(ii) *μ_n is σ-additive*, $(n \in \mathbf{N})$.

III. An approximation theorem for vector measures

Let E be a Fréchet space, $\mathscr{U}$ a fundamental system of neighbourhoods of zero in E (consisting of closed and absolutely convex sets) and $(P_v)_{v \in \mathscr{U}}$ the family of the Minkowski functionals.

The function f: $S \to E$ is called λ-integrable with respect to the measure λ: $\Sigma \to [0, +\infty)$, if f is strongly measurable and, for every $v \in \mathscr{U}$, we have $\int_s P_v(f)\, d\lambda < \infty$. We denote $L^1(S, \lambda, E)$ the quotient space $\mathscr{L}^1(S, \lambda, E)/n$, where $\mathscr{L}^1(S, \lambda, E)$ is the space of all λ-integrable functions $f{:}L\ S \to E$ and $n = \{f \in \mathscr{L}^1(S, \lambda, E)$ such that $q_v(F) = 0,\ v \in \mathscr{U}\}$. Note that $L^1(S, \lambda, E)$ is a Fréchet space with the topology defined by the family of seminorms q_v, $v \in \mathscr{U}$, where $q_v(f) = \int_s P_v(f)\, d\lambda$. Let μ: $\Sigma \to E$ be a vector measure. We say that μ is of bounded variation if

$$V(\mu, v)(S) = \sup\left\{ \sum_{i=1}^{n} P_v(\mu(S_i)),\ S_i \in \Sigma,\ S_i \subset S \text{ disjoint} \right\} < \infty$$

for every $v \in \mathscr{U}$.

We define the measure $\lambda_f(S) = \int_s f\, d\lambda$, for all $f \in L^1(S, \lambda, E)$, satisfying $V(\lambda_f, v)(S) = \int_s P_v(f)\, d\lambda$. It is a measure of bounded variation and satisfies ([3], page 372)

$$P_v(\lambda_f(S)) \leqslant \int_S P_v(f)\, d\lambda$$

We are able to state and prove the second main theorem.

7. THEOREM. *Let (S, Q, λ) be a finite (positive) measure space, E a Fréchet space with the Radon-Nikodym property and μ: $Q \to E$ an additive vector measure of bounded variation with $\mu \ll \lambda$. Then, there exist a sequence $\{\phi_n\}$ of simple functions ϕ_n: $S \to E$ such that*

$$P_v\left(\int_A \phi_n\, d\lambda - \mu(A) \right) \underset{n}{\to} 0$$

for every $A \in Q$ and for all $v \in \mathscr{U}$.

PROOF. By Stone's theorem ([5], Theorem 1) there exists a totally disconnected compact Hausdorff space K, for which the algebra $\hat{Q}$ of all open-closed subsets of K is isomorphic to the algebra Q. Let ϕ be the above isomorphism. We define $\hat{\mu}$: $\hat{Q} \to E$ by $\mu(\phi(A)) := \mu(A)$ and $\hat{\lambda}$: $\hat{Q} \to [0, +\infty)$ by $\hat{\lambda}(\phi(A)) := \lambda(A)$. $\hat{\lambda}$ is regular ([1], Theorem 2); therefore, $\hat{\lambda}$ is σ-additive ([6], Theorem 13, page 138), Hahn's extension theorem now implies that exists a unique extension of $\hat{\lambda}$ (denoted also by $\hat{\lambda}$) to the σ-algebra Σ_0 generated by $\hat{Q}$. We consider the standard metric on Σ_0, $d(E_1, E_2) = \hat{\lambda}(E \Delta E_2)$ and we denote the resulting metric space by $\Sigma_0(\hat{\Lambda})$. Recall that $\hat{Q}$ is then a dense subset of $\Sigma_0(\hat{\lambda})$ ([10], [13], Theorem D). Therefore, the function $\hat{\mu}$: $Q \to \Sigma_0(\hat{\lambda}) \to E$ is continuous (since $\mu \ll \lambda$ implies $\hat{\mu} \ll \hat{\lambda}$) and it has an extension, denoted also by $\hat{\mu}$, $\hat{\mu}$: $\Sigma_0(\hat{\lambda}) \to E$. Now, from Radon-Nikodym's theorem, there exists $\hat{f} \in L_1(\hat{\lambda}, \Sigma_0, E)$ such that

$$\hat{\mu}(A) = \int_A \hat{f} \, d\hat{\lambda} \quad (\text{for all } A \in \Sigma_0).$$

(This is denoted by $\hat{\mu} = \hat{f}\hat{\lambda}$.) Hence there exists a sequence $\hat{\phi}_n$ of simple functions converging to the function $\hat{f}$, that is,

$$q_v(\hat{\phi}_n - \hat{f}) = \int_S P_v(\hat{\phi}_n - \hat{f}) \, d\hat{\lambda} \to 0 \quad \text{for all } v \in \mathscr{U}. \tag{1}$$

We also have that

$$\hat{\lambda}_{(\hat{\phi} - \hat{f})}(S) = \int_S (\hat{\phi}_n - \hat{f}) \, d\hat{\lambda} = (\hat{\phi}_n - \hat{f})\hat{\lambda}(S) \tag{2}$$

is a vector measure of bounded variation.

From (1) and (2) we obtain

$$P_v\left(\hat{\lambda}_{\hat{\phi}_n - \hat{f}}(S)\right) = P_v\left(\int_S (\hat{\phi}_n - \hat{f}) \, d\hat{\lambda}\right) \leqslant \int_S P_v(\hat{\phi}_n - \hat{f}) \, d\hat{\lambda}.$$

Hence $P_v(\hat{\lambda}_{\hat{\phi}_n - \hat{f}}(A)) \to 0$, for all $A \in Q$, therefore $P_v[(\hat{\phi}_n - \hat{f})\hat{\lambda}(A)] \to 0$. So $P_v[\hat{\phi}_n \hat{\lambda}(A) - \hat{f}\hat{\lambda}(A)] \to 0$ and

$$P_v[\phi_n \lambda(A - \mu)(A)] \underset{n}{\to} 0, \quad \text{for all } A \in Q \text{ and } v \in \mathscr{U}.$$

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