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UNIFORM CONTRACTIFICATION

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ABSTRACT. Let (X, τ) be a metrizable space and $\{f_n : n = 1, 2, \ldots\}$ be a commuting family of continuous mappings on X with a common fixed point $\xi \in X$ such that (I) for each $k = 1, 2, \ldots, f_k^n[X] \to \{\xi\}$ as $n \to \infty$ and (II) $\cup_{n \geq k} f_n[X] \to \{\xi\}$ as $k \to \infty$. Then for each $c \in (0, 1)$, there exists a metric d on X inducing the topology τ such that $d(f_n(x), f_n(y)) \leq c\, d(x, y)$, for all $x, y \in X$ and $n = 1, 2, \ldots$. The above result is also generalized to Tychonoff spaces.

Let (X, τ) be a Tychonoff space and $\not p(\tau)$ be the collection of all families of pseudometrics on X inducing τ. If $A \subset X$, $\bar{A}$ denotes the closure of A. If $D \in \not p(\tau)$, a sequence $(x_n)_{n=1}^{\infty}$ in X is said to be a Cauchy sequence with respect to D if for each $d \in D$, $d(x_n, x_m) \to 0$ as $n, m \to \infty$. (X, τ) is sequentially complete with respect to D if every Cauchy sequence with respect to D converges in (X, τ).

THEOREM 1. *Let (X, τ) be a Thchonoff space and $\{f_n : n = 1, 2, \ldots\}$ a commuting family of continuous mappings on X with a common fixed point $\xi \in X$ such that*

(i) *for each $k = 1, 2, \ldots, f_k^n[X] \to \{\xi\}$ as $n \to \infty$*
(ii) *$\cup_{n \geq k} f_n[X] \to \{\xi\}$ as $k \to \infty$.*

Then for each $D \in \not p(\tau)$, there exists $D^* \in \not p(\tau)$ such that (i) Card D = Card D^*, *and* (ii) *for each $\rho \in D^*$, $\rho \leq 1$ and $\rho(f_n(x), f_n(y)) \leq \rho(x, y)$, for all $x, y \in X$ and $n = 1, 2, \ldots$. Moreover, if (X, τ) is sequentially complete with respect to D, (X, τ) is also sequentially complete with respect to D^*.*

Proof. Let $D \in \not p(\tau)$. We may assume that $d \leq 1$ for each $d \in D$, otherwise we replace d by the equivalent pseudometric $d/1 + d$. For each $d \in D$, define

$$d^*(x, y) = \sup\{d(f_1^{k_1} \cdots f_n^{k_n}(x), f_1^{k_1} \cdots f_n^{k_n}(y)) : k_i \geq 0, n = 1, 2, \ldots\}$$

for all $x, y \in X$. Let $D^* = \{d^* : d \in D\}$. Then D^* satisfies all the required properties. (See the proof of Theorem 1 in [1].) The last assertion follows from the fact that $d \leq d^*$ for all $d \in D$.

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COROLLARY 2. *Let (X, τ) be a metrizable space and $\{f_n : n = 1, 2, \ldots\}$ be a commuting family of continuous mappings on X with a common fixed point $\xi \in X$ such that the conditions (i) and (ii) in Theorem 1 are satisfied. Then there exists a bounded metric $d(\leq 1)$ on X inducing τ such that $d(f_n(x), f_n(y)) \leq d(x, y)$, for all $x, y \in X$ and $n = 1, 2, \ldots$. Moreover, the metric d can be chosen complete if (X, τ) is completely metrizable.*

In the proof of the next theorem, we use the idea first developed in the proof of theorem 2 in [4].

THEOREM 3. *Let (X, τ) be a metrizable space and $\{f_n : n = 1, 2, \ldots\}$ be a commuting family of continuous mappings on X with a common fixed point $\xi \in X$ such that the conditions (i) and (ii) in Theorem 1 are satisified. Then $\{f_n : n = 1, 2, \ldots\}$ is uniformly contractifiable under a bounded metric on X, i.e., for each $c \in (0, 1)$, there exists a bounded metric $d(\leq 1)$ on X inducing τ such that $d(f_n(x), f_n(y)) \leq cd(x, y)$, for all $x, y \in X$ and $n = 1, 2, \ldots$.*

Proof. By Corollary 2, there exists a bounded metric $\rho(\leq 1)$ on X inducing τ such that $\rho(f_n(x), f_n(y)) \leq \rho(x, y)$, for all $x, y \in X$ and $n = 1, 2, \ldots$. For $x, y \in X$, let

$$n(x) = \sup\{m : m = n_1 + \cdots + n_k, \text{ where } n_i \geq 0,\ k \geq 1 \text{ and } x \in \overline{f_1^{n_1} \cdots f_k^{n_k}[X]}\},$$

$$n(x, y) = \min\{n(x), n(y)\},$$

$$\lambda(x, y) = c^{n(x, y)} \rho(x, y), \text{ with the convention that } c^\infty = 0,$$

$$d(x, y) = \inf\{\textstyle\sum_{i=1}^n \lambda(x_i, x_{i+1}) : x_1, \ldots, x_{n+1} \in X \text{ with } x_1 = x \text{ and }$$

$$x_{n+1} = y\ ,\ n = 1, 2, \ldots\}.$$

Then we can show that (*a*) d is a bounded (by 1) metric on X such that $d(f_n(x), f_n(y)) \leq cd(x, y)$, for all $x, y \in X$ and $n = 1, 2, \ldots$ and (*b*) for all $x, y \in X$ with $x \neq \xi$, $d(x, y) \geq c^{c(x)} \cdot \min\{L_x, \rho(x, y)\}$, where $L_x = \inf\{\rho(x, \overline{f_1^{t_1} \cdots f_q^{t_q}[X]}) : q \geq 1$ and $t_1 + \cdots + t_q > n(x)\} > 0$. (See the proof of Theorem 3 in [1].) To complete the proof, it remains to prove that d and ρ are equivalent. Since $d \leq \rho$, it suffices to show that for $x_n, x \in X$, $n = 1, 2, \ldots, d(x, x_n) \to 0$ as $n \to \infty$ implies $\rho(x, x_n) \to \infty$ as $n \to \infty$.

CASE 1. Suppose $x \neq \xi$. Then by (*b*), $d(x, x_n) \geq c^{n(x)} \cdot \min\{L_x, \rho(x, x_n)\}$, where $L_x > 0$ depends only on x. Since $d(x, x_n) \to 0$ as $n \to \infty$, we must have $\rho(x, x_n) \to \infty$.

Before we prove the other case, we shall prove the following:

(*) for any $\varepsilon > o$, there exists a positive integer $N(\varepsilon)$ such that for all $y \in X$, $\rho(y, \xi) > \varepsilon$ implies $d(y, \xi) \geq c^{N(\varepsilon)} \cdot \varepsilon/2$. Indeed, let $\varepsilon > 0$. Since $\overline{\bigcup_{n \geq k} f_n[X]} \to \{\xi\}$ as $k \to \infty$, there exists a positive integer $N_0 > 1$ such that $\overline{\bigcup_{n \geq N_0} f_n[X]} \subset B_\rho(\xi; \varepsilon/2) = \{z \in X : \rho(\xi, z) < \varepsilon/2\}$. For each $k = 1, \ldots, N_0 - 1$, since $f_k^n[X] \to \{\xi\}$ as $n \to \infty$, there exists a positive integer n_k such that $\overline{f_k^{n_k}[X]} \subset B_\rho(\xi; \varepsilon/2)$. Define $N(\varepsilon) = n_1 + \cdots + n_{N_0-1}$. Note that for any $z \in X$, $z \notin \overline{\bigcup_{j \geq N_0} f_j[X] \cup \bigcup_{i=1}^{N_0-1} f_i^{n_i}[X]}$ implies $n(z) < N(\varepsilon)$. Now suppose $\rho(y, \xi) > \varepsilon$. Let $\eta > 0$ be given. Then there exists $x_1 = y, x_2, \ldots, x_{M+1} = \xi \in X$ such that $d(y, \xi) + \eta > \sum_{i=1}^M c^{n(x_i, x_{i+1})} \rho(x_i, x_{i+1})$. Define $k = \min\{i : \rho(x_i, \xi) < \varepsilon/2\}$, then $k \geq 2$ since $\rho(y, \xi) > \varepsilon$. It follows that for $i = 1, \ldots, k-1$, $\rho(x_i, \xi) \geq \varepsilon/2$ and hence $n(x_i) < N(\varepsilon)$. Thus

$$\begin{aligned} d(y, \xi) + \eta &\geq c^{N(\varepsilon)} \rho(y, x_2) + \cdots + c^{N(\varepsilon)} \rho(x_{k-1}, x_k) + c^{n(x_k, x_{k+1})} \rho(x_k, x_{k+1}) \\ &\quad + \cdots + c^{n(x_M, \xi)} \rho(x_M, \xi) \\ &\geq c^{N(\varepsilon)} \cdot \{\rho(y, x_2) + \cdots + \rho(x_{k-1}, x_k)\} \\ &\geq c^{N(\varepsilon)} \cdot \rho(y, x_k) \\ &\geq c^{N(\varepsilon)} \cdot \frac{\varepsilon}{2}. \end{aligned}$$

Therefore $d(y, \xi) \geq c^{N(\varepsilon)} \cdot \varepsilon/2$ as $\eta > 0$ is arbitrary.

CASE 2. Suppose $x = \xi$, i.e. $d(x_n, \xi) \to 0$ as $n \to \infty$. Thus for $\varepsilon > 0$, there exists a positive integer N such that $d(x_n, \xi) < c^{N(\varepsilon)} \cdot \varepsilon/2$ for all $n \geq N$. By (*), $\rho(x_n, \xi) \leq \varepsilon$ for all $n \geq N$. Hence $\rho(x_n, \xi) \to 0$ as $n \to \infty$. This completes the proof of the theorem.

The above result solves the problem posted at the end of section 1 in [2].

EXAMPLE 4. Let $X = [0,1]$ equipped with the usual topology τ. For $n = 1, 2, \ldots$, define $f_n(x) = \left(1 - \frac{1}{n}\right)x$, for all $x \in X$. Then $\{f_n : n = 1, 2, \ldots\}$ is a commuting family of continuous mappings on X with a common fixed point $\xi = 0$ and satisfies the condition (i) but not the condition (ii) in Theorem 3. Since $f_k^n[X] \to \{0\}$ as $n \to \infty$ is not uniform in k, $\{f_n : n = 1, 2, \ldots\}$ cannot be uniformly contractifiable, i.e. there does not exist a metric d on X equivalent to the usual metric on X such that each f_n is a d-contraction with the same Lipschitz constant.

If X is the real line, the above example was given by A. J. Goldman and P. R. Meyers in [2]. Since [0, 1] is compact, the above examples shows that even if (X, τ) is compact, the condition (ii) in Theorem 3 (and hence also Theorem 3 in [1]) cannot be omitted.

Observing that if $\{f_n : n = 1, 2, \ldots\}$ is uniformly contractifiable under a bounded metric, then $f_k^n[X] \to \{\xi\}$ as $n \to \infty$ uniformly in k, Theorem 3 can be rephrased to give us the following characterizations:

THEOREM 5. *Let (X, τ) be a metrizable space and $\{f_n : n = 1, 2, \ldots\}$ be a commuting family of continuous mappings on X with a common fixed point $\xi \in X$ such that $\bigcup_{n \geq k} f_n[X] \to \infty$. Then the following are equivalent:*

(1) *For each $k = 1, 2, \ldots, f_k^n[X] \to \{\xi\}$ as $n \to \infty$.*

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(2) $f_k^n[X] \to \{\xi\}$ *as* $n \to \infty$, *uniformly in* k,

(3) $\{f_n : n = 1, 2, \ldots\}$ *is uniformly contractifiable under a bounded metric on* X.

Theorems 1 and 3 together with necessary modification in the proof of Theorem 4 in [1] give us the following:

THEOREM 6. *Let* (X, τ) *be a Tychonoff space and* $\{f_n : n = 1, 2, \ldots\}$ *be a commuting family of continuous mappings on* X *with a common fixed point* $\xi \in X$ *such that* $\bigcup_{n \geq k} f_n[X] \to \{\xi\}$ *as* $k \to \infty$. *Then the following are equivalent*:

(1) *for each* $k = 1, 2, \ldots, f_k^n[X] \to \{\xi\}$ *as* $k \to \infty$,

(2) $f_k^n[X] \to \{\xi\}$ *as* $n \to \infty$, *uniformly in* k,

(3) $\{f_n : n = 1, 2, \ldots\}$ *is topologically uniformly contractifiable under bounded pseudometrics on* X, *i.e. for each* $c \in (0, 1)$ *and for each* $D \in \not{p}(\tau)$, *there exists* $D^* \in \not{p}(\tau)$ *such that* (*i*) Card $D =$ Card D^*, *and* (*ii*) *for each* $\rho \in D^*$, $\rho \leq 1$, *and* $\rho(f_n(x), f_n(y)) \leq c\rho(x, y)$, *for all* $x, y \in X$ *and* $n = 1, 2, \ldots$.

If f is a mapping on X, define $f_k = f^k$ for $k = 1, 2, \ldots$, we see that the conditions (i) and (ii) coincide. This observation gives us the following:

COROLLARY 7. *Let* (X, τ) *be a Tychonoff space and* $f : X \to X$ *be continuous with a fixed point* $\xi \in X$. *Then the following are equivalent*:

(1) $f^n[X] \to \{\xi\}$ *as* $n \to \infty$,

(2) *for each* $c \in (0, 1)$ *and for* $D \in \not{p}(\tau)$ *there exists* $D^* \in \not{p}(\tau)$ *such that* (*i*) Card $D =$ Card D^* *and* (*ii*) *for each* $\rho \in D^*$, $\rho \leq 1$ *and* $\rho(f(x), f(y)) \leq c\rho(x, y)$ *for all* $x, y \in X$.

The above result answers the question raised at the end of [3].

We conclude here with the following two open problems:

Problem 1. In Theorem 3, if (X, τ) is completely metrizable, can the metric d be so chosen to be also complete?

We remark that the above question remains open even if $\{f_n : n = 1, 2, \ldots\}$ is finite.

Problem 2. In Theorem 5 (resp. Theorem 6), if we drop the condition "$\bigcup_{n \geq k} f_n[X] \to \{\xi\}$ as $k \to \infty$," do conditions (2) and (3) remain equivalent?

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