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ON WEIGHTED NORM INEQUALITIES FOR FRACTIONAL AND SINGULAR INTEGRALS

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0. Introduction. In a recent paper [**12**] Muckenhoupt and Wheeden have established necessary and sufficient conditions for the validity of norm inequalities of the form $\| |x|^\alpha Tf \|_q \leqq C \| |x|^\alpha f \|_p$, where Tf denotes a Calderón and Zygmund singular integral of f or a fractional integral with variable kernel. The purpose of the present paper is to prove, by somewhat different methods, similar inequalities for more general weight functions.

In what follows, for $p \geqq 1$, p' is the exponent conjugate to p, given by $1/p + 1/p' = 1$. Ω will always denote a locally integrable function on $\mathbf{R}^n$ which is homogeneous of degree 0, $\Omega^\sim$ will denote a measurable function on $\mathbf{R}^n \times \mathbf{R}^n$ such that for each $x \in \mathbf{R}^n$, $\Omega^\sim(x, .)$ is locally integrable and homogeneous of degree 0. $\|\Omega\|_u$ is the L^u norm of Ω, restricted to the unit sphere $S^{n-1} = \{x \in \mathbf{R}^n: |x| = 1\}$, with respect to Euclidean surface measure σ on S^{n-1}. If $u = 1$

$$\|\Omega\|_1{}^* = 1 + \|\Omega_0\|[L \log^+ L(S^{n-1})] + \|\Omega_1\|_1,$$

where Ω_0, Ω_1 denote the even and odd parts of Ω, respectively (see [**3**, Theorem 1]). $\|\Omega^\sim\|_u$ will denote ess sup$\{\|\Omega^\sim(x, .)\|_u: x \in R^n\}$. w_0, w_1 and ω_0, ω_1 denote non-negative measurable functions on $\mathbf{R}^n$ and $\mathbf{R}_+ = (0, \infty)$, respectively. For $x \in \mathbf{R}^n$, $\omega_0(x)$, for instance, has the same meaning as $\omega_0(|x|)$.

Let χ denote the characteristic function of the interval $(\frac{1}{2}, 2)$. $\mathbf{Z}$ will denote the set of integers. For any integer z, the quantities $M_r(w_0, w_1, \Omega, z)$, $M^*{}_{r,v}(w_0, w_1, z)$, $N_r(w_0, \Omega^\sim, z)$, $N^*{}_{r,v}(w_0, z)$ are defined as follows:

$$(1) \quad M_r(w_0, w_1, \Omega, z) = \operatorname*{ess\,sup}_{2^{z-1}<|x|<2^z} w_0(x)^{-1} \cdot \left[\sup_{\rho>0} \rho^{-n} \int_{|y|<|\Omega(y)|^{r/n}\rho} \chi(|x-y|/|x|) w_1(x-y)dy\right]^{1/r};$$

$$(2) \quad M^*{}_{r,v}(w_0, w_1, z) = \operatorname*{ess\,sup}_{2^{z-1}<|x|<2^z} w_0(x)^{-1} \cdot \left[\int_{S^{n-1}} \left(\sup_{\rho>0} \rho^{-n} \int_0^\rho \chi(|x+ty'|/|x|) w_1(x+ty')t^{n-1}dt\right)^{v/r} d\sigma(y')\right]^{1/v};$$

$$(3) \quad N_r(w_0, \Omega^\sim, z) = \operatorname*{ess\,sup}_{2^{z-1}<|x|<2^z} \sup_{\alpha>0} \alpha \cdot \left(\int_{w_0(x-y)<|\Omega^\sim(x,y)||y|^{-n/r}\alpha^{-1}} \chi(|x-y|/|x|) w_0(x-y)dy\right)^{1/r};$$

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$$(4)\quad N^*_{r,v}(w_0, z) = \operatorname*{ess\,sup}_{2^{z-1}<|x|<2^z} \left[\int_{S^{n-1}} \left(\sup_{\alpha>0} \alpha^r \int_{w_0(x+ty')<t^{-n/r}\alpha^{-1}} \cdot \chi(|x+ty'|/|x|) w_0(x+ty') t^{n-1} dt \right)^{v/r} d\sigma(y') \right]^{1/v}.$$

If $\Omega = 1$ or $\Omega^{\sim} = 1$, the notation will be abbreviated to $M_r(w_0, w_1, z)$, $N_r(w_0, z)$, respectively. For any real numbers r_1, r_2, let $r_1 \vee r_2 = \max(r_1, r_2)$, $r_1 \wedge r_2 = \min(r_1, r_2)$, and $r_1^+ = r_1 \vee 0$. C denotes a positive constant, not necessarily the same at each occurrence.

The following results will be proved.

PROPOSITION 1. *For $u_0, u_1 > 0$, define*

$$(5)\qquad B^{u_0u_1}(w_0, w_1) = \sup_{s>0} \left(\int_{|x|<s} w_0(x)^{u_0} dx \right)^{1/u_0} \left(\int_{|x|>s} w_1(x)^{u_1} dx \right)^{1/u_1}.$$

Suppose that $1 < r \leqq \infty$, $1 < p < r'$, $1/q = 1/p - 1/r'$, and set

$$Tf(x) = \int |x-y|^{-n/r} f(y) dy.$$

Then

$$(6)\quad ||w_1Tf||_q/||w_0f||_p \leqq \left\{ C[B^{p'q}(w_0^{-1}, |\cdot|^{-n/r}w_1) + B^{qp'}(w_1, |\cdot|^{-n/r}w_0^{-1})] + C_{p,q} \sup_{|z_1-z_2|\leqq 1} M_r(w_0^p, w_1^q, z_1)^{r/q} N_r(w_0^p, z_2)^{r/p'} \right\}.$$

On the other hand,

$$(7)\quad B^{p'q}(w_0^{-1}, |\cdot|^{-n/r}w_1) + B^{qp'}(w_1, |\cdot|^{-n/r}w_0^{-1}) \leqq C \sup_f (||w_1Tf||_q/||w_0f||_p).$$

PROPOSITION 2. *For $u_0, u_1, v_0, v_1 > 0$, and any real α, define*

$$(8)\quad B_\alpha^{u_0u_1v_0v_1}(w_0, w_1) = \sup_{z\in\mathbf{Z}} \left(\sum_{k=-\infty}^{z} \left(2^{-nk} \int_{2^{k-1}<|x|<2^k} w_0(x)^{v_0} dx \right)^{u_0/v_0} 2^{\alpha k u_0} \right)^{1/u_0} \cdot \left(\int_{2^z}^{\infty} \left(\int_{S^{n-1}} w_1(t\xi)^{v_1} d\sigma(\xi) \right)^{u_1/v_1} t^{-\alpha u_1 - 1} dt \right)^{1/w_1}.$$

Suppose that $1 < r < \infty$, $1 < p < r'$, $1/q = 1/p - 1/r'$, and set

$$(9)\qquad Tf(x) = \int \Omega(x-y)|x-y|^{-n/r} f(y) dy.$$

Then

$$(10)\quad ||w_1Tf||_q/||w_0f||_p \leqq C||\Omega||_u \Big[B_{\alpha_1}^{p'qa_0a_1}(w_0^{-1}, w_1) + B_{\alpha_0}^{qp'b_0b_1}(w_1, w_0^{-1}) + C_{p,q} \sup_{|z_1-z_2|\leqq 1} M^*_{r,v}(w_0^p, w_1, z_1)^{r/q} N^*_{r,v}(w_0^p, z_2)^{r/p'} \Big],$$

provided that

$$1/u + 1/v = 1/r,\ 1/a_0 + 1/a_1 = 1/b_0 + 1/b_1 = 1/v,\ 1/q \leqq 1/u + 1/a_1 \leqq 1/r,$$
$$1/p' \leqq 1/u + 1/b_1 \leqq 1/r,$$
$$\alpha_1 = n/p' + (n-1)(1/a_0 - 1/p')^+,$$
$$\alpha_0 = n/q + (n-1)(1/b_0 - 1/q)^+.$$

COROLLARY 1. *Suppose that* $1 \leqq r < \infty$, $1 < p < r'$, $1/q = 1/p - 1/r'$, $u \geqq r$. *If* $r = 1$, *suppose further that* Ω *has mean value* 0 *on* S^{n-1}. *Let* T *be defined by* (9) *or by*

$$Tf(x) = p.v. \int \Omega(x-y)|x-y|^{-n} f(y)dy = \lim_{\epsilon \to 0} \int_{|y|>\epsilon} \Omega(y)|y|^{-n} f(x-y)dy,$$

according as $r > 1$ *or not. Finally, suppose that*

(11) $$\omega_i(s)/\omega_i(t) \leqq B, \quad \text{for } 1/2 < s/t < 2, \quad i = 0, 1$$

and that for any $s > 0$,

(12) $$\left(\int_0^s \omega_0(t)^{-p'} t^{p'\alpha_1 - 1} dt\right)^{1/p'} \left(\int_s^\infty \omega_1(t)^q t^{-q\alpha_1 - 1} dt\right)^{1/q} \leqq A,$$

(13) $$\left(\int_0^s \omega_1(t)^q t^{q\alpha_0 - 1} dt\right)^{1/q} \left(\int_s^\infty \omega_0(t)^{-p'} t^{-p'\alpha_0 - 1} dt\right)^{1/p'} \leqq A,$$

where

$$\alpha_1 = n/p' - (n-1)(1/u - 1/q)^+,$$
$$\alpha_0 = n/q - (n-1)(1/u - 1/p)^+.$$

Then

(14) $$||\omega_1 Tf||_q \leqq CA(1 + C_{p,q}B^2)||\Omega||_u||\omega_0 f||_p,$$

where $||\Omega||_u$ *on the right-hand side must be replaced by* $||\Omega||_1^*$ *if* $u = 1$.

(13), (12) are, in particular, satisfied for some $A < \infty$ if (11) holds and as $s \to 0$ or $+\infty$,

(15) $$\left(\int_0^s \omega_1(t)^r t^{r\alpha_0 - 1} dt\right)^{1/r} = O(\omega_0(s)s^{\alpha_0}),$$

(16) $$\left(\int_s^\infty \omega_0(t)^{-r} t^{-r\alpha_0 - 1} dt\right)^{1/r} = O(\omega_1(s)^{-1} s^{-\alpha_0}),$$

(17) $$\left(\int_0^s \omega_0(t)^{-r} t^{r\alpha_1 - 1} dt\right)^{1/r} = O(\omega_1(s)^{-1} s^{\alpha_1}),$$

(18) $$\left(\int_s^\infty \omega_1(t)^r t^{-r\alpha_1 - 1} dt\right)^{1/r} = O(\omega_0(s)s^{-\alpha_1}).$$

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Conditions (15), (18) are of a weaker form than those of [**4**; **5**] for the case $\omega_0 = \omega_1$.

Remark 1. For $r < u \leqq p' \wedge q$, $||\Omega||_u$ in (14) can be replaced by $||\Omega||_{uv}$ where $v^{-1} = (r^{-1} - u^{-1})(q/p' \vee p'/q)$. (For the definition of Lorentz norms see, e.g., [**2**; **7**].)

PROPOSITION 3. *For $r(>1)$, p, q as above, set*

$$\tilde{T}f(x) = \int \Omega^{\sim}(x, x-y)|x-y|^{-n/r}f(y)dy.$$

Suppose that $p' < u < \infty$, $1/a = 1/p' - 1/u$, $\beta = n/p' - (n-1)/u$, $r/v < 1 - p'/u$. Then

$$(19)\quad ||w_1\tilde{T}f||_q/||w_0 f||_p \leqq C|||\Omega^{\sim}|||_u\Big[B_\beta^{p'qaq}(w_0^{-1}, w_1) + B_{n/q}^{qp'qa}(w_1, w_0^{-1}) + C_{p,q,u,v}\sup_{|z_1-z_2|\leqq 1} M_r(w_0^p, w_1^q, z_1)^{r/q}N^*_{r,v}(w_0^p, z_2)^{r/p'}\Big].$$

COROLLARY 2. *Suppose that $1 \leqq r < \infty$, $1 < p < r'$, $1/q = 1/p - 1/r'$, $u \geqq p'$. If $r = 1$, suppose further that $\Omega^{\sim}(\mathrm{x}, .)$ has mean value* 0 *on S^{n-1} for any $x \in \mathbf{R}^n$. Define*

$$\tilde{T}f(x) = (p.v.)\int \Omega^{\sim}(x, x-y)|x-y|^{-n/r}f(y)dy.$$

Suppose that (11) *is satisfied and that for any $s > 0$,*

$$(20)\qquad \left(\int_0^s \omega_0(t)^{-p'}t^{\beta p'-1}dt\right)^{1/p'}\left(\int_s^\infty \omega_1(t)^q t^{-\beta q-1}dt\right)^{1/q} \leqq A,$$

$$(21)\qquad \left(\int_0^s \omega_1(t)^q t^{n-1}dt\right)^{1/q}\left(\int_s^\infty \omega_0(t)^{-p'}t^{np'/q-1}dt\right)^{1/p'} \leqq A,$$

where $\beta = n/p' - (n-1)/u$. Then

$$(22)\qquad ||\omega_1 Tf||_q \leqq CA(1 + B^2C_{p,q})|||\Omega^{\sim}|||_u||\omega_0 f||_p.$$

As always, the proof of these results starts with the decomposition $T = T_1 + T_2 + T_3$, where

$$T_1f(x) = \int_{|y|\leqq|x|/2}\Omega(x-y)|x-y|^{-n/r}f(y)dy,$$

$$T_2f(x) = \int_{|y|\geqq 2|x|}\Omega(x-y)|x-y|^{-n/r}f(y)dy,$$

with a similar decomposition $\tilde{T}_1 + \tilde{T}_2 + \tilde{T}_3$, in the case of $\tilde{T}$. The major part of the present paper is concerned with proving that T_1 and $T_2(\tilde{T}_1, \tilde{T}_2)$ satisfy

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(6) or, equivalently (for positive $\Omega, \Omega^{\sim}$), that S_1, S_2 defined by

$$(23) \quad S_1(\Omega^{\sim}, w_0, w_1)(f)(x) = w_1(x)|x|^{-n/r} \int_{|y| \leqq |x|/2} \Omega^{\sim}(x, x-y) w_0(y)^{-1} f(y) dy,$$

$$(24) \quad S_2(\Omega^{\sim}, w_0, w_1)(f)(x) = w_1(x) \int_{|y| \geqq 2|x|} \Omega^{\sim}(x, x-y)|y|^{-n/r} w_0(y)^{-1} f(y) dy,$$

are bounded from L^p to L^q.

The proof is by interpolation between two cases. In the first case, the conditions on w_0, w_1 are as weak as possible compared to those satisfied by Ω. In the second case, no additional condition beyond those required for the boundedness of T between unweighted L^p and L^q spaces is imposed on Ω, and it is found that the conditions obtained in the first case for the dimension n equal to 1 are nearly sufficient.

In Propositions 1, 2, 3, the required inequality for T_3 is obtained by simplification of the conditions for T_3 to be of restricted weak type at the end points $p = 1$ and $p = r'$ with respect to the measures $w_0 \mathscr{L}^n$ and $w_1 \mathscr{L}^n$, where $\mathscr{L}^n$ denotes Lebesgue measure on $\mathbf{R}^n$, and application of the Marcinkiewicz Interpolation Theorem.

In Corollaries 1 and 2, the required norm inequalities for T_3, $\tilde{T}_3$ follow from well known results except possibly for the case $r > 1$ in Corollary 2. Corollary 2 also provides an answer to a question left open in [**12**].

1. An extension of Hardy's inequality. If T is an operator from L^p of some measure space Y to the space of measurable functions on some measure space X, define the (L^p, L^q) norm of T by

$$||T||_{p,q} = \sup\{||Tf||_q/||f||_p : f \in L^p(Y)\}.$$

LEMMA 1. *Suppose that (X, μ), (Y, ν) are σ-finite measure spaces, that $\mathscr{F}$, $\mathscr{G}$ are classes of measurable subsets of X and Y, respectively, which are linearly ordered by inclusion, and that R is a relation with domain $\mathscr{F}$ and range $\mathscr{G}$ which is order-reversing in the sense that if $F_i R G_i$, $i = 1, 2$, then $F_1 \subset F_2$ implies $G_1 \supseteq G_2$ and $G_1 \subset G_2$ implies $F_1 \supseteq F_2$. (Unless otherwise indicated, the containment is strict.) Define an initial segment $\mathscr{F}'$ of $\mathscr{F}$ as a subset such that for every element F_1 of $\mathscr{F}'$ and every element F_2 of $\mathscr{F} \sim \mathscr{F}'$, it is true that $F_1 \subseteq F_2$. Suppose that $\mathscr{F}$ contains a dense countable subset $\mathscr{F}_0$ in the sense that for every initial segment $\mathscr{F}'$ of $\mathscr{F}$ and $\mathscr{F}'' = \mathscr{F} \sim \mathscr{F}'$*

$$(25) \quad \mu(\bigcup\{F : F \in \mathscr{F}'\} \sim \bigcup\{F : F \in \mathscr{F}' \cap \mathscr{F}_0\}) = \mu(\bigcap\{F : F \in \mathscr{F}'' \cap \mathscr{F}_0\} \sim \bigcap\{F : F \in \mathscr{F}''\})$$

and that this property is shared by $\mathscr{G}$.

For $u, v > 0$ set

$$(26) \quad B^{uv}(R) = \sup\{\mu(F)^{1/u} \nu(G)^{1/v} : FRG\}$$

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(where $0^0 = 0, 0 \cdot \infty = 0$). Define the operator H on non-negative measurable functions on Y by

$$Hf(x) = \sup\left\{ \int_G f(y)d\nu(y) : x \in F, FRG \right\}.$$

Then for $1 \leqq p \leqq q \leqq \infty$, $1/p' + 1/q = 1/r$

$$1 \leqq ||H||_{p,q}/B^{p'q}(R) \leqq (p')^{1/p'}q^{1/q}r^{-1/r}. \tag{27}$$

This can be considered as a (self-dual) generalization of Hardy's inequality ($X = Y = \mathbf{R}_+$, $d\mu(x) = x^{\alpha-1}dx$, $d\nu(x) = x^{-\beta-1}dx$ for $\alpha, \beta > 0$, $p = q$, $\alpha/p' = \beta/p$, $\mathscr{F} = \{[x, \infty): x > 0\}$, $\mathscr{G} = \{(0, x]: x > 0\}$). The inequality (27) for the real line, intervals, and $p = q$ has been established by several authors (see [**10**]). The present proof although similar to that of Muckenhoupt makes the result appear as a natural consequence of the semi-trivial end point results (for $p = 1$ or $q = \infty$) and the following simple inequality.

LEMMA 2. *Suppose that (X, μ) is a totally finite measure space and that Φ is a function from X to the set of measurable subsets of X such that for each x, $x \in \Phi(x)$, the range of Φ is linearly ordered by inclusion, the union of any subset $\mathscr{F}'$ of the range of Φ differs from the union of a countable subset of $\mathscr{F}'$ by a set of measure 0, and $\mu(\Phi)$ is measurable. Then for any $\alpha > 0$,*

$$\int_X \mu(\Phi(x))^{\alpha-1}d\mu(x) \leqq \alpha^{-1}\mu(X). \tag{28}$$

Equality holds if and only if the range of $\mu(\Phi)$ is dense in the interval $(0, \mu(X))$.

Proof. The point is that $\mu(\Phi)^{-1}$ is in weak L^1 ($L^{1\infty}(X, \mu)$) and hence in $L^{1-\alpha}(X, \mu)$, since $\mu(X) < \infty$. More precisely, let λ denote the distribution function of $\mu(\Phi)^{-1}$; i.e., for $t > 0$, $\lambda(t) = \mu(E_t)$, where $E_t = \{x: \mu(\Phi)^{-1} > t\}$. Let $F_t = \bigcup\{\Phi(x): x \in E_t\}$; then $E_t \subseteq F_t$, and the hypotheses further imply that

$$\mu(F_t) = \sup\{\mu(\Phi(x)): x \in E_t\}.$$

Hence, $\lambda(t) = \mu(E_t) \leqq t^{-1}$. Clearly, $\lambda(t) = \mu(X)$ for $0 < t < \mu(X)^{-1}$. Moreover (see, e.g., [**20**, p. 117]),

$$\begin{aligned}\int_X \mu(\Phi(x))^{\alpha-1}d\mu(x) &= -\int_0^\infty t^{1-\alpha}d\lambda(t)\\ &= \mu(X)^{\alpha-1}\mu(X) + (1-\alpha)\int_{\mu(X)^{-1}}^\infty \lambda(t)t^{-\alpha}dt\\ &\leqq \mu(X)^\alpha + (1-\alpha)\int_{\mu(X)^{-1}}^\infty t^{-1-\alpha}dt\\ &= \alpha^{-1}\mu(X)^\alpha.\end{aligned}$$

Since λ is monotonic, strict inequality holds in (28) if and only if $\lambda(t) < t^{-1}$ for some $t \in (\mu(X)^{-1}, \infty)$. It is easy to see that this occurs if and only if $\mu(\Phi)$ does not assume any value in some subinterval $(\alpha, \beta)(\alpha < \beta)$ of $(0, \mu(X))$.

For $0 < u, v \leqq \infty$ and any measurable function K on $X \times Y$, define the double norm $X^u Y^v K$ by

$$X^u Y^v K = ||Y^v K||_u, \text{ where } Y^v K(x) = ||K(x, .)||_v.$$

It is well known that for S defined by $Sf(x) = \int K(x, y) f(y) d\nu(y)$,

$$(29) \qquad ||S||_{p,q} \leqq X^q Y^{p'} K \quad \text{if } 1 \leqq p \leqq \infty, q > 0,$$

$$(30) \qquad ||S||_{p,q} \leqq Y^{p'} X^q K \quad \text{if } 1 \leqq p, q \leqq \infty,$$

with equality holding in (29) if $q = \infty$ and in (30) if $p = 1$ (see, e.g., [**18**, Lemma 2]). Furthermore, if $K = K_0^{1-t} K_1^t$, where $K_0, K_1 \geqq 0, 0 \leqq t \leqq 1$, then by interpolation (or Hölder's inequality),

$$(31) \qquad ||S||_{p,q} \leqq (X^{q_0} Y^{p_0'} K_0)^{1-t} (Y^{p_1'} X^{q_1} K_1)^t,$$

provided that $1/p = (1 - t)/p_0 + t/p_1$, $1/q = (1 - t)/q_0 + t/q_1 (p_0, p_1, q_1 \geqq 1, q_0 > 0)$.

Note that the kernel K of H is the characteristic function χ_E of the set $E = \bigcup\{F \times G: FRG\}$. Hence,

$$(32) \qquad X^\infty Y^r K = \operatorname*{ess\,sup}_x \sup\{\nu(G)^{1/r}: x \in F, FRG\},$$

$$(33) \qquad Y^\infty X^r K = \operatorname*{ess\,sup}_y \sup\{\mu(F)^{1/r}: y \in G, FRG\}.$$

It is easy to see that $X^\infty Y^r K = B^{\infty r}(R)$. Thus, by (29) and (30), the right-hand inequality of (27) holds if $p = r'$ or $p = 1$.

In general, the idea is to write

$$(34) \qquad K = K_0^{r/p'} K_1^{r/q},$$

and to determine K_0, K_1 in such a way that $X^\infty Y^r K_0$ and $Y^\infty X^r K_1$ agree with each other as closely as possible. For this purpose, define two functions Φ and Ψ on X and Y, respectively, by

$$\Phi(x) = \bigcap\{F: x \in F \in \mathscr{F}\}, \quad \Psi(y) = \bigcap\{G: y \in G \in \mathscr{G}\}.$$

The hypotheses on $\mathscr{F}$, $\mathscr{G}$ imply that Φ and Ψ have the properties stipulated in Lemma 2. Next, let

$$K_0(x, y) = \chi_E(x, y) \mu(\Phi(x))^{1/q} \nu(\Psi(y))^{1/p' - 1/r},$$
$$K_1(x, y) = \chi_E(x, y) \mu(\Phi(x))^{1/q - 1/r} \nu(\Psi(y))^{1/p'},$$

so that (34) is satisfied. Moreover,

$$X^\infty Y^r K_0 = \operatorname*{ess\,sup}_x \mu(\Phi(x))^{1/q} \left(\int_{\{y: y \in G, FRG, x \in F\}} \nu(\Psi(y))^{r/p' - 1} d\nu(y) \right)^{1/r},$$

$$Y^\infty X^r K_1 = \operatorname*{ess\,sup}_y \nu(\Psi(y))^{1/p'} \left(\int_{\{x: x \in F, FRG, y \in G\}} \mu(\Phi(x))^{r/q - 1} d\mu(x) \right)^{1/r}.$$

Hence, by Lemma 2,

$$X^\infty Y^r K_0 \leqq (q/r)^{1/r} \operatorname*{ess\,sup}_x \mu(\Phi(x))^{1/q} \sup\{\nu(G)^{1/p'}: FRG, x \in F\},$$
$$\leqq (q/r)^{1/r} \sup_{FRG} \mu(F)^{1/q}\nu(G)^{1/p'}.$$

Analogously,

$$Y^\infty X^r K_1 \leqq (q/r)^{1/r} \sup_{FRG} \mu(F)^{1/p'}\nu(G)^{1/q}.$$

Thus, the right-hand inequality in (27) now follows from (31) and (34).

The left-hand inequality in (27) follows by evaluation of the ratio $||Hf||_q/||f||_p$ for $f = \chi_F$, the characteristic function of any $F \in \mathscr{F}$, in which case $Hf \geqq \mu(F)\chi_G$ for any G such that FRG.

2. Inequalities for T_1, T_2.

LEMMA 3. *Suppose that* $1 \leqq r \leqq \infty$ *and that* $S_1(w_0, w_1)$ *is defined by*

$$S_1(w_0, w_1)(f)(x) = w_1(x)|x|^{-n/r} \int_{|y|\leqq|x|} f(y)w_0(y)^{-1}d\nu(y). \tag{35}$$

Then, for $1 \leqq p \leqq r'$, $1/q = 1/p - 1/r'$,

$$1 \leqq ||S_1(w_0, w_1)||_{p,q}/B^{p'q}(w_0^{-1}, |\cdot|^{-n/r}w_1) \leqq (p')^{1/p'}q^{1/q}r^{-1/r}. \tag{36}$$

By duality, if

$$S_2(w_0, w_1)(f)(x) = w_1(x) \int_{|y|\geqq|x|} |y|^{-n/r}w_0(y)^{-1}f(y)dy, \tag{37}$$

then

$$1 \leqq ||S_2(w_0, w_1)||_{p,q}/B^{qp'}(w_1, |\cdot|^{-n/r}w_0^{-1}) \leqq (p')^{1/p'}q^{1/q}r^{-1/r}. \tag{38}$$

Proof. Inequalities (36) follow from Lemma 1, if $X = Y = \mathbf{R}^n$, $d\mu(x) = w_1(x)^q|x|^{-nq/r}dx$, and $d\nu(y) = w_0(y)^{-p'}dy$. $\mathscr{F}$ consists of all closed balls with centre at the origin, $\mathscr{G}$ of their complements, and FRG if $G = \sim F$. Hence, $S_1 f = w_1|\cdot|^{-n/r}H(w_0^{p'-1}f)$, and the L^p and L^q norms of f, $S_1 f$ with respect to $\mathscr{L}^n$ are equal to the norms of $w_0^{p'-1}f$, $H(w_0^{p'-1}f)$ with respect to ν, μ, respectively.

Inequalities (38) follow similarly, or because $S_2(w_0, w_1)$ is the adjoint of $S_1(w_1^{-1}, w_0^{-1})$.

LEMMA 4. *Define*

$$A_u(w)(t) = \left(\int_{S^{n-1}} w_1(t\xi)^u d\sigma(\xi)\right)^{1/u}, \quad t > 0,$$

$$b^{uv}_{\alpha,a}(\omega_0, \omega_1) = \sup_{s>0}\left(\int_0^{as} \omega_0(t)^u t^{\alpha u-1}dt\right)^{1/u}\left(\int_s^\infty \omega_1(t)^v t^{-\alpha v-1}dt\right)^{1/v},$$

and for $a, b > 0$

$$S_{1,a,b}(\Omega, w_0, w_1)(f)(x) = w_1(x)|x|^{-n/r} \int_{|y|\leqq a|x|,\,|y-x|\geqq b|x|} \Omega(x-y)w_0(y)^{-1}f(y)dy.$$

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Then, for $1 \leqq p \leqq r'$, $1/q = 1/p - 1/r'$, $q \leqq u \leqq \infty$, $1/v_1 + 1/u = 1/q$,

$$(39) \qquad ||S_{1,a,b}(\Omega, w_0, w_1)||_{p,q} \leqq C_{a,b}||\Omega||_u b^{p'q}_{n/p',a}(A_{p'}(w_0^{-1}), A_{v_1}(w_1)).$$

If

$$S_{2,a,b}(\Omega^{\sim}, w_0, w_1)(f)(x) = w_1(x) \int_{|y-x|\geqq b|x|, |y| \geqq |x|/a} \Omega^{\sim}(x, x-y)|y|^{-n/r} w_0(y)^{-1} f(y)dy,$$

then for the same p, q *and* $p' \leqq u \leqq \infty$, $1/v_2 + 1/u = 1/p'$,

$$(40) \qquad ||S_{2,a,b}(\Omega^{\sim}, w_0, w_1)||_{p,q} \leqq C_{a,b}|||\Omega^{\sim}|||_u b^{qp'}_{n/q,a}(A_q(w_1), A_{p'}(w_0^{-1})).$$

Proof. Consider (40) first. Define the isomorphism τ, from the space of functions on $\mathbf{R}^n \sim \{0\}$ onto that of functions on $\mathbf{R}_+$ with values in the space of functions on S^{n-1}, by $\tau(f)(t)(y') = f(ty')$, $t > 0$, $y' \in S^{n-1}$. Note that

$$S_{2,a,b}(\Omega^{\sim}, w_0, w_1)(f)(sx') = w_1(sx') \int_{|x|/a}^{\infty} \int_{S^{n-1}} \Omega^{\sim}(sx', sx' - ty') w_0(ty')^{-1} \varphi(bs/|sx' - ty'|) f(ty') d\sigma(y') t^{n-1} dt,$$

where φ is the characteristic function of the interval $[0, 1]$.

The diffeomorphism $\psi_{x,t}$, defined by $\psi_{x,t}(y') = |y' - t^{-1}x|^{-1}(y' - t^{-1}x)$, of the subset of $S^{n-1}, D_{x,t} = \{y' : |y' - t^{-1}x| \geqq b|x|t^{-1}\}$, into S^{n-1}, has the property that $\psi_{x,t}{}^*\sigma$, the image of the measure σ under the mapping $\psi_{x,t}$, satisfies $C^{-1}\sigma \leqq \psi_{x,t}{}^*\sigma \leqq C\sigma$ on $D_{x,t}$ for any $t \geqq a^{-1}|x|$. It follows that

$$(41) \quad ||\tau S_{2,a,b}(\Omega^{\sim}, w_0, w_1)(f)(s)||_q \leqq C|||\Omega^{\sim}|||_u A_q(w_1)(s) \int_{s/a}^{\infty} A_{v_2}(w_0^{-1})(t)||\tau f(t)||_p t^{n-1} dt.$$

Also, (40) is equivalent to

$$|| \; ||\tau S_{2,a,b}(\Omega^{\sim}, w_0, w_1)(f)||[L^q(S^{n-1})]||[L^q(\mathbf{R}_+, S^{n-1}ds)] \leqq C b^{p'q}_{n/p',a}|||\Omega^{\sim}|||_u|| \; ||\tau f||[L^p(S^{n-1})]||[L^p(\mathbf{R}_+, t^{n-1}dt)].$$

But this follows from (41) and Lemma 1 applied to the case $X = Y = \mathbf{R}_+$, $d\mu(s) = A_q(w_1)(s)s^{n-1}ds$, $d\nu(t) = A_{v_2}(w_0^{-1})t^{n-1}dt$, $R = \{((0, as] \times [s, \infty)) : s > 0\}$.

To prove (39), observe that if $\Omega^{\sim} = \Omega$ is independent of the first variable and $\Omega \geqq 0$, then $S_{1,a,b}$ is bounded by the adjoint of $S_{2,a,b/a}(\Omega^{\vee}, w_1^{-1}, w_0^{-1})$ (on the set of positive measurable functions), where $\Omega^{\vee}(x) = \Omega(-x)$, because $|y| \leqq a|x|$ and $|x - y| \geqq b|x|$ imply that $|x - y| \geqq b|y|/a$.

LEMMA 5. *Suppose that* X_1, X_2, Y_1, Y_2 *are measurable spaces, that* μ_1, ν_1 *are (totally* σ*-finite) measures on* X_1, Y_1, *that* $\mathscr{A}_i$, $\mathscr{B}_i$ *denote the* σ*-algebras of measurable subsets of* X_i, Y_i, *respectively, and that* $M(N)$ *is a non-negative real valued function on* $X_1 \times \mathscr{A}_2(Y_1 \times \mathscr{B}_2)$ *such that for any* $x_1 \in X_1(y_1 \in Y_1)$,

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$M(x_1, .)(N(y_1, .))$ is a (totally σ-finite) measure on $X_2(Y_2)$ and for any set $A_2 \in \mathscr{A}_2(B_2 \in \mathscr{B}_2)$, $M(., A_2)(N(., B_2))$ is a measurable function on $X_1(Y_1)$ (see, e.g., [**14**, p. 73]). *Denote by μ the measure on $X = X_1 \times X_2$ determined by*

$$(41) \qquad \mu(A_1 \times A_2) = \int_{A_1} M(x_1, A_2) d\mu_1(x_1), \quad A_i \in \mathscr{A}_i.$$

The measure ν on $Y = Y_1 \times Y_2$ is defined analogously.

Let $K(= K(x_1, x_2; y_1, y_2))$ be a locally integrable function on $X \times Y$ and let $||K||[L^p Y, L^q(X)]$ denote the norm of the integral operator S defined by

$$Sf(x_1, x_2) = \int_{Y_1 \times Y_2} K(x_1, x_2; y_1, y_2) f(y_1, y_2) d\nu(y_1, y_2),$$

between $L^p(Y)$ and $L^q(X)$ (with respect to the measures μ, ν). Then

$$||K||[L^p(Y), L^q(X)] \leqq || \, ||K||[L^p(Y_2), L^q(X_2)]||[L^p(Y_1), L^q(X_1)] \, (p > 0, q \geqq 1),$$

where $||K||[L^p(Y_2), L^q(X_2)](x_1, y_1)$ denotes the norm (quasi-norm if $p < 1$) of the integral operator with kernel $K(x_1, .; y_1, .)$ from $L^p(Y_2, N(y_1, .))$ to $L^q(X_2, M(x_1, .))$.

Proof. By Minkowski's inequality for integrals, since $q \geqq 1$,

$$\begin{aligned} ||Sf(x_1, .)||_q = X_2{}^q Sf(x_1) &= \left\| \int \int K(x_1, .; y_1, y_2) f(y_1, y_2) dN(y_1, y_2) d\nu_1(y_1) \right\|_q \\ &\leqq \int \left\| \int K(x_1, .; y_1, y_2) f(y_1, y_2) dN(y_1, y_2) \right\|_q d\nu_1(y_1) \\ &\leqq \int ||K||[L^p(y_2), L^q(X_2)](x_1, y_1) ||f||_p(y_1) d\nu_1(y_1), \end{aligned}$$

where $||f||_p(y_1)$ denotes the norm of $f(y_1, .)$ with respect to the measure $N(y_1, .)$ on Y_2. Hence,

$$||Sf||_q = X_1{}^q X_2{}^q Sf \leqq || \, ||K||[L^p(Y_2), L^q(X_2)]||[L^p(Y_1), L^q(X_1)]||f||_p.$$

Remark 2. More generally, if $u \leqq p$, $v \geqq q$, it follows similarly, by use of the obvious generalization of [**19**, Lemma 3 and Corollary] from the case of product measures to the more general types of measure defined in (41), that

$$||K||[L^{pu}(y), L^{qv}(X)] \leqq C|| \, ||K||[L^{pu}(Y_2), L^{qv}(X_2)]||[L^p(y_1), L^q(X_1)].$$

LEMMA 6. *Define*

$$b_\alpha^{uvw}(\omega_0, \omega_1) = \sup_{z \in \mathbf{Z}} \left(\sum_{k=-\infty}^{z} \left(2^{-k} \int_{2^{k-1}}^{2^k} \omega_0(t)^w dt \right)^{u/w} 2^{\alpha k u} \right)^{1/u} \left(\int_{2^z}^{\infty} \omega_1(t)^v t^{-\alpha v - 1} dt \right)^{1/v}.$$

Suppose that $1 \leqq r \leqq \infty$, $1/q = 1/p - 1/r'$. Then for S_1 defined by (23),

$$(42) \qquad ||S_1(\Omega, \omega_0, \omega_1)||_{p,q} \leqq C||\Omega||_{r\infty} b_{1/p'}^{p'q\infty}(\omega_0{}^{-1}, \omega_1), \quad \text{for } 1 < p < r'.$$

For $p = 1$ *or* $q = \infty$, *this is still valid provided that the left-hand side is replaced by* $||S_1(\Omega_1, \omega_0, \omega_1)||_{p1,q\infty}$ *or if, instead, the right-hand side is replaced by* $||\Omega||_r$.

Dually,

$$(43) \qquad ||S_2(\Omega, \omega_0, \omega_1)||_{p,q} \leqq C||\Omega||_{r\infty} b_{1/q}^{qp'\infty}(\omega_1, \omega_0^{-1}),$$

with analogous results if $p = 1$ *or* $q = \infty$.

Proof. This is by application of the preceding lemma. By duality, it suffices to consider S_2. Let $X_1 = \mathbf{Z}$, provided with the measure ν_1 such that $\nu_1(\{z\}) = 2^z$ for any $z \in \mathbf{Z}$. X_2 is the subset of $\mathbf{R}^n$, $\{x: 1/2 < |x| \leqq 1\}$, which together with the σ-algebra $\mathscr{A}_2$ of Lebesgue measurable subsets becomes a measure space. For $z \in \mathbf{Z}$ and $A_2 \in \mathscr{A}_2$, let $M(z, A_2) = 2^{(n-1)z}\mathscr{L}^n(A_2)$. Next, let $Y_1 = \mathbf{R}_+$, $Y_2 = S^{n-1}$, $\mathscr{B}_1$, $\mathscr{B}_2$ be the σ-algebras of measurable subsets with respect to $\mathscr{L}^1$ or σ, respectively, and let $N(t, B_2) = t^{n-1}\sigma(B_2)$ for $B_2 \in \mathscr{B}_2$.

Note that if the measures μ, ν on $X = \mathbf{Z} \times X_2$, $Y = \mathbf{R}_+ \times S^{n-1}$, are as in Lemma 5, then there are isomorphisms F_1, F_2 between the measure spaces (X, μ), (Y, ν) and $(\mathbf{R}^n, \mathscr{L}^n)$ defined by $F_1(z, x) = 2^z x$, $F_2(t, y) = ty$, respectively. Therefore, (43) is equivalent to the boundedness between $L^p(Y)$ and $L^q(X)$ of the integral operator whose kernel is

$$K(z, x; t, y) = \varphi(2^{z+1}|x|t^{-1})\Omega(2^z x - ty)t^{-n/r}\omega_1(2^z|x|)\omega_0(t)^{-1}.$$

To deduce the latter, it will be shown that if

$$k_0(z, t) = ||K||[L^{r'1}(S^{n-1}), L^\infty(X_2)],\ k_1(z, t) = ||K||[L^1(S^{n-1}), L^{r\infty}(X_2)],$$

then for $i = 0, 1$,

$$(44) \qquad k_i(z, t) \leqq C||\Omega||_{r\infty}\varphi(2^z t^{-1})t^{-1/r}\omega_0(t)^{-1} \operatorname*{ess\,sup}_{1/2<u<1} \omega_1(2^z u).$$

Now

$$k_0(z, t) = \operatorname*{ess\,sup}_{x \in X_2} ||K(z, x; ., .)||_{r\infty}(t)$$

(see, e.g., [**19**, Lemma 1]). It is easy to see that the $L^{r\infty}$ norm with respect to the measure $N(t, .) = t^{n-1}\sigma$ is $t^{(n-1)/r}$ times the $L^{r\infty}$ norm with respect to σ. Thus, for $2^{z+1}|x| \leqq t$,

$$(45) \quad k_0(z, t) \leqq Ct^{-1/r}\omega_0(t)^{-1} \operatorname*{ess\,sup}_{x \in X_2} \omega_1(2^z|x|)||\Omega(2^z x - t.)||[L^{r\infty}(S^{n-1})].$$

But for $2^z t^{-1}|x| \leqq \frac{1}{2}$,

$$||\Omega(2^z x - t.)||[L^{r\infty}(S^{n-1})] = ||\Omega(2^z t^{-1}x - .)||[L^{r\infty}(S^{n-1})] \leqq C||\Omega||_{r\infty},$$

and (44), for $i = 0$, follows by substituting this in (45). To establish (44) for $i = 1$, note that

$$\begin{aligned} k_1(z, t) &\leqq C \operatorname*{ess\,sup}_{y \in S^{n-1}} ||K(., .; t, y)||_{r\infty}(z) \\ &= Ct^{-n/r}2^{(n-1)z/r}\omega_0(t)^{-1} \operatorname*{ess\,sup}_{1/2<u<1} \omega_1(2^z u) \sup_{y \in S^{n-1}} ||\Omega(. - 2^{-z}ty)||[L^{r\infty}(X_2)]. \end{aligned}$$

But the last norm is at most equal to

$$||\Omega||[L^{r\infty}(\{x:t2^{-z} - 1 < |x| < t2^{-z} + 1\})] \leqq C||\Omega||_{r\infty},$$

for $t \geqq 2^{z+1}$. Hence,

$$k_1(z, t) \leqq C||\Omega||_{r\infty}t^{-1/r}\omega_0(t)^{-1} \operatorname*{ess\,sup}_{1/2<u<1} \omega_1(2^z u).$$

Inequality (44) and the Marcinkiewicz Interpolation Theorem for Lorentz spaces imply that

$$||K||[L^p(S^{n-1}), L^q(X_2)](z, t) \leqq C_{p,q}||\Omega||_{r\infty}\varphi(2^z t^{-1})t^{-1/r}\omega_0(t)^{-1} \operatorname*{ess\,sup}_{1/2<u<1} \omega_1(2^z u).$$

Hence, by Lemma 5, the proof of (43) will be finished if it can be shown that for

$$k(z, t) = \varphi(2^z t^{-1})t^{-1/r}\omega_0(t)^{-1} \operatorname*{ess\,sup}_{1/2<u<1} \omega_1(2^z u),$$

$$||k||[L^p(\mathbf{R}_+, \mathscr{L}^1), L^q(\mathbf{Z}, \mu_1)] \leqq Cb_{1/q}^{qp'\infty}(\omega_1, \omega_0^{-1}).$$

This is a consequence of Lemma 1. For, replace X, Y by $\mathbf{Z}$, $\mathbf{R}_+$, respectively, μ by the measure assigning mass $2^z \operatorname{ess\,sup}_{1/2<u<1} \omega_1(2^z u)$ to the one-point set $\{z\}$, $d\nu$ by $t^{-1/r}\omega_0(t)^{-1}dt$, and $\mathscr{F}$, $\mathscr{G}$ by the collection of intervals of the form $\Phi(z) = \{z_1:z_1 \in \mathbf{Z}, z_1 \leqq z\}$ and $\Psi(t) = [t, \infty)$ for $z \in \mathbf{Z}$, $t > 0$, respectively. The relation R is defined by $R = \{(\Phi(z), \Psi(t)):2^z \leqq t < 2^{z+1}, z \in \mathbf{Z}, t > 0\}$.

The restricted weak type results for S_2 mentioned in Lemma 4 follow similarly if use is made of Remark 2. It follows, similarly, that

$$||K||[L^{r'}(S^{n-1}), L^\infty(X_2)](z, t) \leqq C||\Omega||_r\varphi(2^z t^{-1})t^{-1/r}\omega_0(t)^{-1} \operatorname*{ess\,sup}_{1/2<u<1} \omega_1(2^z y).$$

Hence, by Lemma 1,

$$||S_2(\Omega, \omega_0, \omega_1)||_{r',\infty} \leqq C||\Omega||_r b_{1/q}^{qp'\infty}(\omega_1, \omega_0^{-1}).$$

The same inequality for $||S_2||_{1,r}$ is proved similarly.

Remark 3. The essentially new result, going beyond Lemma 4, is (44) for $i = 1$. The preceding argument is just a possible way of interpolating between this result and Lemma 4 in case $\Omega \in L^{r\infty}$. It was obtained in an attempt to apply Lemma 5 with $X_1 = Y_1 = \mathbf{R}_+$, $\mu_1 = \nu_1 = \mathscr{L}^1$, $X_2 = Y_2 = S^{n-1}$, and $M(t, E) = N(t, E) = t^{n-1}\sigma(E)$. This, however, presents the difficulty that the $L^{r\infty}$ norm of $\Omega(s - ty)$ on $S^{n-1}(y \in S^{n-1})$ for $s < t$ need no longer be finite, due to the contribution from a neighbourhood of the $(n - 2)$ dimensional sphere on S^{n-1} defined by $\{\xi:\xi \in S^{n-1}, \xi \cdot (s\xi - ty) = 0\}$. If ω_1 is not essentially bounded locally, then $k_1(z, .)$, for suitable $\Omega \in L^{r\infty}$, $z \in \mathbf{Z}$, will be infinite for t in a set of positive measure. It is in applying Lemma 5 that accuracy is lost even at the end points $p = 1$, $p = r'$; for $f \in L^{r\infty}(X_1 \times X_2)$ does not require that $X_1{}^r X_2{}^{r\infty} f < \infty$.

Interpolation between Lemma 4 and Lemma 6 for fixed p, q yields:

LEMMA 7. *Suppose that* $1 < p < r'$, $1/q = 1/p - 1/r'$, $1/u + 1/v_0 + 1/v_1 = 1/r$,

$$1/q \leqq 1/u + 1/v_1 \leqq 1/r,\ 1/s = 1/u - (1 - p'/v_0)(1/r). \tag{46}$$

Then (see [**8**])

$$||S_1(\Omega, w_0, w_1)||_{p,q} \leqq C||\Omega||_{us}B^{p'qv_0v_1}_{1/p'+(n-1)/v_0}(w_0^{-1}, w_1). \tag{47}$$

If instead of (46),

$$1/p' \leqq 1/u + 1/v_1 \leqq 1/r,\ 1/s = 1/u - (1 - q/v_0)(1/r), \tag{48}$$

then

$$||S_2(\Omega, w_0, w_1)||_{p,q} \leqq C||\Omega||_{us}B^{qp'v_0v_1}_{1/q+(n-1)/v_0}(w_1, w_0^{-1}). \tag{49}$$

Proof. It suffices to consider S_2. Since $|y| \geqq 2|x|$ implies that $|y - x| \geqq |x|$, by Lemma 4, for $1/u + 1/v_1 = 1/p'$,

$$||S_2(\Omega, w_0, w_1)||_{p,q} \leqq C||\Omega||_u B^{qp'qv_1}_{n/q}(w_1, w_0^{-1}). \tag{50}$$

By Lemma 6,

$$||S_2(\Omega, w_0, w_1)||_{p,q} \leqq C||\Omega||_{r\infty} B^{qp'\infty\infty}_{1/q}(w_1, w_0^{-1}). \tag{51}$$

Inequality (49) follows from (50), (51), by interpolation.

In fact, let $\lambda = 1 - q/v_0$ and $1/u_0 = (v_0/q)(1/u) - (v_0/q - 1)(1/r)$. Then $L^{us} = (L^{u_0})^{1-\lambda}(L^{r\infty})^{\lambda}$. Further, $w_i = w_{i0}{}^{1-\lambda}w_{i1}{}^{\lambda}$, $i = 0, 1$; for

$$\begin{aligned}
w_{00}(y) &= w_0(y)^{v_0/q}w_0(y)^{1-v_0/q}|y|^{\gamma_0},\\
w_{01}(y) &= [A_{v_1}(w_0^{-1})(|y|)]^{-1}|y|^{\gamma_1},\\
\gamma_0 &= (1 - q/v_0)\gamma,\\
\gamma_1 &= -(q/v_0)\gamma,\\
\gamma &= (n - 1)(1/p' - 1/r),
\end{aligned}$$

and for $2^{k-1} < |x| \leqq 2^k$,

$$\begin{aligned}
w_{10}(x) &= w_1(x)^{v_0/q}w_{11}(x)^{1-v_0/q}2^{k\delta_0},\\
w_{11}(x) &= \left(2^{-kn}\int_{2^{k-1}<|y|<2^k} w_1(y)^{v_1}dy\right)^{1/v_1} 2^{k\delta_1}\\
\delta_0 &= -(n - 1)(1/q - 1/v_0),\\
\delta_1 &= (n - 1)/v_0,
\end{aligned}$$

and

$$B^{qp'q(v_0v_1/q)}_{n/q}(w_{10}, w_{00}{}^{-1}),\ B^{qp'\infty\infty}_{1/q}(w_{11}, w_{01}{}^{-1}) \leqq CB^{qp'v_0v_1}_{1/q+(n-1)/v_0}(w_1, w_0^{-1}).$$

LEMMA 8. *Suppose that* $1/r' \leqq 1/p \leqq 1$, $1/q = 1/p - 1/r'$. *Then*

$$||S_1(\Omega^{\sim}, \omega_0, w_1)||_{p,q} \leqq C|||\Omega^{\sim}|||_{p'}b^{p'q\infty}_{1/p'}(\omega_0^{-1}, A_q(w_1)). \tag{52}$$

Proof. The proof is similar to that of Lemma 6. Let $\mathbf{Z}$, X_2, M, N be as there, and let

$$K(t, y; z, x) = \varphi(2^{z+1}|x|t^{-1})t^{-n/r}w_1(ty)\Omega^{\sim}(ty, ty - 2^z x)\omega_0(2^z|x|)^{-1}.$$

Then, by the proof of Lemma 6, it suffices to show that for

$$k_p(t, z) = ||K||[L^p(X_2), L^q(S^{n-1})], \quad 1 \leqq p \leqq r',$$

$$(53) \qquad k_p(t, z) \leqq C|||\Omega^{\sim}|||_{p'}\varphi(2^z t^{-1})t^{-1/r}A_q(w_1)(t) \operatorname*{ess\,sup}_{1/2<u<1} \omega_0(2^z u)^{-1}.$$

In fact, for $2^z \leqq t$,

$$\begin{aligned} k_1(t, z) &= ||K||[L^1(X_2), L^r(S^{n-1})](t, z) \\ &\leqq C|||\Omega^{\sim}|||_\infty t^{-n/r} \operatorname*{ess\,sup}_{1/2<u<1} \omega_0(2^z u)^{-1}A_r(w_1)(t)t^{(n-1)/r} \\ &\leqq C|||\Omega^{\sim}|||_\infty t^{-1/r}A_r(w_1)(t) \operatorname*{ess\,sup}_{1/2<u<1} \omega_0(2^z u)^{-1}. \end{aligned}$$

On the other hand, for $p = r'$ and $2^z \leqq t$, similarly, as in the proof of Lemma 4,

$$\begin{aligned} &A_\infty(w_1)(t)^{-1}\left[\operatorname*{ess\,sup}_{1/2<u<1} \omega_0(2^z u)^{-1}\right]^{-1} \\ &\qquad \leqq Ct^{-n/r}2^{z(n-1)/r} \sup_{y\in S^{n-1}} ||\Omega^{\sim}(ty, ty - 2^{\cdot}.)||_r(z) \leqq C|||\Omega^{\sim}|||_r t^{-1/r}. \end{aligned}$$

Thus, (53) holds for $p = 1$ and r'. The general case then follows by interpolation.

LEMMA 9. *Suppose that* $1 \leqq p \leqq r'$, $1/q = 1/p - 1/r'$, $0 \leqq 1/u \leqq 1/p'$. *Then*

$$(54) \qquad ||S_1(\Omega^{\sim}, w_0, w_1)||_{p,q} \leqq C|||\Omega^{\sim}|||_u B^{p'qv_0q}_{n/p'-(n-1)/u}(w_0^{-1}, w_1),$$

where $1/v_0 = 1/p' - 1/u$.

Proof. This is by interpolation between Lemmas 4 and 8. In fact, as a consequence of Lemma 4 (or Lemma 3),

$$||S_1(\Omega^{\sim}, w_0, w_1)||_{p,q} \leqq C|||\Omega^{\sim}|||_\infty B^{p'qp'q}_{n/p'}(w_0^{-1}, w_1).$$

By Lemma 8,

$$||S_1(\Omega^{\sim}, w_0, w_1)||_{p,q} \leqq C|||\Omega^{\sim}|||_{p'} B^{p'q\infty q}_{1/p'}(w_0^{-1}, w_1).$$

Inequality (54) then follows by interpolation between the preceding two inequalities, similarly, as in the proof of Lemma 7.

3. Inequalities for T_3.

LEMMA 10. *If w is a non-negative measurable function on $\mathbf{R}^n$, let $||.||_{uv,w}$ denote the L^{uv} norm with respect to the measure $w\mathscr{L}^n$ on $\mathbf{R}^n$. Suppose that $r > 1$. Then for*

$\tilde{T}_3$, *defined in the Introduction,*

(55) $$C^{-1} \leqq \sup_f (\|T_3 f\|_{r\infty, w_1}/\|w_0 f\|_1) \Big/ \sup_{z \in \mathbf{Z}} M_r(w_0, w_1, \Omega, z) \leqq C,$$

(56) $$C^{-1} \leqq \sup_f (\|\tilde{T}_3 f\|_\infty/\|f\|_{r'1, w_0}) \Big/ \sup_{z \in \mathbf{Z}} N_r(w_0, \Omega^\sim, z) \leqq C,$$

and for $1 < p < r'$, $1/q = 1/p - 1/r'$,

(57) $$\|w_1 T_3 f\|_q \leqq C_{p,q} \sup_{|z_1 - z_2| \leqq 1} M_r(w_0^p, w_1^q, \Omega, z_1)^{r/q} N_r(w_0^p, \Omega, z_2)^{r/p'} \|w_0 f\|_p.$$

Proof. Observe that (since $r > 1$)

$$\sup_f (\|\tilde{T}_3 f\|_{r\infty, w_1}/\|w_0 f\|_1) = Y^\infty X^{r\infty} K,$$

where $X = Y = \mathbf{R}^n$ and X, Y are provided with the measures $w_1 \mathscr{L}^n$, $w_0 \mathscr{L}^n$, respectively, and

$$K(x, y) = \chi(|x|/|y|)\Omega^\sim(x, x - y)|x - y|^{-n/r} w_0(y)^{-1}.$$

But for $\Omega^\sim = \Omega$,

$$Y^\infty X^{r\infty} K = \operatorname*{ess\,sup}_y \|\chi(|.|/|y|)\Omega(. - y)|. - y|^{-n/r} w_0^{-1}\|_{r\infty, w_1},$$

which is equivalent to (see [**7**])

$$\operatorname*{ess\,sup}_y \sup_{\alpha > 0} \left(\int_{|\Omega(x-y)||x-y|^{-n/r} w_0(y)^{-1} > \alpha} \chi(|x|/|y|) w_1(x) dx \right)^{1/r}.$$

Hence, (55) follows if, for $w_0(y) \neq 0$, $w_0(y)^{-1}\rho^{-n/r}$ is substituted for α. Similarly,

$$X^\infty Y^{r\infty} K = \operatorname*{ess\,sup}_x \|\chi(|.|/|x|)\Omega^\sim(x, x - .)|x - .|^{-n/r} w_0^{-1}\|_{r\infty, w_0}.$$

The latter is equivalent to

$$\operatorname*{ess\,sup}_x \sup_\alpha \left(\int_{|\Omega^\sim(x, x-y)||x-y|^{-n/r} w_0(y)^{-1} > \alpha} \chi(|y|/|x|) w_0(y) dy \right)^{1/r} =$$
$$\operatorname*{ess\,sup}_x \sup_{\alpha > 0} \left(\int_{w_0(x-y) \leqq |\Omega^\sim(x,y)||y|^{-n/r} \alpha^{-1}} \chi(|x - y|/|x|) w_0(x - y) dy \right)^{1/r}.$$

Inequality (57) can be proved by means of Lemma 5. For, let $X_1 = Y_1 = \mathbf{Z}$, $\mu_1 = \nu_1$ and such that $\mu_1(\{z\}) = 1$ for any $z \in \mathbf{Z}$. Further, let $X_2 = Y_2 = S = \{x : 1/2 < |x| \leqq 1\}$. Then for μ, ν defined by

$$\mu(\{z\} \times E) = M(z, E) = \int_{2^z E} w_1(x) dx, \ \nu(\{z\} \times E)$$
$$= N(z, E)$$
$$= \int_{2^z E} w_0(x) dx,$$

$(Z \times S, \mu)$, $(Z \times S, \nu)$ are isomorphic to $(\mathbf{R}^n, w_1\mathscr{L}^n)$, $(\mathbf{R}^n, w_0\mathscr{L}^n)$, respectively, and T_3 is equivalent to an integral operator with kernel

$$(58) \quad K(z_1, x; z_2, y) = \chi(2^{z_1 - z_2}|x_1|/|x_2|)\Omega^{\sim}(2^{z_1}x_1, 2^{z_1}x_1 - 2^{z_2}x_2) \cdot |2^{z_1}x_1 - 2^{z_2}x_2|^{-n/r}w_0(2^{z_2}x_2)^{-1}$$

on $(Z \times S)^2$.

By the preceding estimates for $|z_1 - z_2| \leqq 1$,

$$||K||[L^1(S), L^{r\infty}(S)](z_1, z_2) \leqq M_r(w_0, w_1, \Omega, z_2),$$
$$||K||[L^{r'1}(S), L^{\infty}(S)](z_1, z_2) \leqq N_r(w_0, \Omega^{\sim}, z_1),$$

while, if $|z_1 - z_2| > 1$, these norms are 0. Hence, by the Marcinkiewicz Interpolation Theorem for Lorentz spaces,

$$(59) \quad ||K||[L^p(S), L^q(S)](z_1, z_2) \leqq C_{p,q}\varphi(|z_1 - z_2|)M_r(w_0, w_1, \Omega, z_2)^{r/q} \times N_r(w_0, \Omega, z_1)^{r/p'}.$$

(To obtain the bound $C_{p,q}M(w_0, w_1)^{r/q}N(w_0)^{r/p'}$, replace w_0, w_1 by $w_0' = N(w_0)^{r'}w_0, w_1' = M(w_0, w_1)^{-r}N(w_0)^{rr'}w_1$. Then $M(w_0', w_1') = N(w_0') = 1$ and, e.g., $||f||_{p,w_0'} = N(w_0)^{r/p}||f||_{p,w_0}$, so, by the form of the Marcinkiewicz Theorem in [7], $||K||[L^p(S), L^q(S)](z_1, z_2) \leqq C_{p,q}\varphi(|z_1 - z_2|)M_r^{r/q}N_r^{r'/p - rr'/q}$, i.e., (59) is satisfied.)

To complete the proof of (57), it remains to observe that for $k(z_1, z_2) = \varphi(|z_1 - z_2|)$, $||k||[L^p(Z), L^q(Z)] \leqq 3$, and to replace w_0, w_1 by w_0^p, w_1^q, respectively.

Lemma 11. *Suppose that* $1 < p < r'$, $1/q = 1/p - 1/r'$, $r \leqq u \leqq \infty$ *and* $1/u + 1/v = 1/r$. *Then*

$$||w_1T_3f||_q \leqq C_{p,q}||\Omega||_u\left(\sup_{|z_1 - z_2| \leqq 1} M^*_{r,v}(w_0^p, w_1^q, z_1)^{r/q}N^*_{r,v}(w_0^p, z_2)^{r/p'}\right)||w_0f||_p,$$

where $M^*_{r,v}$, $N^*_{r,v}$ *are defined by* (2), (4).

Proof. $M_r(w_0, w_1, \Omega, z)$ is defined as the essential supremum in $\{x: 2^{z-1} < |x| < 2^z\}$ of

$$w_0(x)^{-1}\left(\sup_{\rho > 0} \rho^{-n} \int_{|y| \leqq |\Omega(y)|^{r/n}\rho} \chi(|x - y|/|x|)w_1(x - y)dy\right)^{1/r} \leqq$$
$$w_0(x)^{-1}\left(\int_{S^{n-1}} |\Omega(y')|^r \sup_{\rho > 0} \rho^{-n} \int_0^{\rho} \chi(|x - ty'|/|x|)w_1(x - ty')t^{n-1}dtd\sigma(y')\right)^{1/r}.$$

By Hölder's inequality, $M_r(w_0, w_1, \Omega, z) \leqq ||\Omega||_u M^*_{r,v}(w_0, w_1, z)$.

Moreover,

$$\operatorname*{ess\,sup}_{2^{z-1}<|x|<2^z} \sup_{\alpha>0} \alpha\left(\int_{w_0(x-y)\leqq|\Omega(y)||y|^{-n/r}\alpha^{-1}} w_0(x-y)dy \right)^{1/r}$$

$$\leqq \operatorname*{ess\,sup}_{2^{z-1}<|x|<2^z} \left(\int_{S^{n-1}} \sup \alpha^r |\Omega(y)|^r \right.$$

$$\left. \times \int_{w_0(x-ty')\leqq t^{-n/r}\alpha^{-1}} \chi(|x-ty'|/|x|) w_0(x-ty') t^{n-1} dt d\sigma(y') \right)^{1/r}$$

$$\leqq ||\Omega||_u N^*_{r,v}(w_0, z).$$

To complete the proof of Proposition 2, it is necessary to consider $\tilde{T}_3$ again.

LEMMA 12. *Suppose that* $0 < 1/u < 1/p'$, $1/v < (1/r)(1 - p'/u)$, $1/r' < 1/p < 1$, $1/q = 1/p - 1/r'$. *Then*

$$||w_1\tilde{T}_3 f||_q \leqq C_{p,q,u,v}|||\Omega^\sim||| \sup_{|z_1-z_2|\leqq 1} (M_r(w_0^p, w_1^q, z_1)^{r/q} N_v^*(w_0^p, z_2)^{r/p'})||w_0 f||_p.$$

Proof. By the proof of Lemma 10, it suffices to show that for K as in (58),

$$(60)\quad ||K||[L^p(S), L^q(S)](z_1, z_2) \leqq C_{p,q,u,v}|||\Omega^\sim|||_u \varphi(|z_1 - z_2|) M_r(w_0, w_1, z_2)^{r/q} \times N^*_{r,v}(w_0^p, z_1)^{r/p'}.$$

By (55) of Lemma 10,

$$||K||[L^1(S), L^{r\infty}(S)](z_1, z_2) \leqq C|||\Omega^\sim|||_\infty M_r(w_0, w_1, z_2),$$

and by (56) and the proof of Lemma 11,

$$(61)\qquad ||K||[L^{r'1}(S), L^\infty(S)](z_1, z_2) \leqq C|||\Omega^\sim|||_{ru/p_0'} N^*_{r,v}(w_0, z_1),$$

where $p'/(ru) + 1/v = 1/r$. Hence, by interpolation (see [**2**]),

$$(62)\quad ||K||[L^{p_01}(S), L^{q_0\infty}(S)] \leqq C|||\Omega^\sim|||_u M_r(w_0, w_1, v_2)^{r/q_0} N^*_{r,v}(w_0, z_1)^{r/p_0'}$$

$(1/q_0 = 1/p_0 - 1/r')$. Since $r < p_0'$, $|||\Omega^\sim|||_{ru/p_0'}$ can be replaced by $|||\Omega^\sim|||_u$ in (61). Inequality (60) then follows from (61), (62) by the Marcinkiewicz Interpolation Theorem.

4. Proof of Propositions 1, 2, 3 and Corollaries 1, 2. Inequality (6) of Proposition 1 follows from (36), (38) of Lemma 3 and (57) of Lemma 10, for $\Omega = 1$. For the proof of (7), notice that $S_1(w_0, w_1)(f)$ and $S_2(w_0, w_1)(f)$ defined by (35), (36), are both at most equal to $Cw_1T(w_0^{-1}f)$. Hence, (7) follows from the left-hand inequalities of (36), (38). Proposition 2 follows from Lemmas 4, 7, 11, and Proposition 3 from Lemmas 8 and 12.

Remark 4. Conversely, there is a constant $C_{p,r,n}$, depending only on the indicated variables, such that for any $\Omega \geqq 0$ and if

$$||T|| = \sup\{||w_1 T f||_q/||w_0 f||_p : w_0 f \in L^p\},$$

then, for $r > 1$,

(63) $$\|\Omega\|_1 w_1 \leqq C_{p,r,n}\|T\| w_0 \text{ a.e.}$$

For, suppose that $a > 0$ and that the set of x where $w_1(x)/w_0(x) > a$ has positive measure. Then there are $\alpha, \beta > 0$ for which $\beta/\alpha > a$ and $w_0 \leqq \alpha$, $w_1 \geqq \beta$ on a set $E_{\alpha\beta}$ of positive measure. Suppose that x_0 is a point of density 1 of $E_{\alpha\beta}$. For $\rho > 0$, let $B(x_0, \rho)$ denote the open ball of radius ρ about x_0. For any $\epsilon > 0$, there exists ρ such that $\mathscr{L}^n(B(x_0, \rho) \sim E_{\alpha\beta}) < \epsilon(\omega_n/n)\rho^n$, where ω_n/n is the volume of the unit ball. Also, for f the characteristic function of $B(x_0, \rho) \cap E_{\alpha\beta}$, and $|x - x_0| < \rho/2$

$$Tf(x) \geqq \int_{|y|<\rho/2} \Omega(y)|y|^{-n/r}dy - g*\chi_{\alpha,\beta}(x),$$

where $g(x) = \Omega(x)|x|^{-n/r}$ and $\chi_{\alpha\beta}$ is the characteristic function of $B(x_0, \rho) \sim E_{\alpha\beta}$. Note that the first term on the right-hand side of the preceding inequality equals $C_n\|\Omega\|_1\rho^{n/r'}$. Suppose first that $\|\Omega\|_r < \infty$. Then $\|g\|_{r\infty} \leqq C_n\|\Omega\|_r$ (see, e.g.,[**13**]). Thus, the non-increasing rearrangement of g on $\mathbf{R}_+$ satisfies $g^*(t) \leqq C_n\|\Omega\|_r t^{-1/r}$. Hence,

$$|g*\chi_{\alpha\beta}| \leqq C_n\|\Omega\|_r \int t^{-1/r}dt \leqq C_n\|\Omega\|_r \epsilon^{1/r'}\rho^{n/r'},$$

where the limits of integration are 0 to $\mathscr{L}^n(E_{\alpha\beta})$.

It follows that for $|x - x_0| < \rho/2$,

$$Tf(x) \geqq C_n\rho^{n/r'}(\|\Omega\|_1 - \epsilon^{1/r'}\|\Omega\|_r).$$

Thus, if $\epsilon < 2^{-n-1}$, then

$$\|w_1 Tf\|_q \geqq C_n\beta\rho^{n(1/q+1/r')}(\|\Omega\|_1 - \epsilon^{1/r'}\|\Omega\|_r),$$

and, also, $\|w_0 f\|_p \leqq \alpha\|f\|_p \leqq C_n\alpha\rho^{n/p}$. Hence,

$$\|w_1 Tf\|_q/\|w_0 f\|_p \geqq C_n(\beta/\alpha)(\|\Omega\|_1 - \epsilon^{1/r'}\|\Omega\|_r),$$

and so

$$(\beta/\alpha)(\|\Omega\|_1 - \epsilon^{1/r'}\|\Omega\|_r) \leqq C_n\|T\|.$$

Since $\epsilon > 0$ may be arbitrarily small, it follows that $(\beta/\alpha)\|\Omega\|_1 \leqq C_n\|T\|$. Hence, $a\|\Omega\|_1 \leqq C_n\|T\|$ and $(w_1/w_0)\|\Omega\|_1 \leqq C_n\|T\|$ a.e. If $\|\Omega\|_r = \infty$, this holds for $\Omega_k = \Omega \wedge k$. Hence, by Fatou's Lemma, for Ω likewise.

In the case of fractional integration ($r > 1$), Corollary 1 is a consequence of Proposition 1; for, if $a_0 = b = v$, $a_1 = b_1 = \infty$, then $1/a_0 - 1/p' = 1/r - 1/u - 1/p' = 1/q - 1/u$; hence, $\alpha_1 = n/p' + (n-1)(1/q - 1/u)^+$ and, similarly, $\alpha_0 = n/q + (n-1)(1/p' - 1/u)^+$. Thus, $B_{\alpha_1}{}^{p'qv\infty}(w_0^{-1}, w_1)$, $B_{\alpha_0}{}^{qp'v\infty}(w_1, w_0^{-1})$ are at most equal to constant multiples of the left-hand sides of (12) and (13).

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Also, (11) and (12), (13) imply that

$$\sup_{1/2<s/t<2} \omega_0(s)^{-1}\omega_1(t) \leqq CAB^2, \tag{64}$$

for $s > 0$.

It follows easily that $M^*_{r,v}(\omega_0{}^p, \omega_1{}^q)^{r/q} N^*_{r,v}(\omega_0{}^p)^{r/p'} \leqq CAB^4$. An examination of the proof of Lemma 10 leads to the conclusion that

$$\|\omega_1 T_3\|_q \leqq CAB^2\|\omega_0 f\|_p. \tag{65}$$

This can be deduced directly from (64). For, the kernel K of $w_1 T_3 w_0{}^{-1}$ satisfies

$$\begin{aligned}|K(x, y)| &= \chi(|y|/|x|)\omega_1(x)|\Omega(x-y)|\,|x-y|^{-n/r}\omega_0(y)^{-1}\\ &\leqq CAB^2|\Omega(x-y)|\,|x-y|^{-n/r}\\ &= CAB^2 g(x-y),\end{aligned}$$

where $\|g\|_{r\infty} \leqq C\|\Omega\|_r$ and $\|g*f\|_q \leqq C\|g\|_{r\infty}\|f\|_p$ (see, e.g., [**7**; **13**]).

If $r = 1$, (65) is a consequence of well known results of Calderón and Zygmund [**3**, Theorem 1] and, e.g., [**19**, Lemma 4]. The required inequalities for T_1, T_2 are, of course, contained in Lemmas 4 and 7. Remark 1 follows from Lemma 7. For if, e.g., in (46), $1/v_0 = 1/a_0 = 1/r - 1/u$, then $1/s = p'/q(1/r - 1/u)$. The fact that, e.g., (15), (16) imply (18) follows from the logarithmic convexity of the function $1/p \to \|f\|_p$ (Hölder's inequality); hence of $1/p \to \|f\|_p\|g\|_q$, for $f(t) = \varphi(t/s)\omega_1(t)t^{\alpha_0}$, $g(t) = \varphi(s/t)\omega_0(t)^{-1}t^{-\alpha_0}$.

Corollary 2 follows similarly from Lemma 8 with $a_1 = \infty$, [**3**, Theorem 2] and, e.g., [**19**, Lemma 4] for the middle part $\tilde{T}_3$ if $r = 1$. If $r > 1$, the proof that

$$\|\omega_1\tilde{T}_3 f\|_q \leqq C_{p,q}AB^2\|\|\Omega^\sim\|\|_u\|\omega_0 f\|_p$$

is completed by the following.

Lemma 13. *Suppose that* $1 < r < \infty$, $1 < p < r'$, $1/q = 1/p - 1/r'$, *and* $\tilde{T}$ *is as defined in Proposition* 3. *Then*

$$\|\tilde{T}f\|_q \leqq C\|\|\Omega^\sim\|\|_{r'}\|f\|_p \quad (C = C_{p,q}). \tag{66}$$

Proof. This is very similar to the argument for [**9**, Theorem 9] in the case that $\Omega^\sim(x, y)$ does not depend on x. In fact, for $0 \leqq \operatorname{Re} z < 1$ and f in the class $C_c{}^1$ of continuously differentiable functions of compact support, define

$$T_z f(x) = c(z)\int \operatorname{sgn}\Omega^\sim(x, y)|\Omega^\sim(x, y)|^{rz}|y|^{-nz}f(x-y)dy,$$

where $c(z) = (z-1)(z-2)^{-2}$, $\operatorname{sgn}\Omega^\sim = \Omega^\sim/|\Omega^\sim|$. For any $x \in \mathbf{R}^n$, $T_z f(x)$ is a holomorphic function in $\{z: 0 < \operatorname{Re} z < 1\}$, continuous in $\{z: 0 \leqq \operatorname{Re} z < 1\}$,

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and has a continuous extension to the closed strip $\{z: 0 \leqq \operatorname{Re} z \leqq 1\}$, which is uniformly bounded for $x \in \mathbf{R}^n$.

For, if $\epsilon > 0$, $T_z f(x)$ can be written

$$T_z f(x) = c(z) \int_{|y|>\epsilon} \Omega^\sim_z(x, y)|y|^{-nz} f(x - y)dy$$
$$- n^{-1}(z - 2)^{-2}\epsilon^{n(1-z)} f(x) \int_{S^{n-1}} \Omega^\sim_z(x, y')d\sigma(y')$$
$$+ c(z) \int_{|y|\leqq\epsilon} \Omega^\sim_z(x, y)(f(x - y) - f(x))dy,$$

where $\Omega^\sim_z(x, y) = \operatorname{sgn} \Omega^\sim(x, y)|\Omega^\sim(x, y)|^{rz}$. The last term on the right-hand side approaches 0 uniformly in z as ϵ goes to 0 due to the integrability of $|\Omega(x, .)|^r$ and since $|f(x - y) - f(x)| \leqq C|y|$, while, for any fixed $\epsilon > 0$, the first and second terms are bounded continuous functions of z in the closed strip and these statements hold uniformly for $x \in \mathbf{R}^n$.

If $T_{1+i\eta} f(x)$ denotes the value of the continuous extension of $T_z f(x)$ at $1 + i\eta$, $-\infty < \eta < \infty$, clearly

$$T_{1+i\eta} f(x) = \lim_{\epsilon\to 0}\Big[c(1 + i\eta) \int_{|y|>\epsilon} \Omega^\sim_{1+i\eta}(x, y)|y|^{-n(1+i\eta)} f(x - y)dy$$
$$- n^{-1}(i\eta - 1)^{-2}\epsilon^{-ni\eta} f(x) \int_{S^{n-1}} \Omega^\sim_{1+i\eta}(x, y')d\sigma(y')\Big].$$

By the results of [**9**] and [**3**],

$$T_{1+i\eta} f(x) = \int_{S^{n-1}} \Omega^\sim_z(x, y')\tilde{f}_{i\eta}(x, y')d\sigma(y'), \tag{67}$$

where for $|y'| = 1$, $\eta \neq 0$,

$$\tilde{f}_{i\eta}(x, y') = c(1 + i\eta) \lim_{\epsilon\to 0}\left(\int_\epsilon^\infty t^{-1-ni\eta} f(x - ty')dt - (in\eta)^{-1}\epsilon^{-in\eta} f(x)\right)$$

(if $\eta = 0$, $\tilde{f}_0(x, y') = -n^{-1}f(x)$). From [**9**, Theorem 6], it follows that

$$\|\tilde{f}_{i\eta}(., y')\|_s \leqq C_s\|f\|_s, \quad 1 < s < \infty. \tag{68}$$

By precisely the same argument as in the proof of [**3**, Theorem 2], (67), (68) imply

$$\|T_{1+i\eta} f\|_s \leqq C\|\|\Omega^\sim_{1+i\eta}\|\|_{s'}\|f\|_s = C\|\||\Omega^\sim|^r\|\|_{s'}\|f\|_s. \tag{69}$$

Furthermore,

$$\|T_{i\eta} f\|_\infty \leqq |c(i\eta)|\ \|f\|_1 \leqq C/(1 + |\eta|)\|f\|_1. \tag{70}$$

Let now $s = q/r$; then $1/p = (1/r)(1/s) + (1 - 1/r)$, $1/q = (1/r)(1/s)$, $1/s' = r/p'$, and (69) becomes

$$\|T_{1+i\eta} f\|_s \leqq C\|\|\Omega^\sim\|\|_{p'}{}^r\|f\|_s. \tag{71}$$

Since T_z is an analytic family of operators of admissible growth on C_c^1 satisfying (70), (71), a theorem of Stein (see [**15**, Theorem 2; **20**, p. 110]) implies that $\tilde{T} = c(1/r)^{-1}T_{1/r}$ satisfies

$$(72) \qquad ||\tilde{T}f||_q \leqq C_r|||\Omega^\sim|||_{p'}||f||_p, \quad f \in C_c^1.$$

It clearly suffices to prove (72), in general, for non-negative f, $\Omega^\sim$. Since any non-negative function f in L^p is the limit a.e. of a sequence $\{f_n\}$ in C_c^1, which is bounded in L^p by $||f||_p$, the general validity of (72) follows from Fatou's Lemma.

Remark 5. It does not seem unlikely that the preceding result on positive kernels can be proved without the use of singular integrals. The weaker result

$$(73) \qquad ||\tilde{T}f||_q \leqq C_{p,q}|||\Omega^\sim|||_u||f||_p, \quad \text{for } u > p',$$

which is [**12**, Lemma 7], follows from the Marcinkiewicz Interpolation Theorem, and the restricted weak type result

$$(74) \qquad ||\tilde{T}f||_{q\infty} \leqq C|||\Omega^\sim|||_{p'}||f||_{p1}.$$

If $p = 1$, this is nothing but a well known result about the fractional integral $\int |x - y|^{-n/r}f(y)dy$. If $p = r'$, $|\tilde{T}f(x)| \leqq C||\Omega^\sim(x, .)||_r||f||_{r'1}$, as a result of the duality between $L^{r\infty}$ and $L^{r'1}$ (see [**6**; **7**; **13**]). It follows by the complex method of interpolation, that (74) is generally valid (see [**2**, § 13]). Suppose now $u > p'$, and let $p_0 = u' < p$ and $p_1 = r' > p$; then (73) follows from (74) for p_0, p_1 and the Marcinkiewicz Interpolation Theorem.

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