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LIFTING PROBLEMS AND THE COHOMOLOGY OF C^*-ALGEBRAS

MAN-DUEN CHOI AND EDWARD G. EFFROS

1. Introduction. Suppose that A and B are C^*-algebras, J is a closed two-sided ideal in B, and that $\eta : B \to B/J$ is the quotient map. Given a linear contraction $\varphi : A \to B/J$, a linear map $\Psi : A \to B$ is a *lifting* of φ if one has a commutative diagram

(1.1)
$$\begin{array}{ccc} & & B \\ & \overset{\Psi}{\nearrow} & \downarrow \eta \\ A & \xrightarrow[\varphi]{} & B/J. \end{array}$$

In § 2 we will show that if A is separable and satisfies Grothendieck's metric approximation property (see § 2 for definition) then φ has a contractive lifting Ψ. This had been proved earlier by T. B. Andersen [**3**, Corollary 8] under the additional hypothesis that $A = B/J$ (for related results see [**3**; **37**; **6**]). Since the authors proved in [**13**, Theorem 3.1] that nuclear C^*-algebras have the metric approximation property, the result applies to C^*-algebras A that are nuclear and separable. In a negative vein, we give an example in § 4 of a diagram (1.1) with $A = B/J$ a separable non-nuclear C^*-algebra, for which the identity map $A \to A$ does not have a completely positive lifting Ψ (this answers a question raised in [**13**]). We suspect that there does not exist any bounded lifting, and more generally we conjecture that for C^*-algebras, nuclearity and the metric approximation property are in fact equivalent.

The bounded lifting problem naturally arises in the cohomology theory of C^*-algebras as formulated by Johnson, Kadison, and Ringrose [**27**; **32**]. Let us suppose that $\theta : A \to B$ is a *-homomorphism. Then we may regard B, J, and B/J as A-bimodules. Slightly generalizing an argument of Johnson [**27**, Proposition 1.7], we deduce from the lifting result of § 2 that if A is separable and satisfies the metric approximation property, then one has the expected exact sequence

(1.2) $$\ldots \to H^n(A, J) \to H^n(A, B) \to H^n(A, B/J) \to \ldots .$$

A case of particular interest is that in which $B = \mathscr{B}(H)$ and $J = \mathscr{K}(H)$, the bounded and compact operators, respectively, on a separable Hilbert space

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H. (Andersen's lifting theory is not applicable since Conway [**16**] has shown that $\mathscr{K}(H)$ is not complemented in $\mathscr{B}(H)$, and thus

$$\begin{array}{ccc} & & \mathscr{B}(H) \\ & & \downarrow \\ \mathscr{B}(H)/\mathscr{K}(H) & \xrightarrow[\mathrm{id}]{} & \mathscr{B}(H)/\mathscr{K}(H) \end{array}$$

does not have a bounded lifting.) The resulting exact sequence

$$\ldots \to H^n(A, \mathscr{K}(H)) \to H^n(A, \mathscr{B}(H)) \to H^n(A, \mathscr{B}(H)/\mathscr{K}(H)) \to \ldots$$

provides information about $H^1(A, \mathscr{B}(H)/\mathscr{K}(H))$ and $H^2(A, \mathscr{B}(H)/\mathscr{K}(H))$. We have reason to believe that if A is nuclear and separable these groups must be zero, a result that would be of interest in the cohomological approach to perturbations of *-homomorphisms (see [**33**, Theorem 2; **29**, Theorem 5.1]). This in turn seems relevant to extension theory. We note that since the lifting theorem of § 2 is developed in the context of Banach spaces, (1.2) is also valid for more general Banach algebras. It has been shown that M-ideals naturally arise in certain non-commutative Banach algebras (see [**25**]) which are apparently not C^*-algebras.

We are indebted to F. Knudsen for a suggestion that considerably simplified § 3. He pointed out that one can disentangle the algebraic and analytic difficulties of the subject by considering short exact sequences of cochain complexes.

We shall use the notations **R** and **C** for the real and complex numbers, and **T** for the unit circle in **C**. We say that a subset D of a real (resp., complex) vector space V is *symmetric* if $\alpha D \subseteq D$ for $\alpha = \pm 1$ (resp., $\alpha \in \mathbf{T}$). If V and W are normed vector spaces we let $\mathscr{B}(V, W)$ (resp., $\mathscr{B}_f(V, W)$) denote the normed vector space of bounded (resp., bounded finite rank) linear maps $V \to W$, and we denote the Banach dual of V by V^*. We let $\mathscr{B}_f{}^\sigma(V^*, W)$ be the weak* continuous maps in $\mathscr{B}_f(V^*, W)$. Given normed vector spaces $V_1, \ldots, V_n, W$, we let $\mathscr{B}(V_1, \ldots, V_n; W)$ be the bounded n-linear maps

$$f : V_1 \times \ldots \times V_n \to W$$

with the norm

$$\|f\| = \sup \{\|f(v_1, \ldots, v_n)\| : \|v_i\| \leqq 1\}.$$

If X is a compact Hausdorff space, we let $C_{\mathbf{R}}(X)$ (resp., $C_{\mathbf{C}}(X)$) denote the Banach space of continuous real (resp., complex) functions on X with the uniform norm.

2. A lifting theorem for Banach spaces. Suppose that V is a (real or complex) Banach space and that K is the closed unit ball in V^* with the weak topology. If V is real, we have a natural isometry

$$V \cong \mathrm{Aff}_0(K)$$

where $\mathrm{Aff}_0(K)$ is the Banach subspace of affine functions in $C_{\mathbf{R}}(K)$ vanishing at 0 (that one obtains all such functions follows from [**31**, Lemma 4.3]). It should be noted that, in this functional representation of V, one has that $\|a - b\| \leqq \epsilon$ if and only if $-\epsilon \leqq a - b \leqq \epsilon$. Similarly if V is a complex Banach space we have the isometry

$$V \cong \mathrm{Aff}_{\mathbf{T}}(K)$$

where $\mathrm{Aff}_{\mathbf{T}}(K)$ consists of the affine functions a in $C_{\mathbf{C}}(K)$ such that $a(\alpha p) = \alpha a(p)$ for all $\alpha \in \mathbf{T}$.

A subspace W of a normed vector space X is an ***L**-summand* provided it is the range of an idempotent linear map $e : X \to X$ such that for all $p \in X$.

$$\|p\| = \|ep\| + \|p - ep\|$$

(see [**19**; **2**, I§ 3]). The map e is unique and is called the *L-projection* onto W. In the following lemma we assume that V is a Banach space, W is a weak* closed L-summand in V^* with L-projection e, $F = K \cap W$, and that D is a weak* closed, convex, symmetric subset of K such that if $p \in D$, then $ep \in D$.

LEMMA 2.1. *Suppose that V is real and that $a \in \mathrm{Aff}_0 F$, $b \in \mathrm{Aff}_0 K$ and $\epsilon > 0$ are such that $\|a\|, \|b\| \leqq 1$, and*

$$(2.1) \qquad \|(a - b)|_{D \cap F}\| \leqq \epsilon.$$

Then a has an extension $a' \in \mathrm{Aff}_0 K$ such that $\|a'\| \leqq 1$ and

$$\|(a' - b)|_D\| \leqq 6\epsilon.$$

If V is complex, the same implication is true for $a \in \mathrm{Aff}_{\mathbf{T}} F$, $b \in \mathrm{Aff}_{\mathbf{T}} K$.

Proof. We first assume that V is real. Consider the function defined on K by

$$(2.2) \qquad f(p) = \begin{cases} a(p) & p \in F \\ b(p) + \epsilon & p \in D \backslash F \\ 1 + \epsilon & p \in K \backslash (D \cup F). \end{cases}$$

Letting $\tilde{f}$ be the usual lower envelope of f (see [**1**, p. 4]), we claim that

$$(2.3) \qquad a(p) \leqq \tilde{f}(p), \quad p \in F.$$

We define a function a_b on K by

$$a_b(p) = a(ep) + b((1 - e)p).$$

From [**2**, I, Corollary 4.2], $a \circ e$ is Borel on K and satisfies the barycentric calculus. Since

$$b \circ (1 - e) = b - b \circ e,$$

the same is true for $b \circ (1 - e)$ and a_b. We have that

$$(2.4) \qquad a_b(p) \leqq f(p), \quad p \in K$$

since if $p \in D \backslash F$, then by assumption $ep \in D \cap F$ and from (2.1)

$$a_b(p) = a(ep) + b((1-e)p) \leqq b(ep) + \epsilon + b((1-e)p) = b(p) + \epsilon$$

and if $p \in K \backslash (D \cup F)$, then

$$a_b(p) = a(ep) + b((1-e)p) \leqq ||ep|| + ||(1-e)p|| = ||p|| \leqq 1.$$

The function f is lower semi-continuous since its supergraph

$$S(f) = \{(p, \alpha) : f(p) \leqq \alpha\}$$

is closed in $K \times \mathbf{R}$. To see this, note that if one is given real functions g_i on closed sets $F_i \subseteq K$, $i = 1, 2$ with $g_1|_{F_1 \cap F_2} \leqq g_2|_{F_1 \cap F_2}$, then

$$g(p) = \begin{cases} g_1(p), & p \in F_1 \\ g_2(p), & p \in F_2 \backslash F_1 \end{cases}$$

has supergraph (relative to $F_1 \cup F_2$)

$$S(g) = S(g_1) \cup S(g_2);$$

hence if the g_i are lower semi-continuous, the same is true for g. Since $a(p) \leqq b(p) + \epsilon$, $p \in D \cap F$, $f|_{F \cup D}$ is lower semi-continuous, and since $f(p) \leqq 1 + \epsilon$, $p \in F \cup D$, f is thus also lower semi-continuous.

Given $p \in F$, let $P_p(K)$ be the probability measures on K with resultant p. Then if $\mu \in P_p(K)$,

$$a(p) = a_b(p) = \mu(a_b).$$

It follows from (2.4) and the lower semi-continuity of f (see [**35**; **13**, proof of Proposition 2.2]) that

$$a(p) \leqq \min\{\mu(f) : \mu \in P_p(K)\} = \tilde{f}(p), \quad p \in F,$$

i.e., we have (2.3).

In particular we have that

$$a(p) < \tilde{f}(p) + \epsilon = (f + \epsilon)^{\sim}(p), \quad p \in F;$$

hence from [**2**, I, Lemma 5.1], a has an extension $d \in \mathrm{Aff}_0(K)$ such that

$$d(p) < (f + \epsilon)^{\sim}(p), \quad p \in K.$$

We have that

$$a(p) = d(p) \leqq g(p), \quad p \in F$$

where

$$g = (d + 4\epsilon) \wedge 1.$$

On the other hand since $d \leqq f + \epsilon \leqq 1 + 2\epsilon$, $||d|| \leqq 1 + 2\epsilon$ and

$$(1 + 3\epsilon)^{-1}(d + \epsilon) = d + (1 + 3\epsilon)^{-1}\epsilon(1 - 3d) \leqq d + 4\epsilon;$$

hence

$$(1 + 3\epsilon)^{-1}(d + \epsilon) \leqq \tilde{g}$$

and evaluating both sides at 0,

$$0 < (1 + 3\epsilon)^{-1}\epsilon \leqq \tilde{g}(0).$$

It follows from [**2**, I, Theorem 5.4] that a has an extension $a' \in \mathrm{Aff}_0 K$ such that $a' \leqq g$ on K. In particular, $a'(p) \leqq 1$ on K. or replacing p by $-p$, $||a'|| \leqq 1$. If $p \in D \backslash F$,

$$a'(p) \leqq d(p) + 4\epsilon \leqq f(p) + 5\epsilon = b(p) + 6\epsilon,$$

whereas if $p \in D \cap F$,

$$a'(p) = a(p) \leqq b(p) + \epsilon \leqq b(p) + 6\epsilon.$$

Replacing p by $-p$ we conclude that

$$||(a' - b)|_D|| \leqq 6\epsilon.$$

Now suppose that V is complex. Letting $V_{\mathbf{R}}$ (resp., $(V^*)_{\mathbf{R}}$) denote the real Banach space underlying V (resp., V^*), we may regard $(V^*)_{\mathbf{R}}$ as the Banach dual of $V_{\mathbf{R}}$ by using the pairing

$$(v, f) \rightarrow \mathrm{Re}\, f(v).$$

In particular the map

$$\mathrm{Re} : \mathrm{Aff}_{\mathbf{T}} K \rightarrow \mathrm{Aff}_0 K : a \mapsto \mathrm{Re}\, a$$

is a real linear isometric surjection. Given $a \in \mathrm{Aff}_{\mathbf{T}} F$ and $b \in \mathrm{Aff}_{\mathbf{T}} K$ satisfying (2.1), let $a_1 = \mathrm{Re}\, a$, $b_1 = \mathrm{Re}\, b$. Then we may use the real case to find an extension $a_1' \in \mathrm{Aff}_0 K$ such that $||a_1'|| = 1$ and

$$||(a_1' - b_1)|_D|| \leqq 6\epsilon.$$

Letting $a_1' = \mathrm{Re}\, a'$, $a' \in \mathrm{Aff}_{\mathbf{T}} K$, we have that

$$||(a' - b)|_D|| = ||(a_1' - b_1)|_D|| \leqq 6\epsilon.$$

The equality follows from the fact that D is assumed closed under multiplication by $\alpha \in \mathbf{T}$, hence we may choose $p \in D$ with

$$||(a' - b)|_D|| = (a' - b)(p) = ||(a_1' - b_1)|_D||. \qquad \textit{q.e.d.}$$

If V and W are vector spaces, we let $V \otimes W$ denote the algebraic tensor product of V and W. If V and W are normed we regard $V \otimes W$ and $V^* \otimes W^*$ as dual vector spaces. We define

$$(V \otimes W)_1 = \text{convex hull } \{v \otimes w : v \in V_1, w \in W_1\},$$

where the subscripts 1 on the right denote the unit balls. The *greatest cross-norm* $|| \quad ||_\gamma$ on $V \otimes W$ is the norm determined by the weak closure of $(V \otimes W)_1$

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(see [**20**, p. 64]). Equivalently one finds that

$$||u|| = \inf \{\sum ||v_i|| \, ||w_i|| : u = \sum v_i \otimes w_i\}.$$

On the other hand the *least cross-norm* $|| \quad ||_\lambda$ (this terminology is misleading—see [**20**, p. 65]) is determined by the polar of $(V^* \otimes W^*)_1$, or equivalently,

$$||u|| = \sup \{p \otimes q(u) : p \in V_1^*, q \in W_1^*\}.$$

We write $V \otimes_\gamma W$ and $V \otimes_\lambda W$ for $V \otimes W$ with the corresponding norms, and $V \bar{\otimes}_\gamma W$, $V \bar{\otimes}_\lambda W$ for the completions. The map

$$\mathscr{L} : (V \otimes_\gamma W)^* = (V \bar{\otimes}_\gamma W)^* \to \mathscr{B}(V, W^*)$$

defined by

$$\mathscr{L}(g)(v)(w) = g(v \otimes w)$$

is an isometric surjection. On the other hand the map

$$L : V \otimes_\lambda W \to \mathscr{B}_f{}^\sigma(V^*, W)$$

defined by

$$L(v \otimes w)(g) = g(v)w$$

is also an isometry. In particular, if V is finite dimensional (but not necessarily W), we have the commutative diagram

$$\begin{array}{ccc} V^* \otimes_\lambda W^* & \xrightarrow{L} & \mathscr{B}_f{}^\sigma(V^{**}, W^*) \\ \downarrow & & \| \\ (V \otimes_\gamma W)^* & \xrightarrow{\mathscr{L}} & \mathscr{B}(V, W^*), \end{array}$$

i.e., we have the natural identification

$$(V \otimes_\gamma W)^* \cong V^* \otimes_\lambda W^*.$$

Interchanging γ and λ we have the following result of Grothendieck (see [**37**, Corollary 5]; we include a non-measure theoretic proof):

LEMMA 2.2. *If V is finite dimensional, then the natural linear map $V^* \otimes_\gamma W^* \to (V \otimes_\lambda W)^*$ is an isometric surjection.*

Proof. It suffices to prove that under the adjoint map

$$(V \otimes_\lambda W)^{**} \to (V^* \otimes_\gamma W^*)^*,$$

the image of the unit ball is weak* dense in that of the range space. Composing with the injection $V \otimes_\lambda W \to (V \otimes_\lambda W)^{**}$, it suffices to prove that the same is true for the resulting map

$$V \otimes_\lambda W \to (V^* \otimes_\gamma W^*)^*.$$

We have the commutative diagram

$$\begin{array}{ccc} V \otimes_\lambda W & \xrightarrow{\quad L \quad} & \mathscr{B}(V^*, W) \\ \downarrow & & \downarrow \\ (V^* \otimes_\gamma W^*)^* & \xrightarrow{\quad \mathscr{L} \quad} & \mathscr{B}(V^*, W^{**}) \end{array}$$

where the rows are isometric surjections. Thus it suffices to prove that each contraction $T : V^* \rightarrow W^{**}$ is a point-weak* limit of contractions $T_\nu : V^* \rightarrow W$. A simple proof of this result using the fact that finite-dimensional real (resp., complex) Banach spaces are approximate quotients of the spaces $\mathbf{R}^n$ (resp., $\mathbf{C}^n$) with the supremum norm may be found in [**21**, § 3].

LEMMA 2.3. *If $\varphi : V \rightarrow W$ and $\Psi : V' \rightarrow W'$ are bounded linear maps, then the linear map*

$$\varphi \otimes \Psi : V \otimes_\gamma W \rightarrow V' \otimes_\gamma W' : v \otimes w \mapsto \varphi(v) \otimes \Psi(w)$$

satisfies

$$\|\varphi \otimes \Psi\| \leqq \|\varphi\| \, \|\Psi\|.$$

Proof. Given $u \in V \otimes W$ with $u = \sum v_i \otimes w_i$, we have

$$(\varphi \otimes \Psi)(u) = \sum \varphi(v_i) \otimes \Psi(w_i)$$

where

$$\sum \|\varphi(v_i)\| \, \|\Psi(w_i)\| \leqq \|\varphi\| \, \|\Psi\| \sum \|v_i\| \, \|w_i\|$$

LEMMA 2.4. *Suppose that V and W are Banach spaces and that W_0 is an L-summand in W. Letting $\iota : W_0 \rightarrow W$ be the inclusion map,*

$$1 \otimes \iota : V \otimes_\gamma W_0 \rightarrow V \otimes_\gamma W$$

is an isometry onto an L-summand.

Proof. Letting $e : W \rightarrow W_0$ be the L-projection onto W_0, we have from Lemma 2.3 that both $1 \otimes e : V \otimes_\gamma W \rightarrow V \otimes_\gamma W_0$ and $1 \otimes \iota : V \otimes_\gamma W_0 \rightarrow V \otimes_\gamma W$ are contractive. Since $(1 \otimes e) \circ (1 \otimes \iota) = 1 \otimes 1$, it follows that $1 \otimes \iota$ is an isometry and $V \otimes_\gamma W_0$ can be regarded as a subspace of $V \otimes_\gamma W$. On the other hand, if $u = \sum v_k \otimes w_k \in V \otimes_\gamma W$,

$$\begin{aligned} \sum \|v_k\| \, \|w_k\| &= \sum \|v_k\| \, \|ew_k\| + \sum \|v_k\| \, \|(1 - e)w_k\| \\ &\geqq \|(1 \otimes e)(u)\| + \|u - (1 \otimes e)u\|; \end{aligned}$$

hence $1 \otimes e$ is an L-projection.

A subspace J of a Banach space B is said to be an *M-ideal* if its annihilator $J^\perp$ is an L-summand in B^* (see [**2**, I, § 5]). In particular, if J is a closed two-sided ideal in a C^*-algebra B, then it is an M-ideal (the argument given for [**2**, I, Theorem 6.12] is also valid for the complex spaces J and B).

LEMMA 2.5. *Suppose that J is an M-ideal in a Banach space B and that $L \subseteq N$ are finite dimensional Banach spaces. Given $\epsilon > 0$ and a diagram of contractions*

$$\begin{array}{ccc} & & B \\ & \overset{\Psi}{\nearrow} & \Big\downarrow \eta \\ L \hookrightarrow N & \xrightarrow[\varphi]{\qquad} & B/J \end{array}$$

where η is the quotient map and

$$||(\eta \circ \Psi - \varphi)|_L|| \leqq \epsilon,$$

there exists a contractive lifting φ' of φ such that

$$||(\varphi' - \Psi)|_L|| \leqq 6\epsilon.$$

Proof. We may isometrically identify the diagram

$$(2.5) \qquad \begin{array}{ccc} \mathscr{B}(L, B/J) & \overset{\circ\iota}{\longleftarrow} & \mathscr{B}(N, B/J) \\ \Big\uparrow \eta\circ & & \Big\uparrow \eta\circ \\ \mathscr{B}(L, B) & \overset{\circ\iota}{\longleftarrow} & \mathscr{B}(N, B) \end{array}$$

where $\iota : L \rightarrow N$ is the inclusion map, with the diagram

$$\begin{array}{ccc} L^* \otimes_\lambda B/J & \xleftarrow{\iota^* \otimes 1} & N^* \otimes_\lambda B/J \\ \Big\uparrow 1 \otimes \eta & & \Big\uparrow 1 \otimes \eta \\ L^* \otimes_\lambda B & \xleftarrow{\iota^* \otimes 1} & N^* \otimes_\lambda B. \end{array}$$

Taking adjoints, we have from Lemma 2.2 the diagram

$$(2.6) \qquad \begin{array}{ccc} L \otimes_\gamma (B/J)^* & \xrightarrow{\iota \otimes 1} & N \otimes_\gamma (B/J)^* \\ \Big\downarrow 1 \otimes \eta^* & & \Big\downarrow 1 \otimes \eta^* \\ L \otimes_\gamma B^* & \xrightarrow{\iota \otimes 1} & N \otimes_\gamma B^* \end{array}$$

The maps $\iota \otimes 1$ are weak* continuous contractive injections. Thus letting D' and K be the closed unit balls of $L \otimes_\gamma B^*$ and $N \otimes_\gamma B^*$, respectively, D' is mapped weak* homeomorphically onto a symmetric compact convex subset D of K. From Lemma 2.4 the maps $1 \otimes \eta^*$ are weak* continuous isometries. Letting $e : B^* \rightarrow B^*$ be the L-projection with range $J^\perp$,

$$W = \text{the image of } (1 \otimes \eta^*) = (1 \otimes e)(N \otimes_\gamma B^*)$$

is a weak* closed L-summand, and we may identify the closed unit ball of $N \otimes_\gamma (B/J)^*$ with $F = K \cap W$. Similarly we identify the closed unit ball of $L \otimes_\gamma (B/J)^*$ with $D' \cap W'$ where $W' = (1 \otimes e)(L \otimes_\gamma B^*)$. We have

$$(2.7) \qquad (\iota \otimes 1)(1 \otimes e) = (1 \otimes e)(\iota \otimes 1);$$

hence

$$(\iota \otimes 1)(W') = (\iota \otimes 1)(L \otimes_\gamma B^*) \cap W,$$

and since $\iota \otimes 1$ is one-to-one $D \subseteq (\iota \otimes 1)(L \otimes_\gamma B^*)$,

$$(2.8) \qquad (\iota \otimes 1)(D' \cap W') = D \cap (\iota \otimes 1)(W') = D \cap W = D \cap F.$$

We have a diagram of inclusion maps

$$\begin{array}{ccc} D \cap F & \hookrightarrow & F \\ \downarrow & & \downarrow \\ D & \hookrightarrow & K \end{array}$$

and we may identify (2.5) with the diagram of restriction maps

$$(2.9) \qquad \begin{array}{ccc} \mathrm{Aff}_{\mathbf{T}}(D \cap F) & \leftarrow & \mathrm{Aff}_{\mathbf{T}}(F) \\ \uparrow & & \uparrow \\ \mathrm{Aff}_{\mathbf{T}}(D) & \leftarrow & \mathrm{Aff}_{\mathbf{T}}(K) \end{array}$$

(for real Banach spaces, replace $\mathrm{Aff}_{\mathbf{T}}$ by Aff_0). Since $1 \otimes e : L \otimes_\gamma B^* \to L \otimes_\gamma B^*$ maps D' onto $D' \cap W'$, it follows from (2.7) and (2.8) that it maps D onto $D \cap F$, and D satisfies the conditions of Lemma 2.1.

Reinterpreting the hypotheses in terms of (2.9), we are given $a \in \mathrm{Aff}_{\mathbf{T}}F$ and $b \in \mathrm{Aff}_{\mathbf{T}}K$ such that $\|a\|_F \leqq 1$, $\|b\|_K \leqq 1$, and

$$\|(a - b)|_{D \cap F}\| \leqq \epsilon.$$

Thus from Lemma 2.1 there is an element $a' \in \mathrm{Aff}_{\mathbf{T}}K$ such that $a'|_F = a$, $\|a'\| = 1$, and

$$\|(a' - b)|_D\| \leqq 6\epsilon.$$

Returning to (2.5), a' corresponds to a contraction $\varphi' : N \to B$ such that $\eta \circ \varphi' = \varphi$ and $\|(\varphi' - \Psi)_L\| \leqq 6\epsilon$. This proves the lemma.

A Banach space V is said to have the *metric approximation property* provided there is a net of finite rank contractions $\varphi_\nu : V \to V$ converging to the identity map in the point-norm topology. The argument used to prove the following result was motivated by that of Andersen for [**3**, Proposition 5].

THEOREM 2.6. *Suppose that A and B are Banach spaces, A is separable and has the metric approximation property, and that J is an M-ideal in B. Then each contraction $\varphi : A \to B/J$ has a contractive lifting $\Psi : A \to B$.*

Proof. We fix a dense sequence $a_1, a_2, \ldots$ in A. We inductively define a sequence of finite dimensional subspaces $L_0 \subseteq L_1 \subseteq \ldots$ with $\bigcup_{k=0}^{\infty} L_k$ dense in A and contractions $\theta_k : A \to L_k$ as follows. We let $L_0 = \{0\}$ and $\theta_0 = 0$. Having defined L_k and θ_k for an integer $k \geqq 0$, we may use the metric approximation property to find a finite rank contraction $\theta_{k+1} : A \to A$ such that

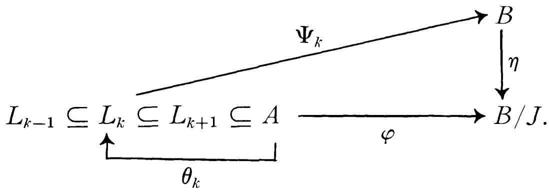
(2.10) $\|(\theta_{k+1} - 1)|_{L_k}\| \leqq 2^{-(k+1)}$.

We then define

$$L_{k+1} = L_k + \theta_{K+1}(A) + Fa_{k+1}$$

(where F is the underlying field $\mathbf{R}$ or $\mathbf{C}$).

We next inductively construct a sequence of contractions $\Psi_k : L_k \to B$ such that $\eta \circ \Psi_k = \varphi|_{L_k}$. We let $\Psi_0 = 0$. Suppose that we have defined Ψ_k with this property for an integer $k \geqq 0$, and consider the diagram (with the convention $L_{-1} = \{0\}$)



We have that

$$\|(\eta \circ \Psi_k \circ \theta_k - \varphi)|_{L_{k-1}}\| = \|(\varphi \circ \theta_k - \varphi)|_{L_{k-1}}\| \leqq 2^{-k}.$$

From Lemma 2.5 we have a contractive lifting

$$\Psi_{k+1} : L_{k+1} \to B$$

of $\varphi|_{L_{k+1}}$ such that

$$\|(\Psi_{k+1} - \Psi_k \circ \theta_k)|_{L_{k-1}}\| \leqq 6 \cdot 2^{-k}.$$

If follows from (2.10) (with k rather than $k + 1$) that

$$\|(\Psi_{k+1} - \Psi_k)|_{L_{k-1}}\| \leqq 6 \cdot 2^{-k} + \|\Psi_k \circ (1 - \theta_k)|_{L_{k-1}}\| \leqq 7 \cdot 2^{-k}.$$

Fixing k_0, it follows that for all $k \geqq k_0$,

$$\|(\Psi_{k+1} - \Psi_k)|_{L_{k0-1}}\| \leqq 7 \cdot 2^{-k}$$

and the maps Ψ_k converge uniformly on L_{k_0-1}. We let $\Psi^{(k_0-1)} : L_{k_0-1} \to B$ be the limit contraction. The maps $\Psi^{(k)}$ are compatible and thus define a contraction $\bigcup L_k \to B$. The latter extends uniquely to a contraction $\Psi : A \to B$ with $\eta \circ \Psi = \varphi$. This completes the proof of the theorem.

We conclude this section with a few additional remarks about the greatest cross norm. If $V_1, \ldots, V_n, W$ are normed vector spaces, we have a natural surjective isometry

$$\theta : \mathscr{B}(V_1, \ldots, V_n; W) \to \mathscr{B}(V_1 \overline{\otimes}_\lambda \ldots \overline{\otimes}_\lambda V_n; W)$$

defined by

$$\theta(f)(v_1 \otimes \ldots \otimes v_n) = f(v_1, \ldots, v_n)$$

(see [**24**, p. 342]). Furthermore, we have:

LEMMA 2.7. *Suppose that A and B are Banach spaces. If A and B are separable or both have the metric approximation property the same is true for $A \overline{\otimes}_\gamma B$.*

Proof. If A and B are separable, let a_m and b_n be dense sequences in A and B, respectively. Then finite linear combinations $\sum a_{m_i} \otimes b_{n_i}$ will be dense in $A \overline{\otimes}_\gamma B$, hence $A \overline{\otimes}_\gamma B$ is separable.

If $\varphi_\nu : A \to A$, $\Psi_\nu : B \to B$ are finite rank contractions converging to the identity maps in the point norm topologies. From Lemma 2.3 the finite rank maps $\varphi_\nu \otimes \Psi_\nu$ are contractions. It is trivial that they converge on $A \otimes_\gamma B$ to the identity map in the point-norm topology. Since they are contractions, they have unique extensions $\varphi_\nu \overline{\otimes} \Psi_\nu$ which also converge in point-norm to the identity on $A \overline{\otimes}_\gamma B$.

3. Cohomology. We recall that a (vector) *cochain complex* C is a sequence of vector spaces C^n, $n \geqq 0$ and linear maps δ^n, $n \geqq -1$ (we will often omit the superscripts on the δ's)

$$0 \xrightarrow{\delta^{-1}} C^0 \xrightarrow{\delta^0} C^1 \xrightarrow{\delta^1} \ldots.$$

such that $\delta^n\delta^{n-1} = 0$. (see [**26**, § IV.1]). For each $n \geqq 0$, we let

$$Z^n(C) = \ker \delta^n$$
$$B^n(C) = \operatorname{im} \delta^{n-1}$$
$$H^n(C) = Z^n(C)/B^n(C).$$

Thus we have that $H^n(C) = 0$ if and only if C is exact at C^n, i.e., $\operatorname{im} \delta^{n-1} = \ker \delta^n$.

Given cochain complexes C and C', a *cochain homomorphism* $\varphi : C' \to C$ is a sequence of linear maps $\varphi^n : C'^n \to C^n$, $n \geqq 0$ such that $\delta\varphi^n = \varphi^{n+1}\delta$. A diagram of cochain complexes

$$\ldots \xrightarrow{\varphi} C \xrightarrow{\Psi} \ldots$$

is said to be *exact* at C if $\ker \Psi^n = \operatorname{im} \varphi^n$ for each n. Given an exact sequence of cochain complexes

(3.1) $\quad 0 \to C' \xrightarrow{\varphi} C \xrightarrow{\Psi} C'' \to 0,$

we obtain a corresponding exact cohomology sequence

$$(3.2)\qquad 0 \longrightarrow H^0(C') \xrightarrow{\varphi_*^{\,0}} H^0(C) \xrightarrow{\Psi_*^{\,0}} H^0(C'') \xrightarrow{\Delta^0} H^1(C') \xrightarrow{\varphi_*^{\,1}} \ldots .$$

(see [**26**, IV 2]). Using brackets to indicate cohomology classes, these maps are defined by

$$\varphi_*^{\,n}([z_n']) = [\varphi^n(z_n')] \quad z_n' \in Z^n(C'),$$
$$\Psi_*^{\,n}([z_n]) = [\Psi^n(z_n)] \quad z_n \in Z^n(C),$$
$$\Delta^n([z_n'']) = [c_{n+1}'] \quad z_n'' \in Z^n(C''),$$

the element $c_{n+1}' \in Z^{n+1}(C')$ being determined by any diagram of the form

$$\begin{array}{ccccc} & & c_n & \xmapsto{\Psi^n} & z_n'' \\ & & \Big\downarrow \delta & & \\ c_{n+1}' & \xmapsto{\varphi^{n+1}} & c_{n+1} & & \end{array}$$

Let us suppose that A is a Banach algebra, and that V is a Banach A-bimodule, i.e., V is a Banach space and an algebraic A-bimodule with

$$||av|| \leqq ||a||\,||v||, \quad ||va|| \leqq ||v||\,||a||, \quad a \in A, v \in V.$$

The corresponding *bounded cohomology chain complex* $C(A, V)$ is defined by letting

$$C^0(A, V) = V$$
$$C^n(A, V) = \mathscr{B}(A, \ldots, A; V) \cong \mathscr{B}(A \mathbin{\overline{\otimes}_\gamma} \ldots \mathbin{\overline{\otimes}_\gamma} A, V)$$

and $\delta^n : C^n(A, V) \rightarrow C^{n+1}(A, V)$ is defined for $n \geqq 0$ by

$$(\delta^0 v)(a) = av - va,$$
$$(\delta^n f)(a_1, \ldots, a_{n+1}) = a_1 f(a_2, \ldots, a_{n+1}) + (-1)^{n+1} f(a_1, \ldots, a_n)a_{n+1}$$
$$+ \textstyle\sum_{i=1}^n (-1)^i f(a_1, \ldots, a_i a_{i+1}, \ldots, a_{n+1}).$$

We will not use the Banach structure on the vector spaces $C^n(A, V)$. We write $H^n(A, V)$ for $H^n(C(A, V))$.

If W is an A-submodule and Banach subspace of V, and $\eta : V \rightarrow V/W$ is the quotient map, we obtain an exact sequence

$$(3.3)\qquad 0 \rightarrow C(A, W) \xrightarrow{\varphi} C(A, V) \xrightarrow{\Psi} C(A, V/W)$$

where

$$\varphi^n : C^n(A, W) \rightarrow C^n(A, V)$$

is the inclusion isometry, and

$$(3.4)\qquad \Psi^n : C^n(A, V) \rightarrow C^n(A, V/W)$$

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is the contraction defined by

$$\Psi^n(f)(a_1, \ldots, a_n) = \eta f(a_1, \ldots, a_n).$$

LEMMA 3.1. *Suppose that A is separable and satisfies the metric approximation property, and that W is an M-ideal in V. Then the sequence*

$$0 \to C(A, W) \overset{\varphi}{\to} C(A, V) \overset{\Psi}{\to} C(A, V/W) \to 0$$

is exact.

Proof. It suffices to prove that (3.4) is surjective. For $n = 0$, this is trivial. For $n > 0$ this is a consequence of Lemma 2.7 and Theorem 2.6.

Applying (3.2) we have

COROLLARY 3.2. *Under the hypotheses of Lemma* 3.1, *one has an exact sequence*

$$0 \to H^0(A, W) \to H^0(A, V) \to H^0(A, V/W) \to H^1(A, W) \to \ldots .$$

If $\varphi : A \to B$ is a homomorphism of Banach algebras and V is a B-bimodule, then we may also regard it as an A-bimodule by letting $a \cdot v = \varphi(a)v$ and $v \cdot a = v\varphi(a)$. In particular, suppose that $A \subseteq \mathscr{B}(H)$ is a C^*-algebra. Then we may regard $\mathscr{B}(H)$, $\mathscr{K}(H)$, and $\mathscr{B}(H)/\mathscr{K}(H)$ as A-bimodules. A von Neumann algebra is said to be *approximately finite-dimensional* if it is generated by an increasing sequence of finite-dimensional subalgebras. We note that in particular, for a C^*algebra $A \supseteq \mathscr{K}(H)$, the weak closure $\bar{A} = \mathscr{B}(H)$ is approximately finite-dimensional.

COROLLARY 3.3. *If $A \subseteq \mathscr{B}(H)$ is a separable nuclear C^*-algebra and the weak closure $\bar{A}$ is an approximately finite-dimensional von Neumann algebra, then*

$$H^n(A, \mathscr{B}(H)/\mathscr{K}(H)) \cong H^{n+1}(A, \mathscr{K}(H))$$

for all $n \geqq 1$.

Proof. From [**30**, Theorem 6.1] or [**34**, Theorem 7.1; **18**, Corollary 4.4—see Math. Rev. for correction], we have that

$$H^n(A, \mathscr{B}(H)) = H^n(\bar{A}, \mathscr{B}(H))$$

and the last group is zero (see [**34**, Theorem 4.5]). On the other hand, since A is nuclear it satisfies the metric approximation property (see § 1). From Corollary 3.2, and the above we have the exact sequence

$$0 = H^n(A, \mathscr{B}(H)) \to H^n(A, \mathscr{B}(H)/\mathscr{K}(H)) \to H^{n+1}(A, \mathscr{K}(H)) \to H^{n+1}(A, \mathscr{B}(H)) = 0;$$

hence $H^n(A, \mathscr{B}(H)/\mathscr{K}(H)) \cong H^{n+1}(A, \mathscr{K}(H))$ as desired.

Remark 3.4. In the preliminary version of Connes' remarkable paper [**15**], it is stated that an injective von Neumann algebra on a separable Hilbert space must be approximately finite-dimensional. We are convinced (as is the referee) that this must be the case, but we do not have access to the details. Assuming that this is true, we may delete the hypothesis of $\bar{A}$ being approximately finite dimensional from Corollary 3.3 and Proposition 3.6 below. To see this, note that from [**12**] the weak closure must be injective, hence approximately finite-dimensional from Connes' assertion.

LEMMA 3.5. *Suppose that $\varphi : A \to B$ is a bounded surjective homomorphism of Banach algebras such that $J = \ker \varphi$ has a bounded approximate identity and that V is a Banach B-bimodule. Then $H^1(B, V) \cong H^1(A, V)$ and one has an isomorphic injection*

$$H^2(B, V) \hookrightarrow H^2(A, V).$$

Proof. Consider the exact sequence

$$0 \to C(B, V) \xrightarrow{\varphi} C(A, V) \xrightarrow{\rho} D \to 0$$

where $\varphi^0 = \mathrm{id} : V \to V$, $D^0 = 0$, and for each $n > 0$

$$\varphi^n(f)(a_1, \ldots, a_n) = f(\varphi a_1, \ldots, \varphi a_n), \; D^n = \operatorname{coker} \varphi^n.$$

ρ^n is the quotient map, and the boundary maps on D are induced by those on $C(B, V)$ and $C(A, V)$. Then we have the exact sequence

$$0 = H^0(D) \to H^1(B, V) \to H^1(A, V) \to H^1(D) \to H^2(B, V) \to H^2(A, V)$$

and it suffices to prove that $H^1(D) = Z^1(D)$ is zero. Letting $J = \ker \varphi^1$ and $\mathscr{B}_e(J, V)$ be the maps in $\mathscr{B}(J, V)$ which extend to elements of $\mathscr{B}(A, V)$, we have an exact sequence

$$0 \to C^1(B, V) \to C^1(A, V) \xrightarrow{\rho} \mathscr{B}_e(J, V) \to 0$$

where ρ is the restriction map. Thus we may identify

$$C^1(A, V) \xrightarrow{\rho^1} D^1 \quad \text{with} \quad C^1(A, V) \xrightarrow{\rho} \mathscr{B}_e(J, V).$$

Given a cycle $f \in \mathscr{B}_e(J, V)$ we may select $\bar{f} \in C^1(A, V)$ with $\rho^1(\bar{f}) = f$. Since $\rho^2(\delta \bar{f}) = 0$, $\delta \bar{f} = \varphi^2(g)$ for some $g \in C^2(B, V)$. It follows that if $k_1, k_2 \in J$ then

$$\begin{aligned} 0 &= g(0, 0) \\ &= \delta \bar{f}(k_1, k_2) \\ &= k_1 \bar{f}(k_2) - \bar{f}(k_1 k_2) + \bar{f}(k_1) k_2 \\ &= -\bar{f}(k_1 k_2). \end{aligned}$$

From the Cohen Factorization Theorem [**14**, Theorem 1], if $k \in J$ then

$k = k_1k_2$, $k_i \in J$; hence

$$f(k) = \bar{f}(k_1k_2) = 0,$$

i.e., $f = 0$, and the proof is complete.

We note that for dual modules V, Sinclair has proved that $H^n(B, V) \cong H^n(A, V)$ for all n (see [**36**; **17**, § 4]).

PROPOSITION 3.6. *If $A \subseteq \mathscr{B}(H)$ is a separable nuclear C^*-algebra and the weak closure $\bar{A}$ is an approximately finite-dimensional von Neumann algebra and B is its image in $\mathscr{B}(H)/\mathscr{K}(H)$, then one has maps*

$$H^1(B, \mathscr{B}(H)/\mathscr{K}(H)) \cong H^2(A, \mathscr{K}(H)),$$
$$H^2(B, \mathscr{B}(H)/\mathscr{K}(H)) \hookrightarrow H^3(A, \mathscr{K}(H)).$$

Proof. This is immediate from Lemma 3.5 and Corollary 3.3.

We conjecture that if $A \subseteq \mathscr{B}(H)$ is a separable nuclear C^*-algebra, then $H^n(A, \mathscr{K}(H)) = 0$, for $n \geqq 2$. If this is the case it will follow that for a *-isomorphism φ from a separable nuclear C^*-algebra B into $\mathscr{B}(H)/\mathscr{K}(H)$, we have $H^1(B, \mathscr{B}(H)/\mathscr{K}(H)) = H^2(B, \mathscr{B}(H)/\mathscr{K}(H)) = 0$, since identifying B with its image and A with the inverse image in $\mathscr{B}(H)$, A will be separable and nuclear from [**12**, Corollary 3.3]. This in turn will imply that given a *-isomorphism $\Psi : B \to \mathscr{B}(H)/\mathscr{K}(H)$ which is *norm* close to φ, then Ψ will be equivalent to φ in the sense of [**7**, **8**] (see [**33**, Theorem 2; **29**, Theorem 5.1]). We note that B. E. Johnson has proved that $H^n(\mathscr{K}(H), \mathscr{K}(H)) = 0$ for $n \geqq 2$ [**28**, Theorem 4.4].

4. Completely positive liftings. A unital C^*-algebra B is said to be *injective* (resp., *separably injective*) if given unital C^*-algebras (resp., separable unital C^*-algebras) $A \subseteq S$ (we assume the unit is the same), any completely positive map $\varphi : A \to B$ has a completely positive extension $\Psi : S \to B$. We note that if $B \subseteq \mathscr{B}(H)$ is injective, then there exists a completely positive projection of $\mathscr{B}(H)$ onto B, hence B is injective in the sense of [**10**].

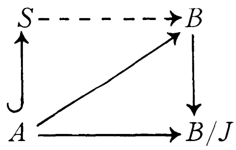
LEMMA 4.1. *Suppose that B is a separably injective unital C^*-algebra and that J is a closed two-sided ideal in B. If one can solve the completely positive lifting problem*

$$\begin{array}{ccc} & & B \\ & & \downarrow \\ A & \to & B/J \end{array}$$

for any separable C^*-algebra A and completely positive map $A \to B/J$, then B/J

is also separably injective.

Proof. This is immediate from the diagram



where the diagonal is a lifting and upper row is an extension of the diagonal.

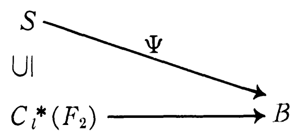
We let $\mathbf{F}_2$ denote the free group on two generators and $C_l^*(\mathbf{F}_2)$ be the C^*-algebra generated by left translations on $H = l^2(\mathbf{F}_2)$.

LEMMA 4.2. *If B is a unital C^*-algebra which has a unital trace and there is a unital *-isomorphism from $C_l^*(\mathbf{F}_2)$ into B, then B is not separably injective.*

Proof. From [**10**, end of § 3] there exist unitary $v_1, v_2 \in C_l^*(F_2)$ and a projection $e \in \mathscr{B}(H)$ such that

$$\begin{aligned} &(4.1) \qquad e + v_1^* e v_1 \geqq 1 \\ &\qquad\qquad e + v_2^* e v_2 + v_2^{*2} e v_2^2 \leqq 1. \end{aligned}$$

Let S be the C^*-algebra generated by $C_l^*(\mathbf{F}_2)$ and e. If B is separably injective, then there exists a completely positive map Ψ such that the diagram



commutes. Since, in particular,

$$1 = \Psi(v_i^* v_i) = \Psi(v_i^*)\Psi(v_i),$$

the operators v_i are in the multiplicative domain of Ψ, i.e.,

$$\Psi(v_i^* s) = \Psi(v_i^*)\Psi(s), \quad \Psi(s v_i) = \Psi(s)\Psi(v_i)$$

for all $s \in S$ (see [**9**, Theorem 3.1]). Apply Ψ to (4.1),

$$\begin{aligned} &\Psi(e) + u_1^*\Psi(e)u_1 \geqq 1, \\ &\Psi(e) + u_2^*\Psi(e)u_2 + u_2^{*2}\Psi(e)u_2^2 \leqq 1, \end{aligned}$$

where the operators $u_i = \Psi(v_i)$ are unitary. Letting τ be a unital trace on B we conclude $2\tau(\Psi(e)) \geqq 1 \geqq 3\tau(\Psi(e))$, a contradiction, completing the proof.

The direct sum M of all matrix algebras M_n ($1 \leqq n < \infty$) is an injective von Neumann algebra. Letting τ_n be the unital trace on M_n, we define a trace τ_ω on M by

$$\tau_\omega(r) = \lim_\omega \tau_n(r_n),$$

where we are using the generalized limit determined by a free ultra-filter ω

on the positive integers (see [**38**, § 1]). We have that

$$J_\omega = \{r \in M : \tau_\omega(r^*r) = 0\}$$

is a norm-closed two-sided ideal in M and M/J_ω is a type II_1 factor. We now appeal to the fact proved by Wassermann [**38**, § 1.6] that there is a unital $*$-isomorphism $\pi : C_l^*(\mathbf{F}_2) \to M/J_\omega$ to conclude

THEOREM 4.3. *The diagram*

$$\begin{array}{ccc} & & M \\ & & \downarrow \\ C_l^*(\mathbf{F}_2) & \xrightarrow{\pi} & M/J_\omega \end{array}$$

does not have a completely positive lifting.

Proof. From Lemma 4.2, M/J_ω is not separably injective, hence the argument of Lemma 4.1 gives the desired result.

We let $C^*(\mathbf{F}_2)$ denote the full group C^*-algebra of $\mathbf{F}_2$ (see [**22**, § 13.9.1]). We are indebted to L. Brown for the following remarkable result.

LEMMA 4.4. *If J is a closed two-sided ideal in a unital C^*-algebra B and one has a $*$-homomorphism π from $C^*(\mathbf{F}_2)$ into B/J, then one can always find a unital completely positive lifting in the diagram*

$$\begin{array}{ccc} & \nearrow & B \\ & & \downarrow \eta \\ C^*(\mathbf{F}_2) & \xrightarrow{\pi} & B/J. \end{array}$$

Proof. Let $u_1, u_2 \in C^*(\mathbf{F}_2)$ be the unitaries determined by the generators of $\mathbf{F}_2$. Then for each i,

$$v_i = \begin{bmatrix} 0 & \pi(u_i) \\ \pi(u_i^*) & 0 \end{bmatrix}$$

is a self-adjoint unitary in

$$M_2(B/J) \cong M_2(B)/M_2(J).$$

We may find unitary pre-images $r_i \in M_2(B)$ for the v_i. To see this, let r_i' be a self-adjoint pre-image of v_i, and let $r_i = f_i(r_i')$ where f_i is any continuous function from $[-\|r_i'\|, \|r_i'\|]$ into the unit circle such that $f(-1) = -1$, $f(1) = 1$. Letting

$$r_i = \begin{bmatrix} a_i & b_i \\ c_i & d_i \end{bmatrix}$$

we have that

$$\begin{bmatrix} \eta(a_i) & \eta(b_i) \\ \eta(c_i) & \eta(d_i) \end{bmatrix} = \begin{bmatrix} 0 & \pi(u_i) \\ \pi(u_i^*) & 0 \end{bmatrix}.$$

The assignment

$$u_i \mapsto \begin{bmatrix} b_i & a_i \\ d_i & c_i \end{bmatrix} = \begin{bmatrix} a_i & b_i \\ c_i & d_i \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

determines a representation of $\mathbf{F}_2$, and thus extends to a *-homomorphism

$$\Psi_1 : C^*(\mathbf{F}_2) \rightarrow M_2(B).$$

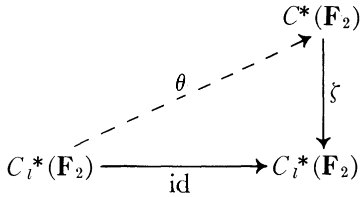
Letting

$$E : M_2(B) \rightarrow B : [b_{ij}] \mapsto b_{11},$$

it is evident that $\Psi = E \circ \Psi_1$, is the desired lifting of π.

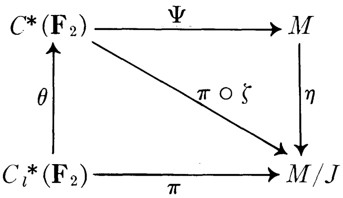
THEOREM 4.5. *The diagram*

(4.2)



*where ζ is the natural surjective *-homomorphism, does not have a completely positive lifting θ.*

Proof. If θ exists, consider the diagram



where the top row is a lifting of $\pi \circ \zeta$ (by Lemma 4.4). Then the map $\Psi \circ \theta$ is a lift of π, contradicting Theorem 4.3.

Remark 4.6. One may also prove Theorem 4.5 by using a result of Berger, Coburn, and Lebow [**5**, Theorem 2.2] completing an argument of Douglas and Howe. Suppose that J is a closed two-sided ideal in a C^*-algebra A and that there is a completely positive lifting for the diagram

$$\begin{array}{ccc} & & A \\ & & \downarrow \\ A/J & \rightarrow & A/J. \end{array}$$

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Then given any C^*-algebra B, the kernel of the map

$$A \otimes_{\min} B \to (A/J) \otimes_{\min} B$$

is the norm closure of $J \otimes B$. Essentially the same argument shows that the kernel of

$$A \otimes_{\min} A \to (A/J) \otimes_{\min} (A/J)$$

is the norm closure of $J \otimes A + A \otimes J$. Thus if the completely positive lifting in Theorem 4.5 existed, then the kernel of

$$C^*(\mathbf{F}_2) \otimes_{\min} C^*(\mathbf{F}_2) \to C_l^*(\mathbf{F}_2) \otimes_{\min} C_l^*(\mathbf{F}_2)$$

would be the norm closure of

$$\ker \zeta \otimes C^*(\mathbf{F}_2) + C^*(\mathbf{F}_2) \otimes \ker \zeta.$$

However, Wassermann has shown this is not the case [**38**, § 2.8].

Remark 4.7. It is tempting to conjecture that $C_l^*(\mathbf{F}_2)$ does not satisfy the metric approximation property. Owing to Theorem 2.6, it would suffice to prove that (4.2) does not have a contractive lifting. Perhaps it is relevant that we can prove that there does not exist even a positive extension Ψ in the proof of Lemma 4.2.

Remark 4.8. It would seem likely that the Calkin algebra $\mathscr{B}(H)/\mathscr{K}(H)$ is not separably injective. If this is the case, we will have a *-isomorphism from a separable C^*-algebra into the Calkin algebra which does not have a completely positive lifting. Such an example would be of considerable interest in extension theory.

Added in proof. Extending the technique of § 4, Joel Anderson has proved that the Calkin algebra is not separably injective.

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University of Toronto,
Toronto, Ontario;
University of Pennsylvania,
Philadelphia, Pennsylvania