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A NOTE ON YOUNG'S RAISING OPERATOR

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Consider the following formula due to Young [**7**] for the calculation of the homogeneous product sum, h_λ, in terms of Schur functions;

$$h_\lambda = \sum[\prod S_{rs}^{\alpha_{rs}}]\{\lambda\}$$

where the operation S_{rs} is defined as follows:

Y_1: S_{rs}, where $r < s$, "represents the operation of moving one letter from the s-th row up to the r-th row; and the resulting term is regarded as zero, when any row becomes less than a row below it, or when letters from the same row overlap; as, for instance, happens when $\lambda_1 = \lambda_2$ in the case of $S_{13}S_{23}$."

The following example of the above is given by Robinson [**4**].

$$\begin{aligned} h_{(3,2,1)} &= [1 + S_{23} + S_{13} + S_{12} + S_{12}S_{23} + S_{12}S_{13} + S_{12}^2S_{23} + S_{12}^2S_{13}]\{3, 2, 1\} \\ &= \{3, 2,1\} + \{3, 3\} + \{4, 2\} + \{4, 1, 1\} + \{4, 2\} \\ &\quad + \{5, 1\} + \{5, 1\} + \{6\}. \end{aligned}$$

Calculation by other means shows that the above analysis of $h_{(3,2,1)}$ is correct; however, it will be noticed that the operator $S_{12}^3S_{23}$ does not appear in the above yet it is not specifically excluded by the rule Y_1. The further condition

$$Y_2: \alpha_{rs} \leqq \lambda_s \text{ for } s = 2, 3, \ldots .$$

is also required, although neither Young nor Robinson mention this fact. (The operators produced by Robinson do in fact satisfy this above condition also.)

It will be shown in this paper that by using a well-established extension of the definition of a Schur function that allows for parts in non-decreasing order of magnitude, both rules Y_1 and Y_2 become unnecessary and the expression

$$h_\lambda = \sum[\prod S_{rs}^{\alpha_{rs}}]\{\lambda\}$$

becomes an unrestricted summation.

1. Raising operators. Let n be a positive integer. A *partition* of n is a set

$$(\lambda) = (\lambda_1, \lambda_2, \ldots, \lambda_N)$$

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of N integers, (positive, negative, or zero), such that

$$\lambda_1 + \lambda_2 + \ldots + \lambda_N = n.$$

If (λ) is such that $\lambda_1 \geqq \lambda_2 \geqq \ldots \geqq \lambda_N \geqq 0$, then (λ) will be called a *proper partition* of n and we shall denote this fact by $\lambda \vdash n$.

We now define an operator δ_{ij} which operates on a partition (λ) by increasing the term λ_i by one and decreasing λ_j by one. We shall be considering the case when $i < j$ in which case we shall call δ_{ij} a *raising operator*.

If we now consider a function $f(\lambda)$, we can allow δ_{ij} to operate on $f(\lambda)$ by defining

$$\delta_{ij}[f(\lambda)] = f(\delta_{ij}(\lambda)).$$

2. Symmetric functions. Let $x_1, x_2, \ldots, x_m$ be a set of m variables or indeterminates. For a given partition $(\lambda) = (\lambda_1, \ldots, \lambda_N)$ we define the *monomial symmetric function*, M_λ, to be the sum

$$\sum x_1^{\lambda_1} x_2^{\lambda_2} \ldots x_N^{\lambda_N}$$

of all different monomial expressions of the form

$$x_{i_1}^{\lambda_1} x_{i_2}^{\lambda_2} \ldots x_{i_N}^{\lambda_N}$$

where $i_1, i_2, \ldots, i_N$ is a selection of N different numbers from the set $1, 2, \ldots, m$ taken in any order.

The *symmetric product sum*, s_r, is simply defined to be the monomial symmetric function for $(\lambda) = (r)$, i.e.,

$$s_r = \sum_{i=1}^{m} x_i^r \text{ for } r = 1, 2, \ldots \text{ and } s_0 = 1.$$

Thence, we define $s_\lambda = s_{\lambda_1} s_{\lambda_2} \ldots s_{\lambda_N}$.

The *homogeneous product sum*, h_r, is defined by

$$h_r = \sum_{\mu} M_\mu$$

where the summation is over all proper partitions (μ) of r. In addition, we define $h_0 = 1$, $h_r = 0$ if $r < 0$, and finally $h_\lambda = h_{\lambda_1} h_{\lambda_2} \ldots h_{\lambda_N}$.

Let $\sigma = \sigma(1)\sigma(2) \ldots \sigma(N)$ be a permutation of the numbers $1\ 2 \ldots N$ and let

$$(\lambda\sigma) = (\lambda_{\sigma(1)}, \lambda_{\sigma(2)}, \ldots, \lambda_{\sigma(N)}).$$

Clearly, we have that

$$h_{\lambda\sigma} = h_\lambda$$

for all permutations σ. Thus, any homogeneous product sum h_ν has a canonical form h_λ in which $\lambda_1 \geqq \lambda_2 \geqq \ldots \geqq \lambda_N$ obtained by rearranging the terms in (ν) into descending order. In addition, $h_\lambda = 0$ if any $\lambda_i < 0$. Thus we have that the set of all homogeneous product sums is given by $\{h_\lambda : (\lambda) \vdash n,\ n = 0, 1, 2, \ldots\ldots\}$.

Finally, we shall define the *Schur function* of a proper partition (λ), which will be denoted by $\{\lambda\}$. There are numerous equivalent definitions of these functions and the reader is referred to [**1**], [**2**], [**3**], and [**5**]. The most usual definition is probably in terms of symmetric group characters as follows:

$$\{\lambda\} = \frac{1}{n!} \sum_\rho g_\rho \chi_\rho{}^\lambda s_\rho$$

where the summation is over all proper partitions (ρ) of n, $\chi_\rho{}^\lambda$ is the characteristic of the conjugacy class (ρ) of $\mathscr{S}_n$ (the symmetric group of degree n) in the representation corresponding to (λ), g_ρ is the order of the conjugacy class (ρ) in $\mathscr{S}_n$, and s_ρ is the symmetric power sum.

3. Results.

THEOREM 1.

$$\{\lambda\} = \prod_{i<j} (1 - \delta_{ij}) h_\lambda.$$

Example. $(\lambda) = (3, 2, 2)$, and so,

$$\begin{aligned}\{\lambda\} &= (1 - \delta_{12})(1 - \delta_{13})(1 - \delta_{23}) h_{322} \\ &= h_{322} - h_{331} - h_{421} + h_{43} - h_{412} + h_{421} + h_{511} - h_{52} \\ &= h_{322} - h_{331} - h_{421} + h_{43} + h_{511} - h_{52}.\end{aligned}$$

Proof. The following expression for $\{\lambda\}$ is well known (e.g. see [**2**]).

$$\{\lambda\} = |h_{\lambda_i - i + j}| = \begin{vmatrix} h_{\lambda_1} & h_{\lambda_1+1} & h_{\lambda_1+2} & \ldots & h_{\lambda_1+N-1} \\ h_{\lambda_2-1} & h_{\lambda_2} & h_{\lambda_2+1} & \ldots & h_{\lambda_2+N-2} \\ \cdot & \cdot & & & \cdot \\ \cdot & \cdot & & & \cdot \\ \cdot & \cdot & & & \cdot \\ h_{\lambda_N-N+1} & h_{\lambda_N-N+2} & & \ldots & h_{\lambda_N} \end{vmatrix}$$

Hence

$$(1) \quad \{\lambda\} = \sum \pm\, h_{(1,\sigma(1))} h_{(2,\sigma(2))} \ldots h_{(N,\sigma(N))}$$

where $h_{(i,\sigma(i))} = h_{\lambda_i + \sigma(i) - i}$ and the summation is over all permutations σ, the $+$ or $-$ being taken depending on whether σ is even or odd.

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However, consider the Vandermonde determinant

$$|x_i^{j-1}| = \begin{vmatrix} 1 & x_1 & x_1^2 & \dots & x_1^{n-1} \\ 1 & x_2 & x_2^2 & \dots & x_2^{n-1} \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ 1 & x_n & x_n^2 & \dots & x_n^{n-1} \end{vmatrix}.$$

Clearly, we have

$$|x_i^{j-1}| = \prod_{j>i} (x_j - x_i)$$

and also

$$|x_i^{j-1}| = \sum \pm x_{\sigma(1)}{}^0 x_{\sigma(2)}{}^1 x_{\sigma(3)}{}^2 \dots x_{\sigma(n)}{}^{n-1}$$

the summation being over all permutations σ, the $+$ and $-$ sign being taken according as σ is even or odd. Hence,

$$\prod_{i<j} \left(1 - \frac{x_i}{x_j}\right) x_1{}^0 x_2{}^1 x_3{}^2 \dots x_n{}^{n-1} = \sum \pm x_{\sigma(1)}{}^0 x_{\sigma(2)}{}^1 \dots x_{\sigma(n)}{}^{x-1}$$

i.e.,

$$\prod_{i<j} (1 - \delta_{ij}) x_1{}^0 x_2{}^1 \dots x_n{}^{n-1} = \sum \pm x_{\sigma(1)}{}^0 x_{\sigma(2)}{}^1 \dots x_{\sigma(n)}{}^{n-1}$$

where δ_{ij} operates on the sequence of suffixes $1\ 2\ 3 \dots n$.

Comparing this with equation (1), we have

$$(2) \quad \sum \pm h_{(1,\sigma(1))} h_{(2,\sigma(2))} \dots h_{(N,\sigma(N))} = \prod_{i<j} (1 - \delta_{ij}) h_{(1,1)} \dots h_{(N,N)}$$

where δ_{ij} operates on the second suffixes.

But, if D is a term in $\prod_{i<j}(1 - \delta_{ij})$ and $D(12 \dots N) = \sigma(1)\sigma(2) \dots \sigma(N)$ then

$$h_{(1,\sigma(1))} \dots h_{(N,\sigma(N))} = D(h_{(1,1)} \dots h_{(N,N)}).$$

But $h_{(i,\sigma(i))} = h_{\lambda_i - i + \sigma(i)}$, so

$$D(h_{(1,1)} \dots h_{(N,N)}) = D(h_{\lambda_1} \dots h_{\lambda_N}) = Dh_\lambda.$$

Thus, from (1), (2), and the above, we have the required result, namely

$$\{\lambda\} = \prod_{i<j} (1 - \delta_{ij}) h_\lambda.$$

COROLLARY.

$$h_\lambda = \prod_{i<j} \frac{1}{(1 - \delta_{ij})} \{\lambda\}.$$

Proof. We have

$$\delta_{ab}\{\lambda\} = \delta_{ab} \prod_{i<j} (1 - \delta_{ij})h_\lambda = \prod_{i<j} (\delta_{ab} - \delta_{ab}\delta_{ij})h_\lambda$$

$$= \prod_{i<j} (1 - \delta_{ij})(\delta_{ab}h_\lambda).$$

We note from (1) that, if for any i, $\lambda_i < i - N$, then $\{\lambda\} = 0$. Thus, the sum $(1 + \delta_{ab} + \delta_{ab}^2 + \delta_{ab}^3 + \ldots\ldots)\{\lambda\}$ contains only a finite number of nonzero terms, and hence

$$\prod_{i<j} \frac{1}{(1 - \delta_{ij})} \{\lambda\} = \prod_{i<j} (1 + \delta_{ij} + \delta_{ij}^2 + \ldots\ldots)\{\lambda\}$$

$$= \prod_{i<j} (1 + \delta_{ij} + \delta_{ij}^2 + \ldots)(1 - \delta_{ij})h_\lambda = h_\lambda.$$

Note that the above expression for h_λ is in terms of Schur functions $\{\mu\}$ where (μ) is not necessarily a proper partition. However, from (1) above, we have that

$$\text{(3)} \quad \{\lambda_1, \ldots, \lambda_i, \lambda_{i+1}, \ldots, \lambda_N\} = -\{\lambda_1, \ldots, \lambda_{i+1} - 1, \lambda_i + 1, \ldots, \lambda_N\}$$

and also $\{\lambda\} = 0$ if $\lambda_{i+1} = \lambda_i + 1$ for any i. Thus, for any partition (μ), the Schur function $\{\mu\}$ is either 0 or equal to $\pm\{\lambda\}$ where (λ) is a proper partition formed by successive applications of (3).

Example.

$$h_1h_1h_1 = (1 + \delta_{12} + \delta_{12}^2 + \ldots\ldots)(1 + \delta_{13} + \delta_{13}^2 + \ldots\ldots)$$

$$\times\ (1 + \delta_{23} + \lambda_{23}^2 + \ldots\ldots)\{1, 1, 1\}$$

$$= (1 + \delta_{12} + \delta_{12}^2 + \ldots\ldots)(1 + \delta_{13} + \delta_{13}^2 + \ldots\ldots)$$

$$\times\ (\{1, 1, 1\} + \{1, 2, 0\})$$

$$= (1 + \delta_{12} + \delta_{12}^2 + \ldots\ldots)(\{1, 1, 1\} + \{1, 2, 0\} + \{2, 1, 0\})$$

$$= \{1, 1, 1\} + \{1, 2, 0\} + \{2, 1, 0\} + \{2, 0, 1\}$$

$$+ \{3, -1, 1\} + \{2, 1, 0\} + \{3, 0, 0\} + \{3, 0,0\}$$

$$= \{1, 1, 1\} + \{2, 1\} - \{3\} + \{2, 1\} + \{3\} + \{3\}$$

$$= \{1, 1, 1\} + 2\{2, 1\} + \{3\}.$$

The above formulae are extensions of Young's formula given in the introduction of this paper. They are of particular interest when compared with the formulae given by Littlewood [**3**] for Hall-Littlewood polynomials, viz.,

$$\{\lambda\}^q = \prod_{i<j} (1 - t\delta_{ij})Q_\lambda(t) \quad \text{and} \quad Q_\lambda(t) = \prod_{i<j} \frac{1}{(1 - t\delta_{ij})} \{\lambda\}^q;$$

(see [**6**]).

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