A GENERALIZATION OF $z!$

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Summary

A generalised factorial function $(z:k)!$ is defined as an infinite product similar to the Euler product for $z!$, but with the sequences of integers replaced by the roots of $F(z) = \sin \pi z + k\pi z$. It is proved that, apart from poles in $\mathscr{R}(z) < 0$, $(z:k)!$ is analytic in both variables, and that $F(z)$ may be expressed in the form $F(z) = \pi z/(z:k)!(-z:k)!$

As $|z| \to \infty$, it is shown that the function satisfies a Stirling formula $(z:k)! \sim \sqrt{2\pi z}\, z^z e^{-z}$.

1. Introduction

Koiter [1] has used certain approximations in order to apply the Wiener-Hopf technique to mixed boundary value problems associated with the infinite strip in plane elasto-statics. It has been pointed out by Noble [2] that it is possible in these cases to obtain an exact solution provided the function

$$H(z) = \sinh z + kz$$

can be factorised into a product $H(z) = zH_+(z)H_-(z)$ where H_+ and H_- are regular and non-zero in the upper and lower half planes, respectively. However, to apply this method it is necessary to know the asymptotic behaviour of the factors H_+ and H_- for large $|z|$.

In this paper, such a factorisation is obtained in terms of a generalised factorial function $(z:k)!$ of two variables, defined by an infinite product somewhat similar to Euler's formula for the gamma function. It will be shown in Theorem 1 that this product represents an analytic function of both z and k. The important result that, as $|z| \to \infty$,

$$(z:k)! \sim \sqrt{(2\pi z)} z^z / e^{-z},$$

is given in Theorem 2.

It will be convenient to consider the function

$$F(z) = \sin \pi z + k\pi z, \tag{1}$$

which is obtained from H by trivial replacements.

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2. The factorisation of $F(z)$

The function

$$G(t) = \frac{\sin \pi\sqrt{t}}{\pi\sqrt{t}} + k$$

is an integral function of order $\frac{1}{2}$. Thus, if $M(r) = \max_{|t|=r} |G(t)|$, then

$$M(r) \leqq \frac{e^{\pi\sqrt{r}}}{\pi\sqrt{r}} + k < 2e^{\pi\sqrt{r}},$$

for large r. Hence, as $r \to \infty$, $\log M(r) = O(r^{\frac{1}{2}})$; and, it is easy to see, $\log M(r) = O(r^{\beta})$ does not hold for any $\beta < \frac{1}{2}$. By theorems due to Hadamard [3], $G(t)$ has an infinity of roots τ; for $\beta > \frac{1}{2}$ the infinite series

$$\sum_{\tau \neq 0} |\tau|^{-\beta}$$

converges; and, if $G(0) \neq 0$, i.e. $k \neq -1$,

$$G(t) = G(0) \prod_{\tau} \left(1 - \frac{t}{\tau}\right).$$

The infinite product converges absolutely, and uniformly in $|t| \leqq R$, for any $R > 0$.

If $G(0) = 0$, the same theorems, applied to $G(t)/t$, give

$$G(t) = G'(0) t \prod_{\tau} \left(1 - \frac{t}{\tau}\right),$$

where the infinite product is over the non-zero roots of $G(t)$.

Replacing t by z^2 and τ by ζ^2, we have

$$F(z) = \begin{cases} (1+k)\pi z \prod_{\zeta} \left(1 - \dfrac{z^2}{\zeta^2}\right), & k \neq -1, \\ -\dfrac{\pi^3}{6} z^3 \prod_{\zeta} \left(1 - \dfrac{z^2}{\zeta^2}\right), & k = -1, \end{cases} \tag{2}$$

where now, the products are taken over the non-zero roots $\zeta = \xi + i\eta$ of $F(z)$ with $\xi \geqq 0$. If k is real and $k < -1$ (and in this case only) $F(z)$ has purely imaginary roots; there are exactly two such roots, they are simple and conjugate. The product in (2) is then understood to contain a factor corresponding to one only of these two roots. The second formula in (2) follows formally from the first by taking the limit $k \to -1$, when one root ζ occurring in the product tends to 0, and

$$\frac{1+k}{\zeta^2} \to \frac{\pi^2}{6}.$$

Now define a function $(z:k)!$ by the limit

$$(3) \qquad \frac{1}{(z:k)!} = h(z,k) = (1+k)^{\frac{1}{2}} \lim_{X\to\infty} X^{-z} \prod_{0\leqq\xi<X} \left(1+\frac{z}{\zeta}\right).$$

It will be proved that this limit exists for all z and all k ($\neq -1$) and represents an analytic function of z and k, provided the k-plane is cut from -1 to $-\infty$. For $k=-1$ the definition is

$$(4) \qquad \frac{1}{(z:-1)!} = h(z:-1) = \frac{\pi z}{\sqrt{6}} \lim_{X\to\infty} X^{-z} \prod_{0<\xi<X} \left(1+\frac{z}{\zeta}\right).$$

We agree to regard (3) as double valued for real k, $k<-1$, i.e. on the cut in the k-plane. For such a $k=k_0$, $F(z)$ has the two roots $\pm\zeta_0$, which are purely imaginary, and we agree that the product (3) contains a factor corresponding to one only of these two roots. This ambiguity in the meaning of (3) corresponds to the two limiting values of (3) as k approaches the value k_0, from one or other of the two sides $\mathscr{I}(k)>0$, or $\mathscr{I}(k)<0$. If $\zeta_0 = i\eta_0$ is the root with positive imaginary part ($\eta_0>0$) it is easy to see that the choice of factor $(1+z/\zeta_0)$ in (3) corresponds to the approach $k\to k_0$ from $\mathscr{I}(k)>0$.

Thus, if ζ be the root near ζ_0 for k near k_0 we find

$$\frac{dk}{d\zeta} = k\left((\pi\cot\pi\zeta - \frac{1}{\zeta}\right).$$

For $k=k_0$, $\zeta=\zeta_0=i\eta_0$ this gives $dk=i\rho d\zeta$, where

$$\rho = -k_0\pi\left(\coth\pi\eta_0 - \frac{1}{\pi\eta_0}\right) > 0.$$

This means that as k moves from k_0 into $\mathscr{I}(k)>0$, so ζ moves from ζ_0 into $\xi>0$.

From (2), (3), (4)

$$(5) \qquad (z:k)!\,(-z:k)! = \frac{\pi z}{\sin\pi z + k\pi z},$$

or, equivalently

$$(6) \qquad F(z) = \pi z\, h(z:k)h(-z:k).$$

Obviously, $h(z)$ has no roots or poles in $\mathscr{R}(z)>0$, and $h(-z)$ has no roots or poles in $\mathscr{R}(z)<0$. This is then an explicit factorisation of the type sought.

In the products (2), (3) and (4) multiple roots of $F(z)$ are allowed for by a corresponding repetition of the factors. In fact (excepting for $k=-1$,

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when the triple root at $z = 0$ is the sole multiple root of $F(z)$), only double roots occur; more precisely, $F(z)$ has multiple roots only for a discrete set of values of k, these being all real and in the range $-1 < k < 1$. For each of these values of k, $F(z)$ has exactly two double roots $\pm\zeta$, and these are real. The product (3) contains then just one repeated factor. To prove these statements, let ζ be a multiple root of $F(z)$. Then $k = -\cos \pi\zeta$, and $\pi\zeta = \tan \pi\zeta$. The last equation has real roots only, and, for its different positive roots, the values of $\cos \pi\zeta$ are all different. Finally $F'''(\zeta) = k\pi^2\zeta \neq 0$, so that the multiple root ζ is actually a double root.

In order to establish the limit (3), it is sufficient to replace the continuous variable X by an increasing sequence of values X_n. We shall select the sequences $X_n = 2n+\frac{1}{2}$, and $X_n = 2n+\frac{3}{2}$, for $n = 0, 1, 2, 3, \cdots$. To treat the complete range of values of k, it will be necessary to consider both these sequence replacements for X. However, for a discussion of the limit (3), it is first necessary to obtain some results concerning the roots of $F(z)$.

3. The roots of $F(z)$

We prove three lemmas concerning the roots $\zeta = \xi+i\eta$. The first is concerned with showing that, for a root ζ, $|\eta|$ is 'not too large' compared with ξ. The others concern the way in which the roots ζ are related to the sequences X_n.

LEMMA 1. *For any* $\alpha > 0$, *as* $\xi \to \infty$,

(7) $$\eta = O(\xi^\alpha).$$

If $G > 0$, *be any positive number, then* (7) *holds uniformly with respect to* k *in* $|k| \leqq G$.

PROOF. Clearly, we may choose a constant $c = c(\alpha, G)$, such that, for $x > \pi c$, $y > x^\alpha$,

$$\tfrac{1}{2}e^y > 2Gy+2,$$

and

$$\tfrac{1}{2}e^{x^\alpha} > 2Gx.$$

Then, for $\pi z = x+iy$,

$$|2i\, F(z)| \geqq |e^{-i\pi z}|-|e^{i\pi z}|-|2k\pi z|,$$

where

$$|e^{-i\pi z}| > \tfrac{1}{2}e^{x^\alpha}+\tfrac{1}{2}e^y > 2Gx+2Gy+2,$$

$$|e^{i\pi z}| = e^{-y} < 1,$$

$$|2k\pi z| < 2Gx+2Gy,$$

so that $|2F(z)| > 1$. This means that, if $\zeta = \xi + i\eta$ is a root of $F(z)$, and $\xi > c$, we must have $\pi\eta < \pi(\xi)^\alpha$. A similar argument shows that $\pi\eta > -(\pi\xi)^\alpha$, and from these two inequalities, $\pi|\eta| < (\pi\xi)^\alpha$ for $\xi > c = c(\alpha, G)$. This proves the lemma.

LEMMA 2. *Suppose* $G > 0$, $0 < \varepsilon < \pi/2$.

(*i*) *Let* $X_n = 2n + \frac{1}{2}$. *We can find* $n_0 = n_0(\varepsilon, G)$ *such that, for* $|k| \leqq G$, $|\arg k| \leqq \pi - \varepsilon$, $F(z)$ *has exactly* $2n$ *roots in*

$$0 < \mathscr{R}(z) < X_n,$$

provided $n \geqq n_0$.

(*ii*) *Let* $X_n = 2n + \frac{3}{2}$. *We can find* $n_0 = n_0(\varepsilon, G)$ *such that, for* $|k| \leqq G$, $|\arg(-k)| \leqq \pi - \varepsilon$, *and* $k+1$ *not a negative number*, $F(z)$ *has exactly* $2n+1$ *roots in*

$$0 < \mathscr{R}(z) < X_n.$$

In both cases, for $n > n_0$, $F(z)$ *has just two roots* ζ, ζ^*, *in*

$$X_{n-1} < \mathscr{R}(z) < X_n.$$

PROOF. Consider the integral

$$I = \frac{1}{2\pi i}\int_\Gamma \frac{F'(z)}{F(z)}\,dz, \tag{8}$$

where Γ is the rectangle $PQRS$, indented at the origin, with sides $\mathscr{R}(z) = 0$,

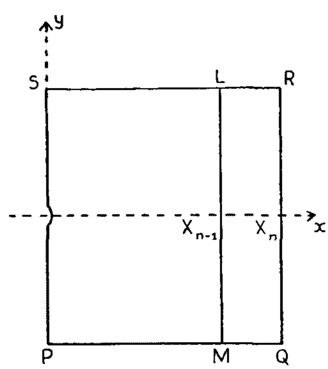

Fig. 1.

X_n, and $\mathscr{I}(z) = \pm y/\pi$, as illustrated in Figure 1. On the sides PQ, RS, as $y \to \infty$,

$$|F(z)| = |\sin \pi z + k\pi z| \sim \tfrac{1}{2}e^y,$$

so that for large y, $F(z)$ does not vanish on these sides. Also $F(z)$ does

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not vanish on SP nor, as we shall see, on QR for n sufficiently large. Then I is the number of roots of $F(z)$ inside Γ.

On PQ, RS, as $y \to \infty$,

$$\frac{F'(z)}{F(z)} = \frac{\pi \cos \pi z + k\pi}{\sin \pi z + k\pi z} = \mp i\pi + O(ye^{-y}), \tag{9}$$

the upper and lower signs corresponding to RS, PQ, respectively. Thus these two sides contribute altogether $X_n + O(ye^{-y})$ to the integral (7). Also the side SP, with indentation, contributes

$$-\frac{1}{2\pi i}\int_P^S \left(\frac{F'(z)}{F(z)} - \frac{1}{z}\right) dz - \frac{1}{2\pi i}\int_P^S \frac{dz}{z} = -\tfrac{1}{2}, \tag{10}$$

since the integrand in the first term changes sign with z, and the integral is therefore zero. The remaining side QR gives a contribution

$$\Big[\log F(z)\Big]_Q^R = \log \frac{F(z)}{F(\bar{z})},$$

where $\pi z = \pi X_n + iy$ and the logarithm is properly interpreted.

(i) Take $X_n = 2n + \frac{1}{2}$, and write $x = \pi X_n$. If

$$w = \frac{\sin \pi z}{\pi z} = \frac{x - iy}{x^2 + y^2} \cosh y = re^{i\theta},$$

we have

$$r = \sin \theta \, \frac{\cosh (x \tan \theta)}{x \tan \theta}, \tag{11}$$

where $y = -x \tan \theta$. For $x = (2n + \frac{1}{2})\pi$, (11) is the polar equation of the path of w when z describes the line QR. Then $y \lessgtr 0$ according as $\theta \gtrless 0$. The curve is symmetrical with respect to the real axis, and cuts it at the point $1/x$.

Suppose $|\arg k| \leqq \pi - \varepsilon$, $|k| \leqq G$, and that in Figure 2 the point K is $w = -k$ in the w plane. Then K lies somewhere in the sector $OACB$ shown in this figure, with $OA = G$, $AOU = \varepsilon$. Draw also the sector $OA'C'B'$, with $OA' = 2G$, and $A'OU = \frac{1}{2}\varepsilon$. Now let y_0 be the value which minimizes the function $y^{-1} \cosh y$. Choose x_0 so that $x_0 \tan \varepsilon/2 > y_0$, and so that, if r_0 is the value given by (11) for $x = x_0$, $\theta = \varepsilon/2$, then $r_0 > 2G$. Now any curve (11) with $x > x_0$ does not meet OA' or OB', and for $\theta > \varepsilon/2$ absolutely, the curve lies entirely outside the sector $OA'C'B'$ of the circle of radius $2G$. Thus we may choose $n_0 = n_0(\varepsilon, G)$ such that for all $n > n_0$, and, therefore, $x > x_0$, both $w = 0$ and $w = -k$ lie on the same side of the curve (11).

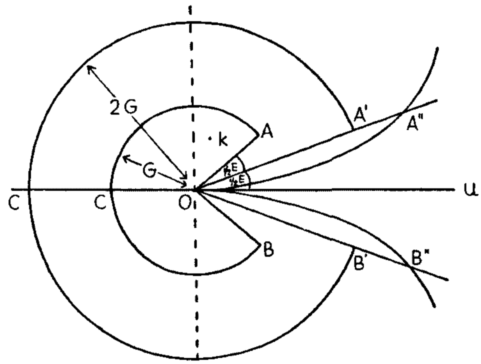

Fig. 2.

Thus, as $y \to \pm\infty$, both $\arg(w+k)$ and $\arg w$ have limits $\pm\pi/2$. Hence $\arg[(w+k)/w] \to 0$ as $y \to \pm\infty$. Then

(12) $$F(z) = \pi z(w+k) = \frac{w+k}{w}\cosh y,$$

so that $F(z)$ does not vanish on QR, and

(13) $$\arg F(z) \to 0 \quad \text{as} \quad y \to \pm\infty,$$

for $n > n_0$. From (12)

(14) $$\log F(z) = \log\cosh y + \log\left(1+\frac{k}{w}\right),$$

where the logarithms on the right are principal values when $|y|$ is sufficiently large. Hence, as $y \to \infty$,

(15) $$\log\frac{F(z)}{F(\bar z)} = \log\left(\frac{1+k/w}{1+k/\bar w}\right) = O\left(\frac{1}{|w|}\right) = O(ye^{-y}).$$

From (9), (10), and (15) we find that

$$I = X_n - \tfrac{1}{2} + O(ye^{-y}).$$

We infer that the 'error' term is zero if y is sufficiently large and $F(z)$ has exactly $X_n - \frac{1}{2} = 2n$ roots in $0 < \mathscr{R}(z) < X_n$.

This result holds for all $n > n_0$, so that under the same conditions, $F(z)$ has exactly two roots ζ, ζ^*, in the strip $X_{n-1} < \mathscr{R}(z) < X_n$.

(ii) The same calculations apply when $X_n = 2n+\frac{3}{2}$. In this case we set

$$re^{i\theta} = w = -\frac{\sin\pi z}{\pi z}.$$

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The path of w is still given by (11), but now

$$F(z) = -\pi z(w-k).$$

In the discussion we suppose $|\arg(-k)| \leqq \pi-\varepsilon$, and refer to the same Figure 2, but now K is the point $w = k$. Equations (12) to (15) still hold, provided that k is replaced by $-k$ and $F(z)$ by $-F(z)$. It follows now also that

$$I = X_n - \tfrac{1}{2}.$$

For $n > n_0$, there are $2n+1$ roots in the strip $0 < \xi < X_n$, and exactly two roots ζ, ζ^* in the strip $X_{n-1} < \xi < X_n$. In the case $k+1$ real and negative, one of the $2n+1$ roots counted lies on the imaginary axis.

LEMMA 3. *If ζ, ζ^* are the two roots of $F(z)$ in the strip $X_{n-1} < \xi < X_n$, then, as $n \to \infty$,*

$$\zeta+\zeta^* = X_{n-1}+X_n+O\left(\frac{\log n}{n}\right), \tag{16}$$

uniformly with respect to k in $|k| \leqq G$ and (i) *for the sequence $X_n = 2n+\frac{1}{2}$, in $|\arg k| \leqq \pi-\varepsilon$;* (ii) *for the sequence $X_n = 2n+\frac{3}{2}$, in $|\arg(-k)| \leqq \pi-\varepsilon$.*

PROOF. (i) Let $X_n = 2n+\frac{1}{2}$, and take $n > n_0(\varepsilon, G)$. In Figure 1, if y is sufficiently large, $F(z)$ is not zero on the rectangular contour $MQRL$, $\mathscr{R}(z) = X_{n-1}, X_n$; $\mathscr{I}(z) = \pm y/\pi$, and, therefore,

$$\zeta+\zeta^* = \frac{1}{2\pi i}\int z\frac{F'(z)}{F(z)}\,dz,$$

taken round this contour. On the horizontal side RL, we find, using (9), that

$$\begin{aligned}\frac{1}{2\pi i}\int_R^L z\frac{F'(z)}{F(z)}\,dz &= -\tfrac{1}{2}\int_R^L zdz+O(y^2e^{-y}),\\ &= \tfrac{1}{2}[X_{n-1}+X_n]+iy/\pi+O(y^2e^{-y}),\end{aligned}$$

and, there is a similar contribution from the lower side MQ. The two sides RL, MQ, together give a contribution to the above integral of

$$X_{n-1}+X_n+O(y^2e^{-y}).$$

Now consider

$$\int_Q^R z\frac{F'(z)}{F(z)}\,dz = \Big[z\log F(z)\Big]_Q^R - \int_Q^R \log F(z)dz. \tag{17}$$

Using (14),

$$\begin{aligned}\Big[z\log F(z)\Big]_Q^R &= X_n\log\frac{F(z)}{F(\bar{z})} + \frac{iy}{\pi}\log F(z)F(\bar{z}),\\ &= 2\log(\cosh y)+O(y^2e^{-y}),\end{aligned}$$

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as $y \to \infty$, where the logarithms take their principal values. Since $X_n - X_{n-1} = 2$, $F(z-2) - F(z) = -2k\pi$, we find that, as $y \to \infty$,

$$\left(\int_Q^R - \int_M^L\right) z \frac{F'(z)}{F(z)} dz = -\int_Q^R \log F(z) dz + \int_Q^R \log [F(z) - 2k\pi] dz + O(y^2 e^{-y}),$$

$$= \int_Q^R \log \left[1 - \frac{2k\pi}{F(z)}\right] dz + O(y^2 e^{-y}). \tag{18}$$

Thus, adding all these contributions to the integral, and letting $y \to \infty$,

$$\zeta + \zeta^* = X_{n-1} + X_n + \frac{1}{2\pi^2} \int_{-\infty}^{\infty} \log \left[1 - \frac{2k\pi}{F(z)}\right] dy, \tag{19}$$

where $\pi z = \pi X_n + iy = x + iy$.

Returning to Figure 2, suppose OA' meets the curve (11) at A'', so that A'' separates the curve into two parts. For one part, the distance from K to this part exceeds G, so, for w on it,

$$|w+k| \geqq G \geqq |k|.$$

The other part is separated from K by the line OA', so the distance from K to any point on it exceeds the distance from K to OA', and hence $|w+k| \geqq |k| \sin \varepsilon/2$. We have supposed that K is, as marked, in the sector AOC. But by symmetry, the same inequalities hold for K in the sector BOC. Thus, for $x > x_0$, and any w on the curve (11),

$$\left|\frac{k}{w+k}\right| < \operatorname{cosec} \tfrac{1}{2}\varepsilon. \tag{20}$$

This inequality holds for $n > n_0(\varepsilon, G)$, and $|k| \leqq G$, $|\arg k| \leqq \pi - \varepsilon$. Thus, in (19),

$$\left|\frac{2k\pi}{F(z)}\right| = \left|\frac{2k}{z(w+k)}\right| < \frac{2}{X_n}\left|\frac{k}{w+k}\right| < \frac{2 \operatorname{cosec} \varepsilon/2}{X_n}. \tag{21}$$

We may suppose X_n sufficiently large in (21) so that $|2k\pi/F(z)| < \frac{1}{2}$. Then the logarithm in (19) must represent the principal value in the whole range $-\infty < y < \infty$. Hence, from the logarithmic series expansion, and (21)

$$\left|\log \left[1 - \frac{2\pi k}{F(z)}\right]\right| < \left|\frac{4k\pi}{F(z)}\right| < \frac{4 \operatorname{cosec} \varepsilon/2}{X_n}. \tag{22}$$

We can now estimate the integral in (19). Take $y > \log x^2$, then

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$$\left|\frac{F(z)}{k}\right| = \left|\frac{\cosh y}{k} + x + iy\right| > \frac{\cosh y}{G} - x - y,$$
$$\geqq \frac{e^y}{2G} - x - y \geqq \frac{e^y}{6G} + \left(\frac{e^{\log x^2}}{6G} - x\right) + \left(\frac{e^y}{6G} - y\right),$$
$$> \frac{e^y}{6G},$$

for $x > c_1$, a suitable constant depending on G only. Then

$$\left|\int_{\log x^2}^{\infty} \log\left[1 - \frac{2k\pi}{F(z)}\right] dy\right| < 24G\pi \int_{\log x^2}^{\infty} e^{-y} dy = \frac{24\pi G}{x^2},$$

if $x > c_1$. The same estimate applies to the integral over the range $-\infty$ to $-\log x^2$.

Also, from (22),

$$\left|\int_{-\log x^2}^{\log x^2} \log\left[1 - \frac{2k\pi}{F(z)}\right] dy\right| < \frac{8 \operatorname{cosec} \varepsilon/2}{X_n} \log x^2.$$

Since $x = \pi X_n$, we obtain from these two inequalities,

$$\int_{-\infty}^{\infty} \log\left[1 - \frac{2k\pi}{F(z)}\right] dy = O\left(\frac{\log n}{n}\right), \tag{23}$$

as $n \to \infty$. Now (16) follows from (19) and (23).

(ii) Let $X_n = 2n + \frac{3}{2}$, $|\arg(-k)| \leqq \pi - \varepsilon$. Now we set

$$w = re^{i\theta} = -\frac{\sin \pi z}{\pi z}.$$

Then the proof of (19) and hence (16) follows exactly as in (i). In figure 2, K is now the point $w = k$ and all the formulae in (i) hold if we replace k by $-k$ and $F(z)$ by $-F(z)$.

4. Some properties of $(z : k)!$

THEOREM 1. *Except for a branch point at $k = -1$, the function $h(z, k) = [(z : k)!]^{-1}$, defined in* (3), *is an analytic function of z, k, for all values of these arguments.*

PROOF. Let

$$\Pi_n = (1+k)^{\frac{1}{2}} X_n^{-z} \prod_{0<\xi<X_n} \left(1 + \frac{z}{\zeta}\right). \tag{24}$$

Take $|z| \leqq R$, $|k| \leqq G$ and $|\arg(\pm k)| \leqq \pi - \varepsilon$ according as $X_n = 2n + \frac{1}{2}$,

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or $X_n = 2n+\frac{3}{2}$. Choose $n_1 > n_0(\varepsilon, G)$, such that $R/X_{n_1} < \frac{1}{2}$. Also, let

$$\sigma_r = \log X_r - \sum_{0<\xi<X_r} \frac{1}{\zeta}. \tag{25}$$

Now we may write, for $q > p \geqq n > n_1$,

$$\log \frac{\Pi_q}{\Pi_p} = -z(\sigma_q - \sigma_p) + \sum_{X_p \leqq \xi < X_q} \left[\log\left(1+\frac{z}{\zeta}\right) - \frac{z}{\zeta}\right]. \tag{26}$$

In the summation $|z/\zeta| \leqq R/X_p < \frac{1}{2}$, and $\log(1+z/\zeta)$ is understood as the principal value; this of course implies the appropriate meaning for a logarithm on the left. By Lemma 2, as $n \to \infty$, and uniformly with respect to z and k,

$$\sum_{X_p \leqq \xi < X_q} \left[\log\left(1+\frac{z}{\zeta}\right) - \frac{z}{\zeta}\right] = \sum_{X_p \leqq \xi < X_q} O\left(\frac{1}{\zeta^2}\right) = O\left(\frac{1}{n}\right). \tag{27}$$

Also, in accordance with lemma 2, if ζ_t and ζ_t^* be the two roots ζ in $X_{t-1} < \xi < X_t$, then

$$\sigma_q - \sigma_p = \sum_{t=p+1}^{q} \left(\frac{1}{t} - \frac{1}{\zeta_t} - \frac{1}{\zeta_t^*}\right) + O\left(\frac{1}{n}\right).$$

By lemma 1, which $0 < \alpha < 1$, ζ_t and ζ_t^* are equal to $2t + O(t^\alpha)$ and hence $\sigma_q - \sigma_p = O(1/n^{1-\alpha})$. Then, from (26) and (27), as $n \to \infty$,

$$\log \frac{\Pi_q}{\Pi_p} \to 0,$$

uniformly for $q > p \geqq n$ and z, k, restricted in the manner specified.

By the general principle of convergence, the sequence $\log(\Pi_n/\Pi_{n_1})$ and hence also the sequence Π_n/Π_{n_1} converges as $n \to \infty$ and uniformly with respect to z and k. Moreover each term of this sequence is an analytic function of z and k; this is obvious for z, and it is clear also for k when we note that Π_n/Π_{n_1} involves symmetrically all the roots ζ of $F(z)$ with $X_{n_1} \leqq \xi < X_n$ and that there are no roots on the bounding line $\xi = X_{n_1}$. Thus the limit Π_∞/Π_{n_1} is an analytic function of z and k.

For $X_n = 2n+\frac{1}{2}$ it is clear that Π_{n_1} is analytic in z, k, in $|z| \leqq R$, $|k| \leqq G$, $|\arg k| \leqq \pi - \varepsilon$.

For $X_n = 2n+\frac{3}{2}$, Π_{n_1} is analytic in z, k, in $|z| \leqq R$, $|k| \leqq G$, $|\arg(-k)| \leqq \pi - \varepsilon$ provided we add the additional restriction that $k+1$ be not zero or a negative number. This is because the first factor in Π_{n_1} changes discontinuously as k crosses the cut from -1 to $-\infty$ in the k plane.

These results, for the two sequences X_n, taken together show that the limit (3) exists and represents an analytic function of z, k as stated in the theorem.

From the proof just given it is clear that σ_r, defined by (25), tends to a limit as $r \to \infty$, which is an analytic function of k in the cut plane. We may therefore define a generalised Euler' 'constant' γ_k by

$$\gamma_k = \lim_{r\to\infty} \left[\sum_{0<\xi<r} \frac{1}{\zeta} - \log r \right]. \tag{28}$$

Then $(z:k)!$ may be represented by the generalised Weierstrass products

$$\begin{aligned} \frac{1}{(z:k)!} &= (1+k)^{\frac{1}{2}} e^{\gamma_k z} \prod_{0<\xi<\infty} \left(1+\frac{z}{\zeta}\right) e^{-z/\zeta}, & k \neq -1, \\ &= \frac{\pi z}{\sqrt{6}} e^{\gamma_{-1} z} \prod_{0<\xi<\infty} \left(1+\frac{z}{\zeta}\right) e^{-z/\zeta}, & k = -1. \end{aligned} \tag{29}$$

From (29),

$$\begin{aligned} (0:k)! &= (1+k)^{-\frac{1}{2}}, & k \neq -1, \\ (z:-1)! &\sim \sqrt{6}/\pi z, & \text{as} \quad z \to 0. \end{aligned} \tag{30}$$

And, if $\psi(z;k) = d/dz(z:k)!$,

$$-\frac{\psi(z:k)}{(z:k)!} = \gamma_k + \sum_{0<\xi<\infty} \left(\frac{1}{z+\zeta} - \frac{1}{\zeta}\right).$$

In particular

$$\psi(0:k) = -\frac{\gamma_k}{(1+k)^{\frac{1}{2}}}. \tag{31}$$

5. Stirling's formula for $(z:k)!$

THEOREM 2. *As* $|z| \to \infty$,

$$(z:k)! \sim \sqrt{2\pi z}\, z^z e^{-z}, \tag{32}$$

uniformly with respect to $\arg z$ *in* $|\arg z| \leqq \pi-\delta$, *and uniformly with respect to* k *in* $|k| \leqq G$.

PROOF. Set $\lambda_n = \frac{1}{2}(X_{n-1}+X_n)$ and define

$$\phi(z) = \lim_{n\to\infty} X_n^{-z} \prod_{r=1}^{n} \left(1+\frac{z}{\lambda_r}\right)^2. \tag{33}$$

Naturally $\phi(z)$ depends on which of the two sequences X_n is taken.

Then as $|z| \to \infty$, uniformly in $|\arg z| \leqq \pi-\delta$,

$$z!\,\phi(z) \sim \begin{cases} [(-\frac{1}{4})!]^2/\sqrt{\pi}, & \text{for} \quad X_n = 2n+\frac{1}{2}, \\ [(\frac{1}{4})!]^2/z\sqrt{\pi}, & \text{for} \quad X_n = 2n+\frac{3}{2}. \end{cases} \tag{34}$$

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From (3),

$$\frac{1}{(z:k)!\,\phi(z)} = (1+k)^{\frac{1}{2}} \lim_{n\to\infty} \frac{\prod_{0<\xi<X_n}\left(1+\frac{z}{\zeta}\right)}{\prod_{r=1}^{n}\left(1+\frac{z}{\lambda_r}\right)^2}. \tag{35}$$

Take $p > n_0(\varepsilon, G)$ and, for the moment, ignore the earlier factors of the products in (35). Thus consider

$$\prod_{p+1}^{n} \frac{\left(1+\frac{z}{\zeta}\right)\left(1+\frac{z}{\zeta^*}\right)}{\left(1+\frac{z}{\lambda_r}\right)^2}, \tag{36}$$

where, in accordance with lemma 2, ζ and ζ^* are the roots of $F(z)$ in $X_{r-1} < \xi < X_r$. In the factor of (36) with $\lambda = \lambda_r$, write

$$\zeta = \lambda + h, \quad \zeta^* = \lambda + h^*.$$

By lemma 1, with any selected α, $0 < \alpha < \frac{1}{2}$, h and h^* are both $O(r^\alpha)$. By lemma 3, $h+h^* = \zeta+\zeta^*-2\lambda = O(\log r/r)$. Here the O-symbols are uniform with respect to k in $|k| \leqq G$ and $|\arg(\pm k)| \leqq \pi-\varepsilon$, according as $X_n = 2n+\frac{1}{2}$ or $X_n = 2n+\frac{3}{2}$.

Now

$$\begin{aligned}\frac{\left(1+\frac{z}{\zeta}\right)\left(1+\frac{z}{\zeta^*}\right)}{\left(1+\frac{z}{\lambda}\right)^2} &= \frac{1+\frac{\lambda}{z+\lambda}\cdot\frac{h+h^*}{\lambda}+\left(\frac{\lambda}{z+\lambda}\right)^2\frac{hh^*}{\lambda^2}}{1+\frac{h+h^*}{\lambda}+\frac{hh^*}{\lambda^2}}, \\ &= 1+O\left(\frac{1}{r^{2-2\alpha}}\right),\end{aligned} \tag{37}$$

since $\lambda/z+\lambda$ is bounded in $\lambda > 0$, $|\arg z| \leqq \pi-\delta$. The O-term is uniform for $|\arg z| \leqq \pi-\delta$ and $|k| \leqq G$, $|\arg(\pm k)| \leqq \pi-\varepsilon$. It follows that the infinite product

$$\prod_{r=p+1}^{\infty} \frac{\left(1+\frac{z}{\zeta}\right)\left(1+\frac{z}{\zeta^*}\right)}{\left(1+\frac{z}{\lambda_r}\right)^2}$$

converges to an analytic limit $\phi_1(z, k)$; and as $|z| \to \infty$ in $|\arg z| \leqq \pi-\delta$

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$$\phi_1(z, k) \to \theta_1(k) = \prod_{r=p+1}^{\infty} \frac{\lambda_r^2}{\zeta\zeta^*},$$

uniformly with respect to k.

Now if we set

$$\theta_2(k) = (1+k)^{\frac{1}{2}} \frac{\prod_{r=1}^{p} \lambda_r^2}{\prod_{0<\xi<X_p} \zeta},$$

and recall that, according to lemma 2, there are either $2p$ or $2p+1$ roots ζ with $< \xi < X_p$, we have from (35), letting $|z| \to \infty$ in $|\arg z| \leqq \pi-\delta$,

$$\frac{1}{(z:k)!\,\phi(z)} \sim \begin{cases} \theta_1(k)\theta_2(k), & X_n = 2n+\frac{1}{2}, \\ z\theta_1(k)\theta_2(k), & X_n = 2n+\frac{3}{2}, \end{cases}$$

uniformly in $|k| \leqq G$, $|\arg(\pm k)| \leqq \pi-\varepsilon$.

Combining this with (34), as $|z| \to \infty$,

$$\frac{z!}{(z:k)!} = \frac{z!\,\phi}{(z:k)!\,\phi} \sim \begin{cases} \dfrac{[(-\frac{1}{4})!]^2}{\sqrt{\pi}}\,\theta_1(k)\,\theta_2(k), \\[2ex] \dfrac{[(\frac{1}{4})!]^2}{\sqrt{\pi}}\,\theta_1(k)\,\theta_2(k), \end{cases} \tag{38}$$

with the same uniformity as that just specified. Since the two statements in (38) hold for a common range of values of arg k so the two expressions in (38) are identical. There is of course no contradiction here since the functions θ_1, θ_2, like ϕ, are defined differently for the two sequences X_n. Thus we may write

$$\frac{z!}{(z:k)!} \sim C(k),$$

as $|z| \to \infty$ uniformly for $|\arg z| \leqq \pi-\delta$, $|k| \leqq G$. And, of course, $C(k)$ is an analytic function of k.

From (5),

$$\frac{\sin \pi z + k\pi z}{\sin \pi z} = \frac{z!\,(-z)!}{(z:k)!\,(-z:k)!},$$

and, letting $|z| \to \infty$ along (say) the imaginary axis, we have

$$1 = [C(k)]^2.$$

Since $C(0) = 1$, so $C(k) \equiv 1$. This shows that $(z:k)!$ behaves asymptotically like $z!$ and Theorem 2 follows from Stirling's formula.

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