THE SECOND DUALS OF CERTAIN SPACES OF ANALYTIC FUNCTIONS

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Let φ be a continuous, decreasing, real-valued function on $0 \leqq r \leqq 1$ with $\varphi(1) = 0$ and $\varphi(r) > 0$ for $r < 1$. Let E_0 be the Banach space of analytic functions f on the open unit disc D, such that $f(z)\varphi(|z|) \to 0$ as $|z| \to 1$, with norm

$$||f|| = \sup \{|f(z)|\varphi(z) : z \in D\},$$

where we write $\varphi(z) = \varphi(|z|)$ for $z \in D$. Let E be the Banach space of analytic functions f on D for which $f\varphi$ is bounded in D, with the same norm as E_0. It is easy to see that E is complete in this norm, and that E_0 is a closed subspace of E.

The dual space of E_0 will be shown to be identifiable with a quotient space of $L^1(D)$. Hence the second dual can be identified with a subspace of $L^\infty(D)$. Our main result is that E may be naturally identified with this second dual, and that the inclusion map of E_0 into E coincides with the natural embedding of E_0 in E_0^{**}. The corresponding result fails, as one can easily see, if the hypothesis of analyticity is replaced by mere continuity in the definitions of E_0 and E. The result bears some similarity to the situation for sequence spaces; the second dual of the space of null sequences is the space of bounded sequences. It should also be compared with results of de Leeuw [2] and of Duren, Romberg, and Shields [3, § 4] for spaces of Lipschitzian functions.

Let $\varphi E_0 = \{\varphi f : f \in E_0\}$; φE is defined similarly. Let

$$N^1 = \left\{g \in L^1(D) : \int \varphi f g dA = 0, \forall f \in E\right\}.$$

Here, dA denotes two-dimensional Lebesgue measure on D. If $g \in L^1(D)$, let $[g]$ denote the coset $g+N^1$ that contains g. Thus, $[g]$ is an element of the

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quotient space L^1/N^1. As usual, we define the quotient norm by

$$\|[g]\| = \inf \{\|g+h\|_1 : h \in N^1\}.$$

THEOREM 1. $(E_0)^* = L^1/N^1$ *and* $(L^1/N^1)^* = E$.

REMARK. More precisely, the first part of the theorem means that each continuous linear functional λ on E_0 has the form $\lambda = \lambda_g$ where $g \in L^1(D)$ and

$$\lambda_g(f) = \int \varphi f g dA; \qquad f \in E_0,$$

and $\|\lambda_g\| = \|[g]\|$.

Similarly, the second part means that each continuous linear functional on L^1/N^1 has the form $\Lambda = \Lambda_f$ where $f \in E$ and

$$\Lambda_f([g]) = \int \varphi f g dA; \qquad g \in L^1(D),$$

and $\|\Lambda_f\| = \|f\|_E$.

The proof will be given via a series of lemmas and connecting comments.

LEMMA 1. *φE is a weak-star closed subspace of $L^\infty(D)$.*

PROOF. By a theorem of Banach (see [1], Chapter VIII, Theorem 5), it is enough to prove that φE is weak-star sequentially closed in $L^\infty(D)$. Suppose now that $f_n \in E$, $n = 1, 2, 3, \cdots$, with φf_n converging weak-star to $h \in L^\infty(D)$, that is,

$$\int \varphi f_n g dA \to \int h g dA, \qquad \forall g \in L^1.$$

By the uniform boundedness principle, the functions φf_n are bounded in norm. Therefore $\{f_n\}$ is uniformly bounded on each compact subset of D and so forms a normal family. By passing to a subsequence if necessary, we may assume that $\{f_n\}$ converges uniformly on compact sets to some analytic function f, which must lie in E since $\{\varphi f_n\}$ is uniformly bounded. Finally, by the Lebesgue dominated convergence theorem,

$$\int \varphi f_n g dA \to \int \varphi f g dA \qquad \text{for all } g \in L^1(D),$$

and so $h = \varphi f$, which proves the lemma.

The second part of the theorem now follows from the general theory of Banach spaces. For N^1 is a closed subspace of $L^1(D)$, and so the dual of the quotient space L^1/N^1 may be identified with $(N^1)^\perp$, the annihilator of N^1 in $L^\infty(D)$. From the definition of N^1 and from the general theory, we see that φE is a weak-star dense subspace of $(N^1)^\perp$. It follows from Lemma 1 that $\varphi E = (N^1)^\perp$.

The first part of the theorem is somewhat harder. First note that φE_0 is a closed subspace of $C_0(D)$, the continuous functions on the closed disc that vanish on the boundary, with the supremum norm. Then $(C_0)^* = M(D)$, the space of bounded complex-valued Borel measures on D, with the variation norm. Let

$$N = \left\{\mu \in M(D) : \int \varphi f d\mu = 0, \forall f \in E_0\right\}.$$

Then the general theory of Banach spaces tells us that the dual space of φE_0 may be identified with the quotient Banach space $M(D)/N$. Thus, our task is to show that this quotient space may be replaced by $L^1(D)/N^1$. We do this in two steps.

LEMMA 2. *If $\mu \in N$ then $\int \varphi f d\mu = 0$ for all $f \in E$.*

PROOF. Fix $f \in E$ and let $f_r(z) = f(rz)$, $0 < r < 1$. Then $f_r \in E_0$ and $f_r \to f$ uniformly on compact subsets of D. Also

$$\varphi(z)|f(rz)| \leqq \varphi(rz)|f(rz)| \leqq ||f||_E,$$

and the result now follows from the bounded convergence theorem.

If $\mu \in M(D)$, then $[\mu]$ will denote the coset $\mu+N$ that contains μ. We will identify $L^1(D)$ with the space of measures $\nu \in M(D)$ that are absolutely continuous with respect to dA. We write $\mu_1 \sim \mu_2$ to mean that $\mu_1+N = \mu_2+N$.

LEMMA 3. *Given $\mu \in M(D)$ and given $\varepsilon > 0$, there exists $\nu \in L^1(D)$ such that $\nu \sim \mu$ and $||\nu|| \leqq (1+\varepsilon)||\mu||$.*

PROOF. This lemma is very similar to a result proved in § 4.1 of [4], for which two proofs were given, the second occurring in § 4.24. Either proof can be adapted to the present situation — we follow the first proof, giving only the main steps. First, let ε_w be the unit point mass at a point $w \in D$. Choose $a = a(w)$ and $b = b(w)$ as continuous functions of w so that $0 < a < b$ and so that the annulus $A_w = \{z : a \leqq |z-w| \leqq b\}$ lies in D. Later, an extra condition will be imposed on b. We define the measure ν_w by

$$\nu_w(E) = \frac{\varphi(w)}{b-a}\int_{t=a}^{t=b}\left(\frac{1}{2\pi i}\int_{|\zeta-w|=t}\frac{\chi_E(\zeta)}{\varphi(\zeta)}\frac{d\zeta}{\zeta-w}\right)dt$$

for all Borel subsets E of D, where χ_E is the characteristic function of E. That $\nu_w \sim \varepsilon_w$ holds is just the Cauchy integral formula averaged over an annulus. Now, for any measure $\mu \in M(D)$, let ν be defined by

$$\nu(E) = \int \nu_w(E)d\mu(w) = \int \left(\int \chi_E(z)d\nu_w(z)\right) d\mu(w).$$

Then $\nu \sim \mu$. To estimate the norm of ν, we have, for any function f that is bounded and continuous in D, say $|f(z)| \leqq 1$,

$$\left|\int f(z)d\nu_w(z)\right| \leqq \frac{\varphi(w)}{\varphi(|w|+b)},$$

so that on choosing b sufficiently small, we have $\|\nu_w\| \leqq 1+\varepsilon$, from which the lemma follows, since it is clear that ν is absolutely continuous with respect to dA.

Using Lemmas 2 and 3, we see that the inclusion map of $L^1(D)$ into $M(D)$ induces an isometric mapping of L^1/N^1 onto M/N. The proof of the theorem is complete.

REMARK. For certain special weight functions φ (e.g. $\varphi(r) = (1-r)^\alpha$, $\alpha > 0$) Shields and Williams [5] have shown that the subspace N^1 is actually a direct summand of L^1, and so the quotient space L^1/N^1 may be replaced, in the statement of Theorem 1, by a subspace of L^1.

One word of caution is in order against trying to generalize the theorem too far. Let A_0 be the space of entire functions f such that $f(z)/z \to 0$ as $z \to \infty$, with norm

$$\|f\| = \sup \{|f(z)|/|z| : |z| \geqq 1\},$$

and let A be the space of entire functions f for which $f(z)/z$ is bounded for $|z| \geqq 1$, with the same norm. It is not true that $(A_0)^{**} = A$, since A_0 is one-dimensional and A is two-dimensional, by Liouville's theorem.

It would be interesting to find analogues of our theorem for non-radial weights φ and for non-circular domains. Williams [6] has obtained an analogue for spaces of entire functions f, assuming that $r^n\varphi(r) \to 0$ as $r \to \infty$ for $n = 0, 1, 2, \cdots$. Our proof will work there, except that in the proof of Lemma 2, the approximating functions f_r must be replaced by the polynomials σ_n which are the Cesàro means of the first n partial sums of the power series for f.

We conclude with a theorem about 'dominating' sets for E, in analogy with what was done for H^∞ in [4], § 4.10, 4.15. We shall not carry out a systematic investigation of this concept here, however.

THEOREM 2. *For each $\varepsilon > 0$ there exists a countable subset $S = S_\varepsilon$ of D, with no limit points in D, such that*

$$\sup \{\varphi(z)|f(z)| : z \in S\} \geqq (1-\varepsilon)\|f\|_E \tag{1}$$

for all $f \in E$.

PROOF. Let B denote the unit ball of E. The functions in B, on each compact subset of D, are uniformly bounded and hence are uniformly equicontinuous. Let

$$K_n = \left\{z : \frac{n-1}{n} \leqq |z| \leqq \frac{n}{n+1}\right\}, \qquad n = 1, 2, 3, \cdots.$$

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If $\varepsilon > 0$ is given, then there exists $\delta = \delta_n > 0$ such that

$$|\varphi(z)f(z)-\varphi(w)f(w)| < \varepsilon$$

for all $z, w \in K_n$ with $|z-w| < \delta$ and all $f \in B$. Hence if S_n is a finite subset of K_n that is δ_n dense, then for all $f \in B$, we have

$$\text{(2)} \qquad \sup \{|\varphi(z)f(z)| : z \in S_n\} \geqq \sup \{|\varphi(z)f(z)| : z \in K_n\}-\varepsilon.$$

Now let $S = \cup S_n$ so that S is a countable subset of D with no limit points in D. Clearly, we have

$$\text{(3)} \qquad \sup \{|\varphi(z)f(z)| : z \in S\} \geqq \sup \{|\varphi(z)f(z)| : z \in D\}-\varepsilon.$$

Finally, let $f \in E$ be arbitrary. Clearly (1) is satisfied if f is the zero function. If f is not the zero function, then $f/||f|| \in B$ and so

$$\sup \{\varphi(z)|f(z)|/||f|| : z \in S\} \geqq 1-\varepsilon,$$

which completes the proof.

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