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TWILLS WITH BOUNDED FLOAT LENGTH

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The simple twills on n harnesses can be classified according to their maximum float length. The number of n-harness twills with specified maximum float length is determined both by Burnside enumeration and, for $n \le 20$, by an adaptation of a sieve algorithm for twills.

Introduction

Every simple twill on n harnesses corresponds to an equivalence class of cyclic binary sequences of length n, where two such sequences $S = (s_1, s_2, \ldots, s_n)$ and $T = (t_1, t_2, \ldots, t_n)$ are equivalent if and only if one can be transformed into the other by a shift $(s_i = t_{i+1})$, by reversal $(s_i = t_{n+1-i})$, by complementation $(s_i = \sim t_i)$, or by some finite sequence of these operations. (Subscripts are added modulo n.) In other words, S and T are equivalent under the action of $D_{2n} \times S_2$, the direct product of the dihedral group of order $2n$ with the symmetric group of degree 2.

Conversely, every equivalence class of cyclic binary sequences of

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length n corresponds to a simple twill on n harnesses, except for the all zero (or all one) sequence.

This correspondence between twills and sequences is explained in Grünbaum and Shephard [2], Hoskins [3] and Hoskins and Street [4], for example; see also the references cited in [4]. It has been used to determine the total number of twills on n harnesses, both by Burnside enumeration [1, p. 191] and by a sieving algorithm, and to determine the number of twills on n harnesses with certain special properties. A further case is now considered.

A sequence with $s_i \neq s_{i+1} = \ldots = s_{i+k} \neq s_{i+k+1}$ is said to have a *float* of length k, that is, a block of k consecutive symbols which are equal. Since the maximum float length is an important property of a twill, the number of twills on n harnesses with given maximum float length are determined here.

The maximum float length is obviously closely related to the number of *breaks* in the sequence, where $(s_1, s_2, \ldots, s_n)$ has m breaks if and only if $s_i \neq s_{i+1}$ for precisely m distinct values of $i = 1, \ldots, m$. For example, the sequences 000111 and 00100111 both have maximum float length three, and have two and four breaks respectively. Note that the number of breaks must always be even.

In order to state our results, we introduce the following notation.

We denote by $F(n, k)$ the number of equivalence classes of binary sequences of length n, with maximum float length k, and by $F(n, k, m, x)$ the number of classes of such sequences with exactly m breaks and exactly x floats of length k. Then

$$F(n, k) = \sum F(n, k, m, x) ,$$

where the summation is over all m and x satisfying the following conditions:

(a) m is even, and

$$\lceil n/k \rceil + \delta \leq m \leq n - k + \varepsilon ,$$

$$\delta, \varepsilon = 0 \text{ or } 1 , \quad \delta \equiv \lceil n/k \rceil \pmod 2 , \quad \varepsilon \equiv n - k \pmod 2 ;$$

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(b) if $n = kq + r$, $0 \le r \le k-1$, then

$$1 \le x \le \begin{cases} q & \text{if } r \ge 2 , \\ & \text{or if } r = 1 , q \text{ odd,} \\ & \text{or if } r = 0 , q \text{ even;} \\ q - 1 & \text{if } r = 1 , q \text{ even,} \\ & \text{or if } r = 0 , q \text{ odd;} \end{cases}$$

(c) $x \le m \le n - kx + x - \varepsilon$, $\varepsilon \equiv n - kx + x \pmod 2$, $\varepsilon = 0$ or 1 , and if $m = x$, then $k|n$, n/k is even, and $x = n/k$.

Similarly, we denote by $F(n, k, -, x)$ the sum of $F(n, k, m, x)$ over all m satisfying the conditions stated above, for given x .

Note that we assume $k < n$, for if $k = n$, then $m = 0$, no corresponding twill exists, and condition (a) becomes $2 \le m \le 0$. Hence $F(n, n)$ is not defined.

Let $S(n, k, m, x)$ denote the set of binary sequences of length n , with maximum float length k , m breaks, and x floats of length k , and let $\mathrm{s} \in S(n, k, m, x)$, where $\mathrm{s} = (s_1, \ldots, s_n)$. We make the convention that $s_n \ne s_1$ (which is always possible since $k < n$), and we can associate with s the sequence of positive integers

$$\mathrm{r(s)} = (r_1, \ldots, r_m) ,$$

where $s_1 = \ldots = s_{r_1} \ne s_{r_1+1} = \ldots = s_{r_1+r_2} \ne s_{r_1+r_2+1} = \ldots$. (In the traditional break notation of weaving, this would be written $\frac{r_1 \quad r_3}{r_2 \quad r_4} \ldots$.) Thus, for example, the sequence

$$\mathrm{s} = (00101) \in S(5, 2, 4, 1)$$

has associated sequence

$$\mathrm{r(s)} = (2, 1, 1, 1) .$$

Let $R(n, k, m, x)$ be the set of positive integer sequences of length m , where

$$\mathrm{r} = (r_1, \ldots, r_m) \in R(n, k, m, x)$$

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satisfies

$$1 \leq r_i \leq k \ , \ \text{for} \ \ i = 1, 2, \ldots, m \ ,$$

$$r_i = k \ \ \text{for exactly} \ \ x \ \ \text{values of} \ \ i \ ,$$

and

$$\sum_{i=1}^{m} r_i = n \ .$$

(Note that $\mathbf{r}$ is actually a composition of n into m parts, where x parts equal k, and $m - x$ parts are less than k.) Then each $\mathbf{s} \in S(n, k, m, x)$ corresponds to $\mathbf{r}(\mathbf{s}) \in R(n, k, m, x)$, and conversely each $\mathbf{r} \in R(n, k, m, x)$ corresponds to exactly two sequences $\mathbf{s}, \mathbf{s}' \in S(n, k, m, x)$, where one is the complement of the other. Thus these two sequences, $\mathbf{s}$ and $\mathbf{s}'$, are equivalent in $S(n, k, m, x)$.

The equivalence relation on sequences in $S(n, k, m, x)$, induced by the action of $D_{2n} \times S_2$, corresponds to an equivalence relation on $R(n, k, m, x)$, where two sequences in $R(n, k, m, x)$ are equivalent if and only if one can be transformed into the other by a cyclic shift, σ, by a reversal ρ, or by some finite sequence of these operations.

If $\mathbf{r} = (r_1, r_2, \ldots, r_{m-1}, r_m)$, then $\mathbf{r}\sigma = (r_m, r_1, r_2, \ldots, r_{m-1})$, and $\mathbf{r}\rho = (r_m, r_{m-1}, \ldots, r_2, r_1)$.

Hence the equivalence relation on $R(n, k, m, x)$ is induced by the dihedral group of order $2m$ defined by

$$G = \langle \sigma, \rho \rangle \ . \tag{1}$$

Thus $F(n, k, m, x)$, the number of equivalence classes of binary sequences in $S(n, k, m, x)$ under the action of $D_{2n} \times S_2$, equals the number of equivalence classes of positive integer sequences in $R(n, k, m, x)$ under the action of G.

Our counting arguments will involve $c(N, M, n)$, the number of compositions of n into exactly M parts, none exceeding N, and we recall that

(2) $$c(N, M, n) = \sum \binom{M}{b_1, \ldots, b_t},$$

where the summation is over all partitions Π of n into b_i parts equal to a_i, $i = 1, \ldots, t$, so that

$$\Pi = a_1^{b_1} \ldots a_t^{b_t},$$

where $\sum_{i=1}^{t} a_i b_i = n$, $\sum_{i=1}^{t} b_i = M$, $1 \leqq a_1 \leqq \ldots \leqq a_t \leqq N$.

We make the convention that

(3) $$c(N, 0, n) = \begin{cases} 1 & \text{if } n = 0, \\ 0 & \text{if } n > 0. \end{cases}$$

Finally, we let ϕ denote Euler's phi function.

We are now ready to state our main result, evaluating $F(n, k, m, x)$ when n, k, m, x are such that $F(n, k, m, x) > 0$. (This rules out, for example, the case where x is odd and $F(n, k, x, x) = 0$.)

THEOREM 1. *(a) If* $n = kx$, *then* $m = x = n/k$ *is even, and*

$$F(n, k, n/k, n/k) = 1.$$

(b) If $n > kx$, *then* $m > x$, *and two cases arise:*

(i) if x *is odd, then*

(α) *either* $m = x + 1$, *so that* $n - kx < k$ *and*

$$F(n, k, m, m-1) = 1,$$

(β) *or* $m > x + 1$ *and*

$$F(n, k, m, x) = \frac{1}{2m} \sum_{d} \phi(d) \cdot \binom{m/d}{x/d} \cdot c\left(k-1, \frac{m-x}{d}, \frac{n-kx}{d}\right) + \tfrac{1}{2}\binom{(m-2)/2}{(x-1)/2} \sum_{u} c\left(k-1, \frac{m-1-x}{2}, \frac{n-kx-u}{2}\right),$$

where the summations are (respectively) over all d *such that* $d \mid \gcd(n, m, x)$ *and all* u *such that*

$$1 \leqq u \leqq \min(k-1, n-(k-1)x-m+1), \quad u \equiv n - kx \pmod 2);$$

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(ii) if x *is even, then*

(α) *either* $m = x + 2$, *so that* $n - kx \le 2k - 2$ *and*

$$F(n,\ k,\ m,\ m-2) = \begin{cases} m(n-kx-1+\delta)/4 & \text{for } n - kx \le k\ , \\ m\big(2k-1+\delta-(n-kx)\big)/4\ , & \text{for } k \le n - kx\ , \end{cases}$$

(β) *or* $m > x + 2$ *and*

$$F(n,\ k,\ m,\ x) = \frac{1}{2m}\sum_{d} \phi(d)\cdot \binom{m/d}{x/d}\cdot c\left(k-1,\ \frac{m-x}{d},\ \frac{n-kx}{d}\right) + \tfrac{1}{2}\binom{(m-2)/2}{x/2}\sum_{u,v} c\left(k-1,\ \frac{m-x-2}{2},\ \frac{n-kx-u-v}{2}\right) + \tfrac{1}{2}\delta\binom{m/2}{x/2} c\left(k-1,\ \frac{m-x}{2},\ \frac{n-kx}{2}\right)\ ,$$

where $\delta = 0$ *or* 1 , $\delta \equiv n - 1 \pmod 2$, *and the summations are (respectively) over all* d *such that* $d \mid \gcd(n,\ m,\ x)$, *and all* $u,\ v$ *such that* $1 \le u < v \le k-1$, $u + v \le n - (k-1)x - m + 2$, $u + v \equiv n - kx \pmod 2$. □

In certain cases, the statement of Theorem 1 can be greatly simplified. For example, if $x = 1$, so that only one float of length k occurs, then by Theorem 1 *(b) (i)*, either $m = 2$, and $F(n,\ k,\ 2,\ 1) = 1$ or $m \geqq 4$, and

$$F(n,\ k,\ m,\ 1) = \tfrac{1}{2}c(k-1,\ m-1,\ n-k) + \tfrac{1}{2}\sum_{u} c\left(k-1,\ \frac{m-2}{2},\ \frac{n-k-u}{2}\right)\ ,$$

where the summation is over all u such that $1 \leqq u \leqq \min(k-1,\ n-k-m+2)$, and $u \equiv n - k \pmod 2$.

In particular, if $k > n/2$, then $x = 1$, giving the following result.

THEOREM 2. *If* $n/2 < k \leqq n-3$, *then*

$$F(n,\ k) = 2^{n-k+3} + 2^{\lfloor (n-k-3)/2 \rfloor}\ .$$

Also $F(n,\ n-1) = F(n,\ n-2) = 1$. □

Again if $n/3 \leqq k \leqq n/2$, then $x = 1$ or $x = 2$. Considering the case $x = 1$ first, we have the following result.

THEOREM 3. *If* $n/3 \leqq k \leqq n/2$, *then*

(α) *either* $n = 2k$ *or* $2k + 1$, *and*

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$$F(n,\ k,\ -,\ 1) = 2^{n-k-3} + 2^{\lfloor (n-k-3)/2 \rfloor} - 1\ ;$$

(β) *or* $n \geqq 2k+2$, *and*

$$F(n,\ k,\ -,\ 1) = 2^{n-k-3} + 2^{\lfloor (n-k-3)/2 \rfloor} - 2^{\lceil (n-2k-3)/2 \rceil} - 2^{n-2k-2} - (n-2k)2^{n-2k-3}\ . \qquad \square$$

Next, if $x = 2$, so that only two floats of length k occur, we may simplify Theorem 1 slightly differently. If $n = 2k$, then by part *(a)*, we have $F(n,\ k,\ 2,\ 2) = 1 = F(n,\ k,\ -,\ 2)$. Otherwise, we apply part *(b) (ii)*, since $n > 2k$. Either $m = 4$, so that in fact $n \geqq 2k+2$, and $F(n,\ k,\ 4,\ 2) = n - 2k - 1 + \delta$, or $m \geqq 6$, so that $n \geqq 2k+4$, and

$$\begin{aligned} F(n,\ k,\ m,\ 2) &= \frac{1}{2m}\left\{\binom{m}{2}c(k-1,\ m-2,\ n-2k) + \delta\cdot\frac{m}{2}\,c\left(k-1,\ \frac{m-2}{2},\ \frac{n-2k}{2}\right)\right\} \\ &\qquad + \frac{m-2}{4}\sum_{u,v} c\left(k-1,\ \frac{m-4}{2},\ \frac{n-2k-u-v}{2}\right) + \frac{\delta m}{4}\cdot c\left(k-1,\ \frac{m-2}{2},\ \frac{n-2k}{2}\right) \\ &= \frac{m-1}{4}\cdot c(k-1,\ m-2,\ n-2k) + \frac{\delta(m+1)}{4}\cdot c\left(k-1,\ \frac{m-2}{2},\ \frac{n-2k}{2}\right) \\ &\qquad + \frac{m-2}{4}\sum_{u,v} c\left(k-1,\ \frac{m-4}{2},\ \frac{n-2k-u-v}{2}\right)\ , \end{aligned}$$

where $\delta = 0$ or 1 , $\delta \equiv n - 1 \pmod 2$, the summation is over all u and v such that $1 \leqq u < v \leqq k-1$, $u + v \leqq n - 2k + 4 - m$, and $u + v \equiv n - 2k \pmod 2$ and this term will occur only for $n \geqq 2k+6$ and $m \leqq n - 2k + 1$.

Now we can finish dealing with the range $n/3 \leqq k \leqq n/2$, with $x = 2$.

THEOREM 4. *If* $n/3 \leqq k \leqq n/2$, *then*

$$F(n,\ k,\ -,\ 2) = \begin{cases} 1 & \text{if } n = 2k\ , \\ 0 & \text{if } n = 2k+1\ , \\ 2 & \text{if } n = 2k+2 \text{ or } 2k+3\ , \\ 7 & \text{if } n = 2k+4\ , \\ 10 & \text{if } n = 2k+5\ , \end{cases}$$

and if $n \geqq 2k+6$, *then*

$$\begin{aligned} F(n,\ k,\ -,\ 2) &= (n-2k-1+\delta) + (n-2k-1)\left(2^{n-2k-5} - \tfrac{3}{4}\right) + 2^{n-2k-3} \\ &\qquad + \delta(n-2k-2)2^{(n-2k-8)/2} + \delta\cdot 5\cdot 2^{(n-2k-6)/2} \\ &\qquad - \tfrac{5}{4}\delta + \left\lfloor\frac{n-2k-1}{2}\right\rfloor\cdot 2^{\lfloor (n-2k-5)/2 \rfloor} - \tfrac{1}{2}\left\lfloor\frac{n-2k-1}{2}\right\rfloor\ , \end{aligned}$$

where $\delta = 0$ *or* 1 , $\delta \equiv n - 1 \pmod 2$. □

Theorems 2, 3 and 4 result from simplifying Theorem 1 for n/k small, which forces all possible x to be small. If k is small instead, Theorem 1 also simplifies.

THEOREM 5. *If* $k = 2$, *then* $F(n, 2, m, x) = 0$ *unless* $n = m + x$, *so that*

$$F(n, 2, -, x) = F(n, 2, n-x, x) .$$

Moreover if $n = m + x$, *then two cases arise:*

(i) *if* x *is odd, then*

(α) *either* $m = x + 1$ *and*

$$F(2x+1, 2, x+1, x) = 1$$

(β) *or* $m \geqq x+3$ *and*

$$F(m+x, 2, m, x) = \frac{1}{2m}\sum_d \phi(d) \cdot \binom{m/d}{x/d} + \tfrac{1}{2}\binom{(m-2)/2}{(x-1)/2} ;$$

(ii) *if* x *is even, then*

(α) *either* $m = x$ *and*

$$F(2x, 2, x, x) = 1$$

(β) *or* $m = x + 2$ *and*

$$F(2x+2, 2, x+2, x) = (x+2)/2$$

(γ) *or* $m \geqq x+4$ *and*

$$F(m+x, 2, m, x) = \frac{1}{2m}\sum_d \phi(d)\binom{m/d}{x/d} + \tfrac{1}{2}\binom{m/2}{x/2} .$$

The summations in both cases are over all d *such that* $d \mid \gcd(n, m, x)$. □

THEOREM 6. *If* $k = 3$ *and* $m > x$, *two cases arise:*

(i) *if* x *is odd, then*

(α) *either* $m = x + 1$, *so that* $n - 3x < 3$ *and*

$$F(n, 3, m, m-1) = 1 ,$$

(β) *or* $m > x + 1$ *and*

$$F(n, 3, m, x) = \frac{1}{2m}\sum_d \phi(d) \cdot \binom{m/d}{x/d}\binom{(m-x)/d}{(n-2x-m)/d} + \tfrac{1}{2}\binom{(m-2)/2}{(x-1)/2}\binom{(m-1-x)/2}{(n-2x-m+1-u)/2}$$

where $u = 1$ *or* 2 , $u \equiv n - x \pmod 2$;

(ii) *if* x *is even, then*

(α) *either* $m = x + 2$, *so that* $n-3x \leq 4$ *and*

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$$F(n,\ 3,\ m,\ m-2) = m(n-3x-1+\delta)/4 ,$$

(β) *or* $m > x + 2$ *and*

$$F(n,\ 3,\ m,\ x) = \frac{1}{2m}\sum_d \phi(d)\begin{Bmatrix} m/d \\ x/d \end{Bmatrix}\begin{Bmatrix} (m-x)/d \\ (n-m-2x)/d \end{Bmatrix}$$

$$+ \tfrac{1}{2}\varepsilon\begin{Bmatrix} (m-2)/2 \\ x/2 \end{Bmatrix}\begin{Bmatrix} (m-x-2)/2 \\ (n-3x-3)/2 \end{Bmatrix} + \tfrac{1}{2}\delta\begin{Bmatrix} m/2 \\ x/2 \end{Bmatrix}\begin{Bmatrix} (m-x)/2 \\ (n-m-2x)/2 \end{Bmatrix} ,$$

where $\delta, \varepsilon = 0$ *or* 1 , $\delta \equiv n - 1 \pmod 2$, $\varepsilon \equiv n \pmod 2$. *The summations in both cases are over all* d *such that* $d \mid \gcd(n, m, x)$. □

Values of $F(n, k, -, x)$ for $n \leqq 20$, and $x = 1$ to 10 , have been calculated by a sieving algorithm, based on that described in [4]. These values have been checked against the results of Theorems 1 to 6. For $4 \leqq k < n/3$, we have been unable to simplify the statement of Theorem 1 into any more convenient form. In checking $F(n, 3, -, x)$ from Theorem 6, we must sum over m , as m runs through all even numbers from $\lceil (n-x)/2 \rceil$ to $2\lceil (n-2x-1)/2 \rceil$.

Details of proofs

We shall use several binomial identities, especially the following:

(B1) $$\sum_{k=0}^{n} \binom{n}{k} = 2^n ,$$

(B2) $$\sum_{m=0}^{\lfloor n/2 \rfloor} \binom{n}{2m} = 2^{n-1} ,$$

(B3) $$\sum_{x=y}^{z} \binom{x}{y} = \begin{Bmatrix} z+1 \\ y+1 \end{Bmatrix} ,$$

(B4) $$(m-1)\begin{Bmatrix} n-2k \\ m-2 \end{Bmatrix} = (m-2)\begin{Bmatrix} n-2k \\ m-2 \end{Bmatrix} + \begin{Bmatrix} n-2k \\ m-2 \end{Bmatrix} = (n-2k)\begin{Bmatrix} n-2k-1 \\ m-3 \end{Bmatrix} + \begin{Bmatrix} n-2k \\ m-2 \end{Bmatrix} .$$

Proof of Theorem 1. Each r-sequence determines uniquely an (unordered) partition

$$\Pi = a_1^{b_1} \dots a_t^{b_t}$$

of $n - kx = \sum_{i=1}^{t} a_i b_i$ into $m - x = \sum_{i=1}^{t} b_i$ parts, where

$1 \leqq a_1 < a_2 < \ldots < a_t \leqq k-1$, and $t \geqq 1$; namely, $\mathbf{r}$ has b_i entries equal to a_i for each $i = 1, \ldots, t$. Conversely, any partition Π of $n - kx$ into $m - x$ parts, each part at most $k - 1$, corresponds to some $\mathbf{r}$-sequences in $R(n, k, m, x)$, and the set $R(\Pi)$ of $\mathbf{r}$-sequences corresponding to a given partition Π is fixed setwise by the group G , defined in (1). If $n = kx$, then $m = x = n/k$. Either m is even, and $F(n, k, n/k, n/k) = 1$, or m is odd and $F(n, k, n/k, n/k) = 0$. We may now assume that $n > kx$, and hence that $m > x$.

Let $\text{fix } g = \{\mathbf{r} \in R(n, k, m, x) \mid \mathbf{r}g = \mathbf{r}\}$, for each $g \in G$. By Burnside's lemma, the number of G-orbits in $R(\Pi)$ is

$$\frac{1}{2m} \cdot \sum_{g \in G} |\text{fix } g \cap R(\Pi)| .$$

Hence for each g in G we evaluate $|\text{fix } g \cap R(\Pi)|$ and sum over all partitions Π . We consider the various elements g in G .

(a) Let $g = \sigma^l$, where $1 \leqq l \leqq m$. Then $\mathbf{r}g = \mathbf{r}$ if and only if, for all i ,

$$r_i = r_{i+(m/d)} ,$$

where $m/d = \gcd(l, m)$, and subscripts are to be taken modulo m , that is, $\mathbf{r}$ consists of d repetitions of a subsequence $\mathbf{r}' = (r_1, \ldots, r_{m/d})$. Also, $x = dx'$, $b_i = db_i'$ where b_i' is an integer, for $i = 1, 2, \ldots, t$, $\sum_{1 \leq i \leq m/d} r_i = n/d = n'$, say, and the subsequence $\mathbf{r}'$ corresponds to a partition

$$(4)\quad \begin{cases} \Pi' = a_1^{b_1'} \ldots a_t^{b_t'} \\ \text{of } (n-kx)/d = n' - kx' = \sum_{i=1}^{t} a_i b_i' \text{ into } (m-x)/d = m/d - x' = \sum_{i=1}^{t} b_i' \\ \text{parts where } 1 \leqq a_1 < a_2 < \ldots < a_t \leqq k-1 , \text{ and } t \geqq 1 . \end{cases}$$

We note of course that $\mathbf{r}'$ determines $\mathbf{r}$ uniquely.

To determine $|\text{fix } \sigma^l|$, we now sum over all partitions Π , bearing

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in mind that $d = m/\gcd(l, m)$. Two cases arise:

(i) $d \nmid \gcd(n, x)$, and $|\text{fix } \sigma^l| = 0$;

(ii) $d \mid \gcd(n, x)$, and

$$|\text{fix } \sigma^l| = \sum \begin{Bmatrix} m/d \\ b_1', \ldots, b_t', x/d \end{Bmatrix} = \sum \binom{m/d}{x/d} \begin{Bmatrix} (m-x)/d \\ b_1', \ldots, b_t' \end{Bmatrix} ,$$

where the sum is over all partitions Π' satisfying (4).

But in terms of compositions (2), this becomes

$$|\text{fix } \sigma^l| = c\left(k-1, \frac{m-x}{d}, \frac{n-kx}{d}\right) \cdot \binom{m/d}{x/d} .$$

Given l , $d = m/\gcd(l, m)$ divides $\gcd(n, m, x)$, and conversely, given a divisor d of $\gcd(n, m, x)$, there are precisely $\phi(d)$ integers l with $\gcd(l, m) = m/d$. Thus altogether

$$\sum_{l=1}^{m} |\text{fix } \sigma^l| = \sum_d \phi(d) \cdot \binom{m/d}{x/d} c\left(k-1, \frac{m-x}{d}, \frac{n-kx}{d}\right)$$

where the summation is again over all d such that $d \mid \gcd(n, m, x)$.

(b) Let $g = \rho$. Then $rg = r$ if and only if, for all i ,

$$r_i = r_{m+1-i} .$$

Thus $\text{fix } \rho \cap R(\Pi) = \emptyset$, unless $x, b_1, \ldots, b_t$ are all even. So we assume that

$$x = 2x'' , \quad b_i = 2b_i'' \text{ for } i = 1, 2, \ldots, t ,$$

and hence that Π corresponds to a partition

$$(5) \quad \begin{cases} \Pi'' = a_1^{b_1''} \cdots a_t^{b_t''} \\ \text{of } (n-x)/2 = \sum_{i=1}^{t} a_i b_i'' \text{ into } (m-x)/2 = \sum_{i=1}^{t} b_i'' \text{ parts, where} \\ 1 \leqq a_1 < a_2 < \ldots < a_t \leqq k-1 , \text{ and } t \geqq 1 . \end{cases}$$

In this case

$$|\text{fix } \rho \cap R(\Pi)| = \begin{Bmatrix} m/2 \\ b_1'', \ldots, b_t'', x'' \end{Bmatrix} = \binom{m/2}{x/2} \cdot \begin{Bmatrix} (m-x)/2 \\ b_1'', \ldots, b_t'' \end{Bmatrix} .$$

Summing over Π , we have

$$|\text{fix } \rho| = \begin{cases} 0 \text{ if } \gcd(n, x) \text{ is odd,} \\ \binom{m/2}{x/2} \sum \binom{(m-x)/2}{b''_1, \ldots, b''_t} \text{ if } \gcd(n, x) \text{ is even,} \\ \text{where the sum is over all partitions } \Pi'' \text{ satisfying (5).} \end{cases}$$

In terms of compositions, this becomes

$$|\text{fix } \rho| = \begin{cases} 0 \text{ if } \gcd(n, x) \text{ is odd,} \\ \binom{m/2}{x/2} c\left(k-1, \frac{m-x}{2}, \frac{n-kx}{2}\right) \text{ if } \gcd(n, x) \text{ is even.} \end{cases}$$

(c) Let $g = \rho\sigma$. Then $\mathbf{r}\rho\sigma = \mathbf{r}$ if and only if for all $i = 2, \ldots, m$, $i \neq (m+2)/2$,

$$r_i = r_{m+2-i} .$$

Again two cases arise:

(i) if $r_1 = r_{(m+2)/2}$, then $x, b_1, \ldots, b_t$ are all even, and the argument of (b) above shows that such sequences make a contribution to $|\text{fix } \rho\sigma|$ of

$$\begin{cases} 0 \text{, if } \gcd(n, x) \text{ is odd,} \\ \binom{m/2}{x/2} \cdot c\left(k-1, \frac{m-x}{2}, \frac{n-kx}{2}\right) \text{ if } \gcd(n, x) \text{ is even;} \end{cases}$$

(ii) if $r_1 \neq r_{(m+2)/2}$, then two cases again arise, depending on the parity of x.

If x is odd, then one of r_1 and $r_{(m+2)/2}$ equals k, and the other equals u, say where $1 \leq u \leq \min(k-1, n-k)$. The rest of the sequence $\mathbf{r}$ is determined by the subsequence $\left(r_2, \ldots, r_{m/2}\right)$ which contains $(x-1)/2$ terms equal to k. Its remaining terms constitute a composition of $(n-kx-u)/2$ into $(m-x-1)/2$ parts, each part at most $k-1$. In this case the contribution to $|\text{fix } \rho\sigma|$ is

$$\begin{cases} 2 \text{, if } m = x+1 \text{, so that } u = n-kx < k \text{,} \\ 2\binom{(m-2)/2}{(x-1)/2} \cdot \sum_u c\left(k-1, \frac{m-1-x}{2}, \frac{n-kx-u}{2}\right) \text{, if } m > x+1 \text{,} \end{cases}$$

where the summation is over all u such that

(6) $\quad 1 \leq u \leq \min\left(k-1, n-(k-1)x-m+1\right)$ and $u \equiv n-kx \pmod 2$.

If x is even, then neither r_1 nor $r_{(m+2)/2}$ equals k; let

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$$\{r_1, r_{(m+2)/2}\} = \{u, v\} ,$$

where $1 \leq u < v \leq k-1$. The rest of the sequence r is determined by the subsequence $\left(r_2, \ldots, r_{m/2}\right)$ which contains $x/2$ terms equal to k . Its remaining terms constitute a composition of $(n-kx-u-v)/2$ into $(m-x-2)/2$ parts, each part at most $k - 1$. In this case the contribution to $|\text{fix } \rho\sigma|$ is

$$\begin{cases} n - kx - 1 - \delta \text{ , if } m = x + 2 \text{ , and } 3 \leq u + v = n - kx \leq k \text{ ,} \\ 2k - 1 - \delta - (n-kx) \text{ , if } m = x + 2 \text{ and} \\ k \leq u + v = n - kx \leq 2k - 3 \text{ ,} \\ 2\binom{(m-2)/2}{x/2} \cdot \sum\limits_{u,v} c\left(k-1, \frac{m-x-2}{2}, \frac{n-kx-u-v}{2}\right) \text{ , if } m > x + 2 \text{ ,} \end{cases}$$

where $\delta = 0$ or 1 , $\delta \equiv n - 1 \pmod 2$, and the summation is over all u, v such that

(7) $\quad 1 \leq u < v \leq k-1$, $u+v \leq n-(k-1)x-m+2$, and $u + v \equiv n - kx \pmod 2$.

Note here that the term for $m = x + 2$ is simply the number of solutions of

$$u + v = n - kx ,$$

where $1 \leq u < v \leq k-1$. For convenience, we denote this number by $s(m, 2)$.

Cases (a), (b), and (c) cover all possibilities, since $G\backslash\langle\sigma\rangle$ consists of $m/2$ elements conjugate to ρ , and $m/2$ elements conjugate to $\rho\sigma$, and conjugate elements fix sets of the same size. Hence we have only to sum the appropriate terms.

First, suppose that x is odd. If $m = x + 1$, then $n - kx < k$ and

$$2m \cdot F(n, k, m, m-1) = \binom{m}{m-1} \cdot c(k-1, 1, n-kx) + m = 2m ;$$

if $m > x + 1$, then

$$2m \cdot F(n, k, m, x)$$
$$= \sum_{d} \phi(d) \cdot \binom{m/d}{x/d} \cdot c\left(k-1, \frac{m-x}{d}, \frac{n-kx}{d}\right) + m\binom{(m-2)/2}{(x-2)/2} \sum_{u} c\left(k-1, \frac{m-1-x}{2}, \frac{n-kx-u}{2}\right) ,$$

where the summations are (respectively) over all d such that $d \mid \gcd(n, m, x)$ and all u satisfying (6).

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This confirms Theorem 1 *(b) (i)*.

Secondly, suppose that x is even. If $m = x + 2$, then $n-kx \leqq 2k-2$ and

$$2m \cdot F(n, k, m, m-2) = \binom{m}{2} \cdot c(k-1, 2, n-kx) + \delta \cdot \frac{m}{2} \cdot c\left(k-1, 1, \frac{n-kx}{2}\right) + \delta \cdot m \cdot \frac{m}{2} c\left(k-1, 1, \frac{n-kx}{2}\right) + \frac{m}{2} \cdot s(m, 2) ,$$

where $\delta = 0$ or 1, $\delta \equiv n - 1 \pmod 2$.

Now

$$c(k-1, 2, n-kx) = \begin{cases} n - kx - 1 , & \text{for } n-kx \leqq k , \\ 2k - 1 - (n-kx) , & \text{for } k \leqq n-kx \leqq 2k-2 , \end{cases}$$

and $s(m, 2) = c(k-1, 2, n-kx) - \delta$. Hence

$$2m \cdot F(n, k, m, m-2) = \frac{m^2}{2} \{c(k-1, 2, n-kx)+\delta\} .$$

If $m > x + 2$, then

$$2m \cdot F(n, k, m, x) = \sum_d \phi(d) \cdot \binom{m/d}{x/d} \cdot c\left(k-1, \frac{m-x}{d}, \frac{n-kx}{d}\right) + m\binom{(m-2)/2}{x/2} \sum_{u,v} c\left(k-1, \frac{m-x-2}{2}, \frac{n-kx-u-v}{2}\right) + \delta m\binom{m/2}{x/2} \cdot c\left(k-1, \frac{m-x}{2}, \frac{n-kx}{2}\right) ,$$

where $\delta = 0$ or 1, $\delta \equiv n - 1 \pmod 2$, and the summations are (respectively) over all d such that $d \mid \gcd(n, m, x)$ and all u, v satisfying (7). This confirms Theorem 1 *(b)*.

Note that in either case *(i)* where $m = x + 1$, or *(ii)* where $m = x + 2$, if we use the convention for $c(N, 0, n)$ defined in (3), then the general formula includes these special cases also.

This completes the proof of the theorem. □

Proof of Theorem 2. If $k > n/2$ and $m \geqq 4$, so that $n \geqq k+3$, then from Theorem 1, we have already that

$$F(n, k, m, 1) = \tfrac{1}{2} \cdot c(k-1, m-1, n-k) + \tfrac{1}{2} \sum_u c\left(k-1, \frac{m-2}{2}, \frac{n-k-u}{2}\right) ,$$

where the summation is over all u such that $1 \leqq u \leqq \min(k-1, n-k-m+2)$

and $u \equiv n - k \pmod 2$. That is,

$$F(n,\ k,\ m,\ 1) = \tfrac{1}{2}\binom{n-k-1}{m-2} + \tfrac{1}{2}\sum_{u}\binom{(n-k-u-2)/2}{(m-4)/2} .$$

Hence if $n/2 < k \leqq n-3$,

$$F(n,\ k) = \sum_{m} F(n,\ k,\ m,\ 1)$$

where the summation is over all even m such that $2 \leqq m \leqq n - k + \varepsilon$, so that

$$F(n,\ k) = 1 + \tfrac{1}{2}\sum_{m'}\binom{n-k-1}{2m'} + \tfrac{1}{2}\sum_{u}\sum_{m''}\binom{(n-k-u-2)/2}{m''}$$

where the summations are (respectively) over all m' such that $m = 2m' + 2$ and $1 \leqq m' \leqq [(n-k-1)/2]$, all u such that $1 \leqq u \leqq n - k - 2$ and $u \equiv n - k \pmod 2$, and all m'' such that $0 \leqq m'' \leqq (n-k-u-2)/2$ and $m = 2m'' + 4 \leqq n - k - u + 2$.

Hence by (B1) and (B2),

$$F(n,\ k) = \tfrac{1}{2} + \tfrac{1}{2}\cdot 2^{n-k-2} + \tfrac{1}{2}\sum_{u} 2^{(n-k-u-2)/2} \quad \text{summing over all}$$

$$u \equiv n - k \pmod 2 \text{ such that } 0 \leqq \frac{n-k-u-2}{2} \leqq \left\lfloor\frac{n-k-3}{2}\right\rfloor$$

$$= \tfrac{1}{2} + 2^{n-k-3} + \tfrac{1}{2}\left(2^{\lfloor(n-k-1)/2\rfloor}-1\right)$$

$$= 2^{n-k-3} + 2^{\lfloor(n-k-3)/2\rfloor} . \qquad \square$$

Proof of Theorem 3. Here we have $n/3 \leqq k \leqq n/2$, and $x = 1$. Then $k \leqq n-k \leqq 2k$, and $m \geqq 4$. Also if $m \geqq n - 2k + 3$, then

$$c(k-1,\ m-1,\ n-k) = c(m-1,\ n-k) = \binom{n-k-1}{m-2} .$$

However if $m \leqq n - 2k + 2$, then $c(k-1,\ m-1,\ n-k) < c(m-1,\ n-k)$, since we must exclude the compositions of $n - k$ into $m - 1$ parts, with one part of size v , $k \leqq v \leqq n - k + m + 2$. Each excluded composition corresponds to $m - 1$ compositions of $n - k - v$ into $m - 2$ parts, so that in this case,

$$c(k-1,\ m-1,\ n-k) = c(m-1,\ n-k) - (m-1)\sum_{v} c(m-2,\ n-k-v)$$

$$= \binom{n-k-1}{m-2} - (m-1)\sum_{v}\binom{n-k-v-1}{m-3} ,$$

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summing over all v such that $k \leqq v \leqq n - k - m + 2$. Hence, from Theorem 1,

$$F(n, k, -, 1)$$

$$= \tfrac{1}{2} \sum_{\substack{m \\ 4 \leqq m \leqq n-2k+2}} \left(\binom{n-k-1}{m-2} - (m-1) \sum_{\substack{v \\ k \leqq v \leqq n-k-m+2}} \binom{n-k-v-1}{m-3} \right)$$

$$+ \tfrac{1}{2} \sum_{\substack{m \\ \max(4, n-2k+3) \leqq m \leqq n-k+\varepsilon}} \binom{n-k-1}{m-2} + \tfrac{1}{2} \sum_{\substack{m \\ 4 \leqq m \leqq n-k+\varepsilon}} \sum_{u} \binom{(n-k-u-2)/2}{(m-4)/2} ,$$

where m is even, $\varepsilon = 0$ or 1 , $\varepsilon \equiv n - k \pmod 2$, and the last summation is over all $u \equiv n - k \pmod 2$ such that $1 \leqq u \leqq \min(k-1, n-k-m+2)$

$$= \tfrac{1}{2} \sum_{\substack{1 \leqq m' \leqq \lfloor (n-k-1)/2 \rfloor \\ m = 2m'+2}} \binom{n-k-1}{2m'}$$

$$- \tfrac{1}{2} \sum_{4 \leqq m \leqq n-2k+2} (m-1) \left(\sum_{k \leqq v \leqq n-k-m+2} \binom{n-k-v-1}{m-3} \right) + \tfrac{1}{2} \sum_{u} \left[\sum_{m''} \binom{(n-k-u-2)/2}{m''} \right]$$

where $u \equiv n - k \equiv \varepsilon \pmod 2$, $1 \leqq u \leqq \begin{cases} k-1 , & \text{if } n \geqq 2k+1 \\ k-2 , & \text{if } n = 2k \end{cases}$,

and m is even, $m = 2m'' + 4$ and $0 \leqq m'' \leqq (n-k-u-2)/2$.

Now

$$\sum_{k \leqq v \leqq n-k-m+2} \binom{n-k-v-1}{m-3} = \sum_{m-3 \leqq n-k-v-1 \leqq n-2k-1} \binom{n-k-v-1}{m-3} = \binom{n-2k}{m-k}$$

by (B3). Hence, by (B4),

(8) $F(n, k, -, 1)$

$$= \tfrac{1}{2}\left(2^{n-k-2}-1\right) - \tfrac{1}{2} \sum_{m} (n-2k) \binom{n-2k-1}{m-3} - \tfrac{1}{2} \sum_{m} \binom{n-2k}{m-2} + \tfrac{1}{2} \sum_{u} 2^{(n-k-u-2)/2}$$

summing over all even m such that $4 \leqq m \leqq n - 2k + 2$, and all

$u \equiv n - k \pmod 2$ such that $1 \leqq u \leqq \begin{cases} k-1 , & \text{if } n \geq 2k+1 \\ k-2 , & \text{if } n = 2k \end{cases}$,

$$= 2^{n-k-3} - \tfrac{1}{2} - \tfrac{1}{2} \sum_{\mu} (n-2k) \binom{n-2k-1}{\mu} - \tfrac{1}{2} \sum_{\nu} \binom{n-2k}{\nu} + \tfrac{1}{2} \sum_{j} 2^{j}$$

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summing over $\mu = m-3$, for m even and $1 \leqq \mu \leqq n-2k-1$, over $\nu = m-2$, for m even and $2 \leqq \nu \leqq n-2k$, and over j an integer, $\frac{n-2k-1}{2} \leqq j \leqq \frac{n-k-3}{3}$

$$= 2^{n-k-3} - \tfrac{1}{2} - \tfrac{1}{2}(n-2k)2^{n-2k-2} - \tfrac{1}{2}\left(2^{n-2k-1}-1\right) + \tfrac{1}{2}\left(2^{\lfloor (n-k-1)/2\rfloor} - 2^{\lceil (n-2k-1)/2\rceil}\right)$$

$$= 2^{n-k-3} + 2^{\lfloor (n-k-3)/2\rfloor} - 2^{\lceil (n-2k-3)/2\rceil} - 2^{n-2k-2} - (n-2k)2^{n-2k-3} .$$

Note that if $n = 2k$ or $2k+1$, the first and second summations in equation (8) are vacuous, and the expression reduces to

$$F(n, k, -, 1) = 2^{n-k-3} + 2^{\lfloor (n-k-3)/2\rfloor} - 1 . \qquad \square$$

Proof of Theorem 4. Again $n/3 \leqq k \leqq n/2$, but now $x = 2$. If $n/3 \leqq k \leqq n/2$, and $n \geqq 2k+4$, then $2k+4 \leqq n \leqq 3k$, and for $m \geq 6$ we have (by our remarks preceding the statement of Theorem 4)

$$(9) \quad F(n, k, m, 2) = \frac{m-1}{4} \cdot \binom{n-2k-1}{m-3} + \delta \cdot \frac{m+1}{4} \cdot \binom{(n-2k-2)/2}{(m-4)/2} + \frac{m-2}{4} \sum_{u,v} \binom{\left(n-2k-(u+v)-2\right)/2}{(m-6)/2} \cdot \left\lfloor \frac{u+v-1}{2} \right\rfloor ,$$

where the summation is over all u, v such that $1 \leqq u < v \leqq k-1$, $u + v \equiv n - 2k \pmod 2$, and

$$(10) \qquad \frac{m-6}{2} \leqq \frac{n-2k-(u+v)-2}{2} \leqq \left\lfloor \frac{n-2k-5}{2} \right\rfloor .$$

Consider the last term

$$L = \frac{m-2}{4} \sum_{u,v} \binom{\left(n-2k-(u+v)-2\right)/2}{(m-6)/2} \cdot \left\lfloor \frac{u+v-1}{2} \right\rfloor .$$

If $m = 6$ and $n = 2k+4$, then $L = 0$. If $m = 6$ and $n = 2k+5$, then the only possible case is $u = 1$, $v = 2$, with n odd, so that

$$L = \frac{6-2}{4} \cdot \binom{(n-2k-5)/2}{0/2} \cdot \left\lfloor \frac{3-1}{2} \right\rfloor = 1 .$$

Letting $u + v = w$, we deal with the two cases that arise, for $n \geq 2k+6$.

(i) If n is odd, then w is odd, and

$$3 \leqq w \leqq n - 2k - m + 4 ,$$

and

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$$\frac{w-1}{2} = \frac{n-2k-1}{2} - \frac{n-2k-w}{2} .$$

Hence

$$L = \sum_{w'} \binom{w'}{(m-6)/2} \binom{\frac{w-1}{2}}{} , \text{ where } w' = (n-2k-w-2)/2 \text{ and the summation is over all } w' \text{ such that } \frac{m-6}{2} \leq w' \leq \frac{n-2k-5}{2}$$

$$= \sum_{w'} -(w'+1)\binom{w'}{(m-6)/2} + \frac{n-2k-1}{2} \sum_{w'} \binom{w'}{(m-6)/2}$$

$$= -\sum_{w'} \frac{m-4}{2} \binom{w'+1}{(m-4)/2} + \frac{n-2k-1}{2} \cdot \binom{(n-2k-3)/2}{(m-4)/2}$$

$$= -\frac{m-4}{2} \cdot \binom{(n-2k-1)/2}{(m-2)/2} + \frac{n-2k-1}{2} \binom{(n-2k-3)/2}{(m-4)/2} ,$$

using the same arguments as in the proof of Theorem 3.

(ii) If n is even, then w is even,

$$4 \leq w \leq n - 2k - n + 4 ,$$

and

$$\left\lfloor \frac{w-1}{2} \right\rfloor = \frac{w-2}{2} = \frac{n-2k-2}{2} - \frac{n-2k-w}{2} .$$

Hence

$$L = \sum_{w'} \binom{w'}{(m-6)/2} \binom{\frac{w-2}{2}}{} , \text{ where } w' = (n-2k-w-2)/2 , \text{ and the summation is over all } w' \text{ such that } \frac{m-6}{2} \leq w' \leq \frac{n-2k-6}{2}$$

$$= \sum_{w'} -(w'+1) \cdot \binom{w'}{(m-6)/2} + \frac{n-2k-2}{2} \sum_{w'} \binom{w'}{(m-6)/2}$$

$$= -\sum_{w'} \frac{m-4}{2} \cdot \binom{w'+1}{(m-4)/2} + \frac{n-2k-2}{2} \binom{(n-2k-4)/2}{(m-4)/2}$$

$$= -\frac{m-4}{2} \binom{(n-2k-2)/2}{(m-2)/2} + \frac{n-2k-2}{2} \binom{(n-2k-4)/2}{(m-4)/2} ,$$

using the same arguments as in case (i) above.

Combining the results of (i) and (ii) with (9) above, we have, for $m \geq 6$,

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$$F(n, k, m, 2) = \frac{m-1}{4}\binom{n-2k-1}{m-3} + \delta \cdot \frac{m+1}{4}\binom{(n-2k-2)/2}{(m-4)/2}$$
$$+ \frac{m-2}{4}\left\{\left\lfloor\frac{n-2k-1}{2}\right\rfloor\binom{\lfloor(n-2k-3)/2\rfloor}{(m-4)/2} - \frac{m-4}{2}\binom{\lfloor(n-2k-1)/2\rfloor}{(m-2)/2}\right\} .$$

For $n \leqq 2k+5$, so that $m \leqq 6$, we confirm the statement of Theorem 4 immediately.

If $n \geqq 2k+6$, then

$$(11) \qquad F(n, k, -, 2) = F(n, k, 4, 2) + \sum_m F(n, k, m, 2)$$
$$= (n-2k-1+\delta) + \sum F(n, k, m, 2) ,$$

where the summations are over all even m such that $6 \leqq m \leqq n - 2k + 2$. We sum the terms in the second expression:

(a) $$\sum_m \frac{m-1}{4}\binom{n-2k-1}{m-3} = \tfrac{1}{4}\sum_m\left\{(m-3)\binom{n-2k-1}{m-3} + 2\binom{n-2k-1}{m-3}\right\}$$
$$= \tfrac{1}{4}\sum_m\left\{(n-2k-1)\binom{n-2k-2}{m-4}+2\binom{n-2k-1}{m-3}\right\}$$
$$= \tfrac{1}{4}(n-2k-1)\left(2^{n-2k-3}-1\right) + \tfrac{1}{2}\left(2^{n-2k-2}-(n-2k-1)\right) ;$$

(b) $$\sum_m \delta \cdot \frac{m+1}{4}\binom{(n-2k-2)/2}{(m-4)/2}$$
$$= \frac{\delta}{2}\sum_m\left\{\frac{m-4}{2}\binom{(n-2k-2)/2}{(m-4)/2} + \frac{5}{2}\binom{(n-2k-2)/2}{(m-4)/2}\right\}$$
$$= \frac{\delta}{2}\sum_m\left\{\frac{n-2k-2}{2}\binom{(n-2k-4)/2}{(m-6)/2} + \frac{5}{2}\binom{(n-2k-2)/2}{(m-4)/2}\right\}$$
$$= \frac{\delta(n-2k-2)}{4} \cdot 2^{(n-2k-4)/2} + \frac{5\delta}{4} \cdot \left(2^{(n-2k-2)/2}-1\right)$$
$$= \delta(n-2k-2) \cdot 2^{(n-2k-8)/2} + 5 \cdot \delta \cdot 2^{(n-2k-6)/2} - \frac{5\delta}{4} ;$$

(c) $$\sum_m \frac{m-2}{4} \cdot \left\lfloor\frac{n-2k-1}{2}\right\rfloor \cdot \binom{(n-2k-3-\delta)/2}{(m-4)/2}$$
$$= \tfrac{1}{2}\left\lfloor\frac{n-2k-1}{2}\right\rfloor\left\{\sum_m \frac{n-2k-3-\delta}{2}\binom{(n-2k-5-\delta)/2}{(m-6)/2} + \binom{(n-2k-3-\delta)/2}{(m-4)/2}\right\}$$
$$= \binom{\lfloor(n-2k-1)/2\rfloor}{2}2^{\lfloor(n-2k-5)/2\rfloor} + \tfrac{1}{2}\left\lfloor\frac{n-2k-1}{2}\right\rfloor\left(2^{\lfloor(n-2k-3)/2\rfloor}-1\right) ;$$

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$$\text{(d)}\quad -\sum_m \frac{(m-2)(m-4)}{8}\binom{\lfloor(n-2k-1)/2\rfloor}{(m-2)/2} = -\sum_m \binom{\lfloor(n-2k-1)/2\rfloor}{2}\binom{\lfloor(n-2k-5)/2\rfloor}{(m-6)/2}$$
$$= -\binom{\lfloor(n-2k-1)/2\rfloor}{2} 2^{\lfloor(n-2k-5)/2\rfloor} .$$

Note that terms in (c) and (d) occur only for $m \leqq n - 2k + 1$.

Adding the results of (a), (b), (c), and (d), and substituting in (11) we have Theorem 4. □

Proof of Theorem 5. If $k = 2$, then $n = m + x$ in order to have any sequences possible at all. Hence

$$F(n,\ 2,\ -,\ x) = F(n,\ 2,\ n-x,\ x) ,$$

and we evaluate $F(n, 2, n-x, x)$ from Theorem 1. If x is odd, then either $n - x = x + 1$, and

$$F(2x+1,\ 2,\ x+1,\ x) = 1$$

or $n-x \geqq x+3$ and

$$F(n,\ 2,\ n-x,\ x) = \frac{1}{2(n-x)} \sum_{d\mid\gcd(n,x)} \phi(d)\binom{(n-x)/d}{x/d} c\left(1,\ \frac{n-2x}{d},\ \frac{n-2x}{d}\right)$$
$$+ \tfrac{1}{2}\binom{(n-x-2)/2}{(x-1)/2} c\left(1,\ \frac{n-2x-1}{2},\ \frac{n-2x-1}{2}\right)$$
$$= \frac{1}{2(n-x)} \sum_{d\mid\gcd(n,x)} \phi(d)\binom{(n-x)/d}{x/d} + \tfrac{1}{2}\binom{(n-x-2)/2}{(x-1)/2} .$$

If x is even, then either $n - x = x$, and

$$F(2x,\ 2,\ x,\ x) = 1$$

or $n - x = x + 2$, and

$$F(2x+2,\ 2,\ x+2,\ x) = (x+2)/2$$

or $n-x \geqq x+4$ and

$$F(n,\ 2,\ n-x,\ x) = \frac{1}{2(n-x)} \sum_{d\mid\gcd(n,x)} \phi(d)\binom{(n-x)/d}{x/d} c\left(1,\ \frac{n-2x}{d},\ \frac{n-2x}{d}\right)$$
$$+ \tfrac{1}{2}\binom{(n-x)/2}{x/2} \cdot c\left(1,\ \frac{n-2x}{2},\ \frac{n-2x}{2}\right) ,$$

since no choice of u and v is possible,

$$= \frac{1}{2(n-x)} \sum_{d\mid\gcd(n,x)} \phi(d)\binom{(n-x)/d}{x/d} + \tfrac{1}{2}\binom{(n-x)/2}{x/2} . \quad \square$$

Proof of Theorem 6. If $k = 3$, and $m > x$, then

$$\frac{n-x}{2} \leqq m \leqq n-2x ,$$

and m can take any even value in this range. Again two cases arise, from Theorem 1.

If x is odd, then either $m = x + 1$, so that $n - 3x < 3$, and

$$F(n, 3, m, m-1) = 1$$

or $m \geqq x+3$, and

$$F(n, 3, m, x) = \frac{1}{2m} \sum_{d|\gcd(n,m,x)} \phi(d) \binom{m/d}{x/d} c\left(2, \frac{m-x}{d}, \frac{n-3x}{d}\right) + \tfrac{1}{2}\binom{(m-2)/2}{(x-1)/2} \sum_{\substack{u=1 \text{ or } 2 \\ u \equiv n-x (\mathrm{mod}\, 2)}} c\left(2, \frac{m-1-x}{2}, \frac{n-3x-u}{2}\right) .$$

Now $c(2, M, n)$ is the number of compositions of n into M parts, where each part is 1 or 2 , so

$$c(2, M, n) = \begin{cases} \binom{M}{n-M} & \text{for } M \leqq n \leqq 2M , \\ 0 & \text{otherwise.} \end{cases}$$

Hence

$$F(n, 3, m, x) = \frac{1}{2m} \sum_{d|\gcd(n,m,x)} \phi(d) \binom{m/d}{x/d} \binom{(m-x)/d}{(n-2x-m)/d} + \tfrac{1}{2}\binom{(m-2)/2}{(x-1)/2} \binom{(m-1-x)/2}{(n-2x-m+1-u)/2}$$

where $1 \leqq u \leqq 2$, $u \equiv n - x \pmod 2$.

If x is even, then either $m = x + 2$, so that $n-3x \leqq 4$, and

$$F(n, 3, m, m-2) = m(n-3x-1+\delta)/4$$

or $m \geqq x+4$, and

$$F(n, 3, m, n) = \frac{1}{2m} \sum_{d|\gcd(n,m,x)} \phi(d) \cdot \binom{m/d}{x/d} c\left(2, \frac{m-x}{d}, \frac{n-3x}{d}\right)$$

$$+ \tfrac{1}{2}\binom{(m-2)/2}{x/2} \sum_{\substack{1 \leqq u < v \leqq 2 \\ u+v \equiv n-3x (\mathrm{mod}\, 2)}} c\left(2, \frac{m-x-2}{2}, \frac{n-3x-u-v}{2}\right) + \tfrac{1}{2}\delta \binom{m/2}{x/2} c\left(2, \frac{m-x}{2}, \frac{n-3x}{2}\right) ,$$

where $\delta = 0$ or 1 , $\delta \equiv n - 1 \pmod 2$. The second summand makes a contribution here only if $u = 1$, $v = 2$, $n \equiv 1 \pmod 2$. Hence

$$F(n, 3, m, x) = \frac{1}{2m} \sum_{d|\gcd(n,m,x)} \phi(d) \cdot \binom{m/d}{x/d} \binom{(m-x)/d}{(n-2x-m)/d} + \tfrac{1}{2}\varepsilon \binom{(m-2)/2}{x/2} \binom{(m-x-2)/2}{(n-2x-m-1)/2} + \tfrac{1}{2}\delta \binom{m/2}{x/2} \binom{(m-x)/2}{(n-m-2x)/2} ,$$

where $\delta, \varepsilon = 0$ or 1 , $\delta \equiv n - 1 \pmod 2$, $\varepsilon \equiv n \pmod 2$. □

Tables of results

The following tables list values of $F(n, k, -, x)$ for $n \leqq 20$, $1 \leqq x \leqq 10$. They were calculated using the algorithm of [4] with appropriate sieving and have been checked against the results of the theorems.

Results for other special cases have also been determined using modified sieving algorithms and will appear subsequently.

TABLE 1. $F(n, k, -, 1)$, the number of twills on n harnesses, with maximum float length k which occurs precisely once per period for $n = 4, \ldots, 20$, $k = 1, \ldots, n-1$.

n \ k	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
4	0	0	1																
5	0	1	1	1															
6	0	0	1	1	1														
7	0	1	2	2	1	1													
8	0	0	3	2	2	1	1												
9	0	1	4	5	3	2	1	1											
10	0	0	7	7	5	3	2	1	1										
11	0	1	10	14	9	6	3	2	1	1									
12	0	0	16	22	17	9	6	3	2	1	1								
13	0	1	25	43	30	19	10	6	3	2	1	1							
14	0	0	40	72	58	33	19	10	6	3	2	1	1						
15	0	1	62	136	106	66	35	20	10	6	3	2	1	1					
16	0	0	101	238	205	122	69	35	20	10	6	3	2	1	1				
17	0	1	159	445	384	242	130	71	36	20	10	6	3	2	1	1			
18	0	0	257	796	740	460	258	133	71	36	20	10	6	3	2	1	1		
19	0	1	410	1476	1406	909	498	266	135	72	36	20	10	6	3	2	1	1	
20	0	0	663	2674	2710	1756	988	514	269	135	72	36	20	10	6	3	2	1	1

TABLE 2. $F(n, k, -, 2)$ = number of twills of n harnesses with maximum float length k which occurs precisely twice per period for $n = 4, \ldots, 20$, $k = 1, \ldots, \lfloor n/2 \rfloor$.

n \ k	1	2	3	4	5	6	7	8	9	10
4	0	1								
5	0	0								
6	0	2	1							
7	0	0	0							
8	0	3	2	1						
9	0	0	2	0						
10	0	4	5	2	1					
11	0	0	6	2	0					
12	0	5	15	7	2	1				
13	0	0	18	8	2	0				
14	0	6	41	23	7	2	1			
15	0	0	58	34	10	2	0			
16	0	7	113	80	25	7	2	1		
17	0	0	174	134	42	10	2	0		
18	0	8	325	291	98	27	7	2	1	
19	0	0	514	524	178	44	10	2	0	
20	0	9	929	1079	392	106	27	7	2	1

TABLE 3. $F(n, k, -, 3)$ = number of twills on n harnesses with maximum float length k which occurs precisely three times per period for $n = 4, \ldots, 20$, $k = 1, \ldots, \lfloor (n-1)/3 \rfloor$.

n \ k	1	2	3	4	5	6
4	0					
5	0					
6	0					
7	0	0				
8	0	0				
9	0	3				
10	0	0	1			
11	0	5	1			
12	0	0	3			
13	0	8	6	1		
14	0	0	11	1		
15	0	12	22	4		
16	0	0	46	6	1	
17	0	16	82	17	1	
18	0	0	163	32	4	
19	0	21	306	77	7	1
20	0	0	572	158	17	1

TABLE 4. $F(n, k, -, 4)$ = number of twills on n harnesses with maximum float length k, which occurs precisely four times per period, for $n = 4, \ldots, 20$, $k = 1, \ldots, \lfloor n/4 \rfloor$.

n \ k	1	2	3	4	5
4	1				
5	0				
6	0				
7	0				
8	0	1			
9	0	0			
10	0	3			
11	0	0			
12	0	8	1		
13	0	0	0		
14	0	16	3		
15	0	0	3		
16	0	29	11	1	
17	0	0	19	0	
18	0	47	49	3	
19	0	0	85	3	
20	0	72	211	14	1

TABLE 5. $F(n, k, -, 5)$ = number of twills on n harnesses with maximum float length k, which occurs precisely five times per period for $n = 6, \ldots, 20$, $k = 1, \ldots, \lfloor (n-1)/5 \rfloor$.

n \ k	1	2	3
6	0		
7	0		
8	0		
9	0		
10	0		
11	0	1	
12	0	0	
13	0	5	
14	0	0	
15	0	16	
16	0	0	1
17	0	38	1
18	0	0	5
19	0	79	12
20	0	0	28

TABLE 6. $F(n, k, -, 6)$ = number of twills on n harnesses with maximum float length k , which occurs precisely six times per period for $n = 6, \ldots, 20$, $k = 1, \ldots, \lfloor n/6 \rfloor$.

n \ k	1	2	3
6	1		
7	0		
8	0		
9	0		
10	0		
11	0		
12	0	1	
13	0	0	
14	0	4	
15	0	0	
16	0	16	
17	0	0	
18	0	50	1
19	0	0	0
20	0	126	4

TABLE 7. $F(n, k, -, 7)$ = number of twills on n harnesses with maximum float length k , which occurs precisely seven times per period for $n = 8, \ldots, 20$, $k = 1, \ldots, \lfloor (n-1)/7 \rfloor$.

n \ k	1	2
8	0	
9	0	
10	0	
11	0	
12	0	
13	0	
14	0	
15	0	1
16	0	0
17	0	8
18	0	0
19	0	38
20	0	0

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TABLE 8. $F(n, k, -, 8)$ = number of twills on n harnesses with maximum float length k, which occurs precisely eight times per period for $n = 8, \ldots, 20$, $k = 1, \ldots, \lfloor n/8 \rfloor$.

n \ k	1	2
8	1	
9	0	
10	0	
11	0	
12	0	
13	0	
14	0	
15	0	
16	0	1
17	0	0
18	0	5
19	0	0
20	0	29

TABLE 9. $F(n, k, -, 9)$ = number of twills on n harnesses with maximum float length k, which occurs precisely nine times per period for $n = 10, \ldots, 20$, $k = 1, \ldots, \lfloor (n-1)/9 \rfloor$.

n \ k	1	2
10	0	
11	0	
12	0	
13	0	
14	0	
15	0	
16	0	
17	0	
18	0	
19	0	1
20	0	0

TABLE 10. $F(n, k, -, 10)$ = number of twills on n harnesses with maximum float length k, which occurs precisely ten times per period for $n = 10, \ldots, 20$, $k = 1, \ldots, \lfloor (n-1)/9 \rfloor$.

n \ k	1	2
10	1	
11	0	
12	0	
13	0	
14	0	
15	0	
16	0	
17	0	
18	0	
19	0	
20	0	1

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