KLEINIAN GROUPS WITH UNBOUNDED LIMIT SETS

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1. The easiest way to construct automorphic functions is by means of the Poincaré series. If G is a Kleinian group with ∞ an ordinary point of G and if $k \geqq 4$, then

$$\sum_{V \in G, V\infty \neq \infty} |c|^{-k} < +\infty, \tag{1}$$

where $Vz = (az+b)/(cz+d)$ and $ad-bc = 1$. The convergence of this series is the crucial step in showing that the Poincaré series converges and is an automorphic form on G. If ∞ is a limit point of G, the series in (1) may diverge and one can derive automorphic forms on G from the Poincaré series of some conjugate group. These constructions are described in greater detail in [**3**, pp. 155–165].

While investigating another problem, the author arrived at the following generalisation of (1).

THEOREM 1. *Let G be any Kleinian group. If $k \geqq 4$, then*

$$\sum_{V \in G} (|a|+|b|+|c|+|d|)^{-k} < +\infty, \tag{2}$$

where $Vz = (az+b)/(cz+d)$ and $ad-bc = 1$.

This result contains (1) and can be used to give a direct construction of automorphic forms, valid for all Kleinian groups. This is indicated below.

The convergence of the series in (2) may also be of use in another context. First, the Hausdorff dimension $d(L)$, of the limit set L of G is invariant under conjugation [**2**, Corollary, p. 735]. Next, Akaza [**1**] and the author (in some unpublished work) have results relating $d(L)$ to

$$\delta(G) = \inf\{t > 0: \sum_{V \in G, V\infty \neq \infty} |c|^{-t} < +\infty\}$$

in the case when ∞ is an ordinary point of G. If ∞ is a limit point of G, $\delta(G)$ loses its significance and in any event, $\delta(G)$ is not invariant under conjugation. We can, however, prove the following result.

THEOREM 2. *Let G be any Kleinian group and (using the notation in Theorem 1) define*

$$\Delta(G) = \inf\{t > 0: \sum_{V \in G} (|a|+|b|+|c|+|d|)^{-t} < +\infty\}.$$

If A is any bilinear transformation, then $\Delta(AGA^{-1}) = \Delta(G)$.

It is easy to see that $\Delta(G) = \delta(G)$ when ∞ is an ordinary point; thus it may well be that $\Delta(G)$ rather than $\delta(G)$ is the relevant function of G.

2. *Proof of Theorem* 1. The proof of (1) is simply to note that, if Q is a closed disc disjoint from the orbit $G(\infty)$ of ∞ (in general, we denote the orbit of z by $G(z)$) and if Q lies

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in some fundamental region, then the discs $V(Q)$ $(V \in G)$ are disjoint and lie in some bounded set. Thus

$$\sum_{V \in G} \text{area} \quad V(Q) < +\infty$$

and this implies (1). To establish (2) we merely replace the area of $V(Q)$ by the area of the image of $V(Q)$ under stereographic projection onto the Riemann sphere. Equivalently, we use the chordal metric with an element of area

$$dA = \frac{dx\,dy}{(1+|z|^2)^2} \qquad (z = x+iy).$$

Thus we have

$$\sum_{V \in G} \int_{V(Q)} \frac{dx\,dy}{(1+|z|^2)^2} < +\infty.$$

A change of variable gives

$$\sum_{V \in G} \int_{Q} \frac{|V'(z)|^2\,dx\,dy}{(1+|Vz|^2)^2} < +\infty$$

which reduces to

$$\sum_{V \in G} \int_{Q} \frac{dx\,dy}{(|az+b|^2+|cz+d|^2)^2} < +\infty. \tag{3}$$

There exists a constant m such that, if $z \in Q$, then

$$|az+b|^2+|cz+d|^2 \leqq m[\lambda(V)]^2,$$

where

$$\lambda(V) = |a|+|b|+|c|+|d| \qquad (ad-bc=1)$$

and this together with (3) implies (2).

If ∞ is an ordinary point of G, there exists a positive constant m such that, for all but a finite set of V,

$$|V\infty| \leqq m, \quad |V^{-1}\infty| \leqq m \quad \text{and} \quad |V^{-1}0| \leqq m.$$

For these V we have

$$|c| \leqq \lambda(V) \leqq (1+m)^2|c|$$

and so with this assumption, (1) and (2) are equivalent.

Proof of Theorem 2. If $V \in G$ and if A is any bilinear transformation, we have

$$\lambda(AVA^{-1}) \leqq K\lambda(V) \leqq K^2\lambda(AVA^{-1}),$$

where K depends only on A (note that $V = A^{-1}AVA^{-1}A$) and so $\Delta(G) = \Delta(AGA^{-1})$.

Before describing the construction of automorphic forms we introduce some notation and one more result. Let R be any fundamental region of G and let E be any open set meeting L, the set of limit points of G. Define

$$\sum(E, t) = \sum_{V(R)\cap E\neq\varnothing} [\lambda(V)]^{-t}$$

and

$$\Delta_E = \inf\{t > 0: \sum(E, t) < +\infty\}.$$

THEOREM 3. *Let G be any Kleinian group and E any open set meeting L. Then $\Delta_E = \Delta(G)$.*

Proof. We need only show that Δ_E is independent of E, for $\Delta_E = \Delta(G)$ when E is the complex plane.

Suppose first that, for some positive t and some z_0 in L,

$$\sum(N, t) = +\infty \tag{4}$$

for every neighbourhood N of z_0. Let z_1 be a limit point and M a neighbourhood of z_1. Then there exists a T in G with $T(N) \subset M$ for some neighbourhood N of z_0 and so

$$\begin{aligned}\sum(M, t) &\geqq \sum_{V(R)\cap N\neq\varnothing} [\lambda(TV)]^{-t} \\ &\geqq \sum_{V(R)\cap N\neq\varnothing} [K\lambda(V)]^{-t} \\ &= +\infty,\end{aligned}$$

where K depends only on T. Thus, if one limit point has the property (4), so does every limit point. As L is compact in the chordal metric, the proof is easily completed.

3. We now sketch the construction of automorphic forms. Let G be any Kleinian group with w an ordinary point of G ($w \neq \infty$) and let E be any compact set of ordinary points that does not meet $G(w)$. Then there exists a positive number k (depending only on G, E and w) such that, for z in E and for all but a finite set of V,

$$\min\{\rho(V^{-1}\infty, z),\quad \rho(V^{-1}0, z),\quad \rho(Vz, w)\} \geqq k, \tag{5}$$

where ρ is the chordal metric. In the following, we use K to denote a positive quantity depending only on G, E and w and not necessarily the same at each occurrence. For z in E and for all but a finite set of V, we have (using (5)),

$$\begin{aligned}|cz+d|^2 &= |c|^2|z-V^{-1}\infty|^2 \\ &\geqq |c|^2k^2(1+|z|^2)(1+|V^{-1}\infty|^2) \\ &\geqq K(|c|^2+|d|^2). \end{aligned}\tag{6}$$

(If $c = 0$, the argument is not valid but (6) still holds.) A similar inequality holds for $|az+b|^2$. Thus, for z in E and for all but a finite set of V,

$$\begin{aligned} |(az+b)-w(cz+d)|^2 &= |cz+d|^2|Vz-w|^2 \\ &\geqq |cz+d|^2k^2(1+|w|^2)(1+|Vz|^2) \\ &\geqq K(|az+b|^2+|cz+d|^2) \\ &\geqq K[\lambda(V)]^2 \end{aligned} \tag{7}$$

by virtue of (5) and (6). Thus we have proved

LEMMA 1. *Let G be any Kleinian group with w ($\neq \infty$) an ordinary point of G. If $k \geqq 4$, the series*

$$\sum_{V \in G} |(az+b)-w(cz+d)|^{-k}$$

converges uniformly on any compact subset of the set of ordinary points that does not meet $G(w)$.

Finally, we prove

THEOREM 4. *Let G be any Kleinian group with an invariant component D of the set of ordinary points, let k be an even integer satisfying $k \geqq 4$ and let H be any rational function whose poles are ordinary points. Then*

$$F(z) = \sum_{V \in G} HV(z)[(az+b)-w(cz+d)]^{-k}$$

is an automorphic form on G of dimension $-k$.

Proof. An argument similar to that given in [3, pp. 159–161] (but using Lemma 1) shows that F is meromorphic in D and satisfies the required functional relation there. It remains to show that F is meromorphic at the parabolic vertices of G lying on the boundary of D, and to do this we need only make slight modifications to the proof of Theorem 3A of [3, pp. 162–163]. (We use the same notation as in this proof.) If z is in the parabolic sector at p cut out of C', then, for all but a finite set of V, (7) holds with $K = k^2(1+|w|^2)$, where we have assumed the sector to be sufficiently small not to meet $G(\infty)$, $G(0)$ or $G(w)$. The parabolic sector at p lies in some disc $|z| \leqq r$. If V is such that $|b| < 2r|a|$ we have

$$|az+b| = |a|\cdot|z-V^{-1}0| \geqq B|a|\cdot|z-p| \geqq B_1(|a|+|b|)|z-p|$$

for some constants B and B_1, whereas, if $|b| \geqq 2r|a|$, then

$$|az+b| \geqq |b|-|az| \geqq (|a|+|b|)|z-p|$$

if z is sufficiently close to p and within the parabolic sector. Similar inequalities hold for $|cz+d|$ and these together with (7) show that

$$|(az+b)-w(cz+d)|^2 \geqq K_1[\lambda(V)]^2|z-p|^2.$$

The proof is easily completed and F is analytic at p.

Let H_1 be any rational function whose poles are ordinary points and suppose that H_1 has a pole of order k ($\geqq 4$) at some ordinary point w ($\neq \infty$). Then $H(z) = H_1(z)(z-w)^k$ is a rational function and so

$$F(z) = \sum_{V \in G} HV(z)[(az+b)-w(cz+d)]^{-k} = \sum_{V \in G} H_1 V(z)(cz+d)^{-k}$$

is an automorphic form on G. This is known when ∞ is an ordinary point of G; the above argument is valid, however, when ∞ is a limit point of G.

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