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AN ESTIMATE OF RAMANUJAN RELATED TO THE GREATEST INTEGER FUNCTION

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If a and n are positive integers and if $\lfloor\ \rfloor$ is the greatest integer function we obtain upper and lower estimates for $\sum_{k=1}^{\infty}\lfloor n/a^k\rfloor$ stated by Ramanujan in his notebooks.

1. INTRODUCTION

Let $\lfloor x\rfloor$ denote the greatest integer not exceeding the real number x. If p is a prime it is well known [3, p.80] that the sum e of the series $\sum_{k=1}^{\infty}\lfloor n/p^k\rfloor$ is the largest exponent such that $n!$ is divisible by p^e. In this note we prove a proposition stated in the third notebook of Ramanujan [4, p.378] which, along with its predecessors are being edited by Berndt [1].

PROPOSITION 1. (Ramanujan). *If a and n are positive integers, then $\sum_{k=1}^{\infty}\lfloor n/a^k\rfloor$ lies between $(n-1)/(a-1)$ and $\{n/(a-1)\}-\{\log(n+1)/\log a\}$.*

We observe that we may assume in the proposition $a\geqslant 2$ and $n\geqslant 2$. Further, on using the binomial theorem, we have

$$(1+n)^{a-1}-a\geqslant 1+(a-1)n-a=(n-1)(a-1)>0.$$

Hence $(1+n)^{a-1}>a$ and this gives, on taking logarithms,

$$\{n/(a-1)\}-\{\log(n+1)/\log a\}<(n-1)/(a-1).$$

In order to establish Proposition 1 we therefore prove the following theorem.

THEOREM 1. (Ramanujan). *Let $a\geqslant 2$ and $n\geqslant 2$ be integers. Then*

$$\{n/(a-1)\}-\{\log(n+1)/\log a\}\leqslant\sum_{k=1}^{\infty}\lfloor n/a^k\rfloor\leqslant(n-1)/(a-1). \tag{1}$$

In the next section we first obtain two lemmas which enable us to prove Theorem 1.

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2. Proof of Theorem 1

Lemma 1. *If $n \geqslant 2$ and $k \geqslant 2$, then*

$$(n+k)^n < (n+1)^{n+k-1}. \tag{2}$$

Further, (2) becomes an equality in the cases

$$\text{(i)} \quad n = 0,\ k \geqslant 1, \quad \text{(ii)} \quad n = 1,\ k = 0, 1, \quad \text{(iii)} \quad n = 2,\ k = 1.$$

Proof: In the case $n = 2$, we have for each $k \geqslant 2$,

$$\begin{aligned}
&(n+1)^{n+k-1} - (n+k)^n \\
&\quad = (1+2)^{k+1} - (2+k)^2 \\
&\quad = \left\{1 + \frac{k+1}{1} \cdot 2 + \frac{(k+1)k}{1 \cdot 2} 2^2 + \cdots + 2^{k+1}\right\} - \left(4 + 4k + k^2\right) \\
&\quad = \left(k^2 - 1\right) + \frac{(k+1)k(k-1)}{1 \cdot 2 \cdot 3} 2^3 + \frac{(k+1)k(k-1)(k-2)}{1 \cdot 2 \cdot 3 \cdot 4} 2^4 + \cdots \\
&\quad = \left(k^2 - 1\right)(1 + \varepsilon), \qquad\qquad (\varepsilon > 0), \\
&\quad > 0,
\end{aligned}$$

since by hypothesis $k \geqslant 2$. Thus (2) is true for $n = 2$ and each $k \geqslant 2$.

Let us assume that (2) holds for each $k \geqslant 2$ and some $m\ (= n) \geqslant 2$. That is

$$(m+k)^m < (m+1)^{m+k-1}. \tag{2'}$$

Now, for each $k \geqslant 2$,

$$\begin{aligned}
(m+1+k)^{m+1} &= (m+k)^m \left(1 + \frac{1}{m+k}\right)^m (m+1+k) \\
&< (m+1)^{m+k-1} \left(1 + \frac{1}{m+k}\right)^m (m+1+k) \qquad \text{(by (2'))} \\
&= (m+2)^{m+k} \left(\frac{m+1}{m+2}\right)^{m+k} \left(1 + \frac{1}{m+k}\right)^m \left(1 + \frac{k}{m+1}\right) \\
&= (m+2)^{m+k} \left(\frac{1}{1 + \frac{1}{m+1}}\right)^k \left(\frac{1 + \frac{1}{m+k}}{1 + \frac{1}{m+1}}\right)^m \left(1 + \frac{k}{m+1}\right) \\
&< (m+2)^{m+k} \left(\frac{1}{1 + \frac{1}{m+1}}\right)^k \left(1 + \frac{k}{m+1}\right) \\
&\leqslant (m+2)^{m+k}.
\end{aligned}$$

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In writing the last inequality we have used the Bernoulli inequality:

$$(1+x)^{\alpha} \geqslant 1+\alpha x \; (\alpha > 1, x \geqslant -1) \tag{3}$$

with $x = 1/(m+1)$ and $\alpha = k$. We have thus shown that (2) is true with $n = m+1$. Thus we have proved Lemma 1 by induction on n. ∎

LEMMA 2. *If $a, b_0, b_1, \ldots, b_k$ are integers such that $a \geqslant 2$ and $0 \leqslant b_i < a$, $i = 0, 1, 2, \ldots, k$, then*

$$a^{b_0+b_1+\cdots+b_k} \leqslant [b_0 a^k + b_1 a^{k-1} + \cdots + b_k + 1]^{a-1}. \tag{4}$$

PROOF: Consider

$$\begin{aligned}
\prod_{i=0}^{k}(1+b_i) &\leqslant 1 + \sum_{m=1}^{k}[\sum_{0 \leqslant i_1 < i_2 < \cdots < i_j \leqslant m} b_{i_1} b_{i_2} \cdots b_{i_j}] \\
&= 1 + b_0 \left[1 + \sum_{m=1}^{k}\left(\sum_{1 \leqslant i_1 < i_2 < \cdots < i_j \leqslant m} b_{i_1} b_{i_2} \cdots b_{i_j}\right)\right] \\
&\quad + b_1 \left[1 + \sum_{m=1}^{k}\left(\sum_{2 \leqslant i_1 < i_2 < \cdots < i_j \leqslant m} b_{i_1} b_{i_2} \cdots b_{i_j}\right)\right] \\
&\quad + \\
&\quad \cdots \\
&\quad + b_{k-1}(1+b_k) + b_k \\
&\leqslant 1 + \sum_{m=0}^{k} b_m \left(1 + \sum_{j=1}^{k-m} \binom{k-m}{j}(a-1)^j\right) \\
&= 1 + \sum_{i=0}^{k} b_i a^{k-i}.
\end{aligned}$$

Taking the $(a-1)$-th power on both sides and using Lemma 1, we have (4). ∎

PROOF OF THEOREM 1: It is convenient to consider the cases $n = a$, $n < a$ and $n > a$ separately.

CASE (I). $n = a \geqslant 2$: We then have $n < a^k$ for each $k \geqslant 2$ and so $\sum_{k=1}^{\infty} \lfloor n/a^k \rfloor = 1$ and the second half of (1) is trivially true with equality prevailing. The first inequality

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of (1) is true in the strict sense, since

$$\begin{aligned}\sum_{k=1}^{\infty}\lfloor n/a^k\rfloor - \{n/(a-1)\} + \{\log(n+1)/\log a\}\\ = \{\log(n+1)/\log n\} - \{1/(n-1)\}\\ = \log\{(n+1)^{n-1}/n\}/(n-1)\log n > 0.\end{aligned}$$

CASE (II). $(2 \leqslant)\, n < a$: We then have $n < a^k$ for each $k \geqslant 1$ and so $\sum_{k=1}^{\infty}\lfloor n/a^k\rfloor = 0$ and the second half of (1) holds trivially in the strict sense. The first half of (1) is also true since

$$\begin{aligned}\sum_{k=1}^{\infty}\lfloor n/a^k\rfloor + \{\log(n+1)/\log a\} - \{n/(a-1)\}\\ = \{\log(n+1)/\log a\} - \{n/(a-1)\} > 0,\end{aligned}$$

on using (2) with $k = a-n$ there, provided $a \geqslant n+2$. In the other subcase $a = n+1$, the left and the middle expressions of (1) both being 0, we indeed have equality in the first half of (1).

CASE (III). $n > a\,(\geqslant 2)$: In this case there exists an integer $k \geqslant 2$ such that

$$k-1 \leqslant \log n/\log a < k,$$

or

$$a^{k-1} \leqslant n < a^k.$$

In the subcase $n = a^{k-1}$, we have

$$\begin{aligned}\sum_{m=1}^{\infty}\lfloor n/a^m\rfloor = \sum_{m=1}^{k-1}\lfloor n/a^m\rfloor\\ = a^{k-2} + a^{k-3} + \cdots + 1\\ = \frac{a^{k-1}-1}{a-1}\\ = \frac{n-1}{a-1}\end{aligned}$$

and thus the second half of (1) holds with equality. The first half of (1) also holds since

$$\begin{aligned}\sum_{m=1}^{\infty}\lfloor n/a^m\rfloor - \{n/(a-1)\} + \{\log(n+1)/\log a\}\\ = \{-1/(a-1)\} + \{\log(n+1)/\log a\}\\ > \{\log(a+1)/\log a\} - \{1/(a-1)\}\\ > 0\end{aligned}$$

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as shown at the end of the concluding step of case (I).

We are thus left with the subcase $a^{k-1} < n < a^k$. In this case we can write

$$n = b_0 a^{k-1} + b_1 a^{k-2} + \cdots + b_{k-2} a + b_{k-1} \tag{5}$$

$(0 \leqslant b_i < a, i = 0, 1, \ldots, k-1, b_0 \geqslant 1)$ and

$$\begin{aligned}\sum_{m=1}^{\infty} \lfloor n/a^m \rfloor &= \sum_{m=1}^{k-1} \lfloor n/a^m \rfloor \\ &= \sum_{m=1}^{k-1} \left(b_0 a^{k-m-1} + \cdots + b_{k-m-1}\right) \\ &= \sum_{j=0}^{k-2} b_j \left(\frac{a^{k-j-1} - 1}{a-1}\right) \\ &= \{(b_0 a^{k-1} + \cdots + b_{k-1})/(a-1)\} - \{(b_0 + \cdots + b_{k-1})/(a-1)\} \\ &= \{n/(a-1)\} - \{(b_0 + b_1 + \cdots + b_{k-1})/(a-1)\}.\end{aligned}$$

The second half of (1) follows immediately since $b_0 \geqslant 1$. The first half is also true since

$$\begin{aligned}\sum_{m=1}^{\infty} \lfloor n/a^m \rfloor - \{n/(a-1)\} + \{\log(n+1)/\log a\} \\ = \{\log(n+1)/\log a\} - \left\{\left(\sum_{i=0}^{k-1} b_i\right)/(a-1)\right\} \geqslant 0,\end{aligned}$$

on using (4) and (5).

This completes the proof of Theorem 1 and hence Proposition 1 is established. ▯

An alternate proof of Proposition 1 due to Berndt [2] is being incorporated in one of the forthcoming volumes of his editions of Ramanujan's Notebooks [1].

References

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