ON THE (f, d_n)-METHOD OF SUMMABILITY

GASTON SMITH

1. Introduction. Let $f(z)$ be a non-constant entire function and let $\{d_n\}$ be a sequence of complex numbers such that

$$d_i \neq -f(1) \quad \text{and} \quad d_i \neq -f(0) \qquad (i \geqslant 1).$$

The set of equations

$$\begin{aligned} a_{00} &= 1, \\ a_{0k} &= 0 \qquad (k \neq 0), \\ \prod_{i=1}^{n} \left[\frac{f(z) + d_i}{f(1) + d_i}\right] &= \sum_{k=0}^{\infty} a_{nk}\, z^k \qquad (n \geqslant 1) \end{aligned} \tag{1.1}$$

defines the elements of a matrix $A = (a_{nk})$, where $n, k = 0, 1, 2, \ldots$.

DEFINITION 1.1. *A sequence $\{t_k\}$, or a series whose kth partial sum is t_k, is said to be (f, d_n)-summable to t if and only if*

$$\lim_{n\to\infty} \sum_{k=0}^{\infty} a_{nk}\, t_k = t,$$

where the a_{nk}'s are defined by (1.1).

We obtain several known methods of summability as special cases of the (f, d_n)-method by placing certain restrictions on $f(z)$ and $\{d_n\}$. If $f(z) = z$ and $d_n = r$, where r is any complex constant, we get the well-known Euler method **(1)**. If $f(z) = \lambda z$, $d_n = n$, and $\lambda > 0$, we obtain the Karamata–Stirling method as defined by Vuckovic **(6)**. If $f(z) = z$ and $d_n = n$ we get the Lototsky method as defined by Agnew **(2)**. If $f(z) = z$ and $\{d_n\}$ is any real sequence, we get a method defined by Jakimovski **(4)**. If $f(z) = z$ and $\{d_n\}$ is any complex sequence, we get a method defined by Cowling and Miracle **(3)**.

In this paper we first obtain some regularity conditions for the (f, d_n)-method. Several necessary conditions and four sufficient conditions are obtained. Then we derive some results concerning the effectiveness of this method for summing power series. The paper is concluded with a discussion of some special cases of the (f, d_n)-method.

Throughout the paper we frequently make use of the following notations. The symbol $f(z)$ denotes an entire function. When $z = x + iy$ we denote $\mathrm{Re}\{f(z)\}$ by $u(x, y)$ or u and $\mathrm{Im}\{f(z)\}$ by $v(x, y)$ or v. The principal argument of d_n is denoted by θ_n. Also, we let $a + ib = f(1)$, $x_n + iy_n = d_n$, and $\rho_n = |d_n|$.

Received November 25, 1963.

2. Regularity conditions for the (f, d_n)-method. It is well known that a linear method of summability defined by the matrix $C = (c_{nk})$ is regular if and only if

$$\sum_{k=0}^{\infty} |c_{nk}| \leqslant M \qquad (n \geqslant 0), \tag{2.1}$$

$$\lim_{n\to\infty} c_{nk} = 0 \qquad (k \geqslant 0), \tag{2.2}$$

$$\lim_{n\to\infty} \sum_{k=0}^{\infty} c_{nk} = 1, \tag{2.3}$$

where M is a constant independent of n.

A. Necessary conditions for the regularity of the (f, d_n)-method.

LEMMA 2.1. *If*

$$\prod_{i=1}^{\infty} (1 - a_i) = 0,$$

a_i is real, and $a_i < 1$ for all i, then there are infinitely many a_i's such that $a_i > 0$.

THEOREM 2.1. *A necessary condition in order that the (f, d_n)-method be regular is that there exist a strictly increasing sequence of natural numbers $\{n_k\}$ such that*

$$\sum_{k=1}^{\infty} \left[1 - \left|\frac{f(0) + d_{n_k}}{f(1) + d_{n_k}}\right|^2\right] = \infty. \tag{2.4}$$

Proof. Suppose that the (f, d_n)-method is regular. Letting $z = 0$ in (1.1) we get

$$a_{n0} = \prod_{i=1}^{n} \left[\frac{f(0) + d_i}{f(1) + d_i}\right]. \tag{2.5}$$

Since regularity condition (2.2) implies that

$$\lim_{n\to\infty} a_{n0} = 0,$$

it follows that

$$\prod_{i=1}^{\infty} \left[\frac{f(0) + d_i}{f(1) + d_i}\right] = 0. \tag{2.6}$$

The relation (2.6) implies that

$$\prod_{i=1}^{\infty} \left|\frac{f(0) + d_i}{f(1) + d_i}\right|^2 = 0 \tag{2.7}$$

so that

$$\prod_{i=1}^{\infty} \left[1 - \left\{1 - \left|\frac{f(0) + d_i}{f(1) + d_i}\right|^2\right\}\right] = 0. \tag{2.8}$$

Let

$$a_i = 1 - \left|\frac{f(0) + d_i}{f(1) + d_i}\right|^2. \tag{2.9}$$

Using (2.9), the relation (2.8) may be written in the form

$$\prod_{i=1}^{\infty} (1 - a_i) = 0, \tag{2.10}$$

where $a_i < 1$ and a_i is real. By Lemma 2.1, there are infinitely many a_i's which are positive. Construct a sequence $\{a_{nk}\}$ which consists of all of the positive a_i's arranged according to increasing magnitude of the subscripts. Consequently (2.10) implies that

$$\prod_{k=1}^{\infty} (1 - a_{nk}) = 0,$$

where $0 < a_{nk} < 1$. Hence by a well-known theorem on infinite products we get

$$\sum_{k=1}^{\infty} a_{nk} = \infty. \tag{2.11}$$

It follows from (2.9) and (2.11) that

$$\sum_{k=1}^{\infty} \left[1 - \left|\frac{f(0) + d_{nk}}{f(1) + d_{nk}}\right|^2\right] = \infty$$

so that the theorem is proved.

COROLLARY 2.1. *If $f(0)$ and $f(1)$ are real, a necessary condition in order that the (f, d_n)-method be regular is that there exist a strictly increasing sequence of natural numbers $\{n_k\}$ such that*

$$\sum_{k=1}^{\infty} \left[\frac{f(0) + f(1) + 2x_{nk}}{|f(1) + d_{nk}|^2}\right] = \pm\infty.$$

COROLLARY 2.2. *A necessary condition in order that the (f, d_n)-method be regular is that*

$$|f(1) + d_n| > |f(0) + d_n|$$

for infinitely many values of n. If $f(0)$ and $f(1)$ are real, a necessary condition for regularity is that

$$x_n > -\tfrac{1}{2}[f(0) + f(1)] > -f(1)$$

for infinitely many values of n when $f(1) > f(0)$ and

$$x_n < -\tfrac{1}{2}[f(0) + f(1)] < -f(1)$$

for infinitely many values of n when $f(1) < f(0)$.

COROLLARY 2.3. *A necessary condition in order that the* (f, d_n)*-method be regular is that* $f(0) \neq f(1)$.

We can now prove the main result concerning necessary conditions for the regularity of the (f, d_n)-method.

THEOREM 2.2. *A necessary condition in order that the* (f, d_n)*-method be regular is that*

$$\sum_{n=1}^{\infty} \frac{1}{|f(0) + d_n|} = \infty. \tag{2.12}$$

Proof. Suppose that

$$\sum_{n=1}^{\infty} \frac{1}{|f(0) + d_n|}$$

is convergent. It follows that

$$\lim_{n\to\infty} \frac{1}{|f(0) + d_n|} = 0. \tag{2.13}$$

The relation (2.13) implies that there exists a positive integer N such that for all $n > N$ we have

$$\frac{1}{|f(0) + d_n|^2} < \frac{1}{|f(0) + d_n|}. \tag{2.14}$$

It follows from Corollary 2.2 that

$$|f(0) + d_n|^2 < |f(1) + d_n|^2$$

holds for the infinitely many values $\{n_k\}$ for which Theorem 2.1 is true. Hence

$$\frac{|f(0) + d_n|}{|f(1) + d_n|^2} < \frac{1}{|f(0) + d_n|} \tag{2.15}$$

for the infinitely many values $\{n_k\}$ for which Theorem 2.1 is true. Let $f(0) = c + di$ and let

$$L_{n_k} = \left| \frac{|f(1) + d_{n_k}|^2 - |f(0) + d_{n_k}|^2}{|f(1) + d_{n_k}|^2} \right|.$$

Now it follows from (2.14) and (2.15) that

$$L_{n_k} \leqslant \frac{(a - c)^2 + 2|a - c| + (b - d)^2 + 2|b - d|}{|f(0) + d_{n_k}|}. \tag{2.16}$$

From the supposition and relation (2.16) we find that

$$\sum_{k=1}^{\infty} L_{nk}$$

converges. Therefore

$$\sum_{k=1}^{\infty} \frac{|f(1)+d_{nk}|^2 - |f(0)+d_{nk}|^2}{|f(1)+d_{nk}|^2}$$

converges, which contradicts Theorem 2.1.

COROLLARY 2.4. *If $d_n \neq 0$, a necessary condition in order that the (f, d_n)-method be regular is that*

$$\sum_{n=1}^{\infty} \rho_n^{-1} = \infty.$$

COROLLARY 2.5. *A necessary condition in order that the (f, d_n)-method be regular is that*

$$\sum_{n=1}^{\infty} \frac{1}{|f(1)+d_n|} = \infty.$$

If we take $f(z) = z$ in relation (1), then Theorem 2.2 becomes a known result **(3)**.

B. Sufficient conditions for the regularity of the (f, d_n)-method.

LEMMA 2.2. *Suppose that d_n is real, $d_n \geqslant 0$, and that the Taylor expansion of $f(z)$ about the origin has non-negative coefficients. Then the (f, d_n)-method is regular if and only if*

$$\sum_{n=1}^{\infty} \frac{1}{f(1)+d_n} = \infty.$$

Proof. The necessity of the condition follows from Corollary 2.5.

Sufficiency. Letting $z = 1$ in relation (1.1), we get

$$\sum_{k=0}^{\infty} a_{nk} = 1;$$

so regularity condition (2.3) holds. Since d_n is real and non-negative and the coefficients of the expansion of $f(z)$ about the origin are non-negative, it follows that $|a_{nk}| = a_{nk}$. Hence regularity condition (2.1) holds. Also since the coefficients of the expansion of $f(z)$ about the origin are non-negative and $f(z)$ is non-constant, it follows that $f(0) < f(1)$. Now choose $\epsilon > 0$ so that $f(0) + 2\epsilon < f(1)$ and let C be a circle with centre at the origin such that

$$|f(t) - f(0)| < \epsilon \qquad \text{for all } t \in C.$$

Hence for $t \in C$ we have that

$$|f(t)| \leqslant f(0) + \epsilon < f(1) - \epsilon.$$

We may represent a_{nk} in the form

$$a_{nk} = \frac{1}{2\pi i} \int_C \prod_{i=1}^{n} \frac{f(t)+d_i}{f(1)+d_i} \cdot \frac{dt}{t^{k+1}}$$

so that

https://doi.org/10.4153/CJM-1965-051-1 Published online by Cambridge University Press

$$a_{nk} = |a_{nk}| \leqslant \frac{1}{2\pi R^k} \int_0^{2\pi} \prod_{i=1}^{n} \left| \frac{f(t) + d_i}{f(1) + d_i} \right| d\theta.$$

Since $1 + x \leqslant e^x$ for x real, we obtain

$$\left| \frac{f(t) + d_i}{f(1) + d_i} \right| \leqslant \exp\left\{ -1 + \frac{|f(t) + d_i|}{f(1) + d_i} \right\}$$

$$\leqslant \exp\left\{ \frac{|f(t)| - f(1)}{f(1) + d_i} \right\}$$

$$\leqslant \exp\left\{ - \frac{\epsilon}{f(1) + d_i} \right\}.$$

Hence it follows that

$$a_{nk} \leqslant R^{-k} \exp\left\{ -\epsilon \sum_{i=1}^{n} \frac{1}{f(1) + d_i} \right\}.$$

Therefore

$$\lim_{n\to\infty} a_{nk} = 0,$$

which proves the lemma.

The preceding lemma generalizes a result of Jakimovski **(4)**.

THEOREM 2.3. *Suppose that*

$$\sum_{i=1}^{\infty} \frac{1}{|f(1) + d_i|} = \infty, \qquad \sum_{i=1}^{\infty} \frac{(\mathrm{Im}\sqrt{d_i})^2}{|f(1) + d_i|^2} < \infty,$$

and that the Taylor expansion of $f(z)$ about the origin has non-negative coefficients. Then the (f, d_n)-method is regular.

Proof. Letting $z = 1$ in relation (1.1), we get

$$\sum_{k=0}^{\infty} a_{nk} = 1;$$

so regularity condition (2.3) holds. Let C be any circle with centre at the origin. The elements a_{nk} are given by the formula

$$a_{nk} = \frac{1}{2\pi i} \int_C \prod_{i=1}^{n} \left[\frac{f(t) + d_i}{f(1) + d_i} \right] \cdot \frac{dt}{t^{k+1}}. \tag{2.17}$$

By expanding the product on the right of (2.17), it follows that

$$\prod_{i=1}^{n} [f(1) + d_i]\, a_{nk} = \frac{1}{2\pi i} \int_C \{ [f(t)]^n + [f(t)]^{n-1}(d_1 + d_2 + \ldots + d_n)$$
$$+ [f(t)]^{n-2}(d_1 d_2 + \ldots + d_{n-1} d_n) + (d_1 d_2 \ldots d_n)\} \frac{dt}{t^{k+1}}.$$

Since $[f(t)]^n$ is an entire function, we may write,

$$[f(t)]^n = \sum_{j=0}^{\infty} p_{jn}\, t^j. \tag{2.18}$$

It follows that

$$\prod_{i=1}^{n} [f(1) + d_i]\, a_{nk} = p_{kn} + (d_1 + d_2 + \ldots + d_n) p_{k,n-1} + (d_1 d_2 + \ldots + d_{n-1} d_n) p_{k,n-2} + \ldots + (d_1 d_2 \ldots d_n) p_{k0}. \tag{2.19}$$

Therefore

$$\sum_{k=0}^{\infty} |a_{nk}| \prod_{i=1}^{n} |f(1) + d_i| \leqslant \sum_{k=0}^{\infty} \{p_{kn} + (\rho_1 + \ldots + \rho_n) p_{k,n-1} + \ldots + (\rho_1 \ldots \rho_n) p_{k0}\}. \tag{2.20}$$

Let $B = (b_{nk})$ be the matrix corresponding to the (f, ρ_n)-method. Hence

$$\sum_{k=0}^{\infty} |b_{nk}| \prod_{i=1}^{n} [f(1) + \rho_i] = \sum_{k=0}^{\infty} \{p_{kn} + (\rho_1 + \ldots + \rho_n) p_{k,n-1} + \ldots + (\rho_1 \ldots \rho_n) p_{k0}\}. \tag{2.21}$$

Now from relations (2.20) and (2.21), it follows that

$$\sum_{k=0}^{\infty} |a_{nk}| \prod_{i=1}^{n} |f(1) + d_i| \leqslant \sum_{k=0}^{\infty} |b_{nk}| \prod_{i=1}^{n} [f(1) + \rho_i]. \tag{2.22}$$

However, since all of the elements of B are non-negative,

$$\sum_{k=0}^{\infty} |b_{nk}| = \sum_{k=0}^{\infty} b_{nk} = 1. \tag{2.23}$$

The relations (2.22) and (2.23) imply that

$$\sum_{k=0}^{\infty} |a_{nk}| \leqslant \prod_{i=1}^{n} \left[\frac{f(1) + \rho_i}{|f(1) + d_i|}\right]. \tag{2.24}$$

Since $1 + x \leqslant e^x$ for all real x and $f(1) + \rho_n \geqslant |f(1) + d_n|$, it follows that

$$\frac{f(1) + \rho_n}{|f(1) + d_n|} \leqslant \left[\frac{f(1) + \rho_n}{|f(1) + d_n|}\right]^2 \leqslant \exp\left\{-1 + \left[\frac{f(1) + \rho_n}{|f(1) + d_n|}\right]^2\right\} \leqslant \exp\left\{4a \cdot \frac{(\operatorname{Im}\sqrt{d_n})^2}{|f(1) + d_n|^2}\right\}. \tag{2.25}$$

From (2.24) and (2.25) we obtain

$$(2.26) \qquad \sum_{k=0}^{\infty} |a_{nk}| \leqslant \exp\left\{4a \cdot \sum_{i=1}^{n} \frac{(\operatorname{Im}\sqrt{d_i})^2}{|f(1) + d_i|^2}\right\}.$$

It follows from (2.26) and the hypothesis that there exists a real number M such that

$$(2.27) \qquad \sum_{k=0}^{\infty} |a_{nk}| \leqslant M$$

for all n. So the regularity condition (2.1) is satisfied.

Now from (2.19) and the analogous relation involving b_{nk} we find that

$$(2.28) \qquad |a_{nk}| \leqslant |b_{nk}| \prod_{i=1}^{n} \frac{f(1) + \rho_i}{|f(1) + d_i|}.$$

The relation

$$\sum_{n=1}^{\infty} \frac{1}{|f(1) + d_n|} = \infty$$

implies that

$$\sum_{n=1}^{\infty} \frac{1}{|f(1) + d_n| + 2f(1)} = \infty.$$

It follows that

$$(2.29) \qquad \sum_{n=1}^{\infty} \frac{1}{f(1) + |d_n|} = \infty$$

since

$$\frac{1}{f(1) + |d_n|} \geqslant \frac{1}{|f(1) + d_n| + 2f(1)}.$$

By Lemma 2.2 and relation (2.29), we find that the (f, ρ_n)-method is regular so that

$$\lim_{n\to\infty} b_{nk} = 0.$$

Moreover (2.25) implies that

$$\prod_{i=1}^{n} \frac{f(1) + \rho_i}{|f(1) + d_i|}$$

is bounded. Therefore we have

$$\lim_{n\to\infty} a_{nk} = 0,$$

which proves the theorem.

COROLLARY 2.6. *If*

$$\sum_{n=1}^{\infty} \frac{1}{|f(1) + d_n|} = \infty, \qquad \sum_{n=1}^{\infty} \theta_n^2 \textit{ converges}$$

and the Taylor expansion of $f(z)$ about the origin has real non-negative coefficients, then the (f, d_n)-method is regular.

Proof. Since the inequality $x^2 + 2\cos x - 2 \geqslant 0$ holds for all real x, it follows that

$$-2a\rho_n \cos\theta_n + 2a\rho_n \leqslant a\rho_n \theta_n^2, \tag{2.30}$$

where $a > 0$ is valid for all n. Using (2.30) and the fact that there exists a positive integer N such that $\cos\theta_n > \frac{1}{2}$ for all $n > N$, we obtain

$$\begin{aligned} 4a\left|\frac{\operatorname{Im}\sqrt{d_n}}{f(1)+d_n}\right|^2 &= \frac{-2a\rho_n\cos\theta_n + 2a\rho_n}{|a+d_n|^2} \\ &\leqslant \frac{a\rho_n\theta_n^2}{a^2 + 2a\rho_n\cos\theta_n + \rho_n^2} \\ &\leqslant \frac{a\rho_n\theta_n^2}{2a\rho_n\cos\theta_n} \leqslant \theta_n^2. \end{aligned} \tag{2.31}$$

Now we can apply Theorem 2.3, which completes the proof.

COROLLARY 2.7. *Suppose that*

$$\sum_{n=1}^{\infty} \frac{1}{|f(1)+d_n|} = \infty, \qquad \sum \frac{\theta_n^2}{\rho_n} \text{ converges}$$

where the sum on the right ranges over all n for which ρ_n is positive, $\operatorname{Re}\{d_n\} \geqslant -f(1)/2$, *and the Taylor expansion of $f(z)$ about the origin has real non-negative coefficients. Then the (f, d_n)-method is regular.*

COROLLARY 2.8. *Suppose that*

$$\sum_{n=1}^{\infty} \frac{1}{|f(1)+d_n|} = \infty, \qquad \sum_{n=1}^{\infty} \rho_n\theta_n^2 \text{ converges},$$

$2|\theta_n| \leqslant \pi$, *and the coefficients of the expansion of $f(z)$ about the origin are real and non-negative. Then the (f, d_n)-method is regular.*

3. Power series.

THEOREM 3.1. *Suppose that*

$$\sum \frac{1}{\rho_n} = \infty, \qquad \lim_{n\to\infty}\theta_n = 0, \qquad \lim_{n\to\infty}\rho_n = \infty,$$

where the sum ranges over all n for which ρ_n is positive. Then the (f, d_n)-method sums the geometric series

$$\sum_{k=0}^{\infty} z^k \tag{3.1}$$

to $(1-z)^{-1}$ for all values of z such that $\operatorname{Re}\{f(z)\} < \operatorname{Re}\{f(1)\}$.

Proof. The partial sums of (3.1) are given by

$$S_k(z) = (1-z)^{-1} - (1-z)^{-1} z^{k+1}. \tag{3.2}$$

Let

$$\sigma_n(z) = \sum_{k=0}^{\infty} a_{nk} S_k(z).$$

We may represent $\sigma_n(z)$ in the form

$$\sigma_n(z) = \frac{1}{1-z} - \frac{z}{1-z} \prod_{i=1}^{n} \left[\frac{f(z)+d_i}{f(1)+d_i}\right]. \tag{3.3}$$

It is sufficient to show that

$$\prod_{i=1}^{\infty} \left[\frac{f(z)+d_n}{f(1)+d_n}\right] = 0$$

for all z such that $\text{Re}\{f(z)\} < \text{Re}\{f(1)\}$.

Since $1 + x \leqslant e^x$ for real x, it follows that

$$\left|\frac{f(z)+d_n}{f(1)+d_n}\right|^2 \leqslant \exp\left\{\frac{2ux_n + 2vy_n - 2ax_n - 2by_n + H}{|f(1)+d_n|^2}\right\} \tag{3.4}$$

$$\leqslant \exp\left\{\frac{\rho_n[(u-a)\cos\theta_n + (v-b)\sin\theta_n]}{|f(1)+d_n|^2}\right\} + \frac{H}{2|f(1)+d_n|^2},$$

where $H = u^2 + v^2 - a^2 - b^2$. Using the hypothesis, we find that

$$\lim_{n\to\infty} \frac{\rho_n^2[(u-a)\cos\theta_n + (v-b)\sin\theta_n]}{|f(1)+d_n|^2} = u - a$$

and

$$\lim_{n\to\infty} \frac{H\rho_n}{2|f(1)+d_n|^2} = 0.$$

Hence there exist a $K > 0$ and an integer $N > 0$ such that for all $n > N$ we have

$$-K\rho_n^{-1} > \frac{\rho_n[(u-a)\cos\theta_n + (v-b)\sin\theta_n]}{|f(1)+d_n|^2} + \frac{H}{2|f(1)+d_n|^2}. \tag{3.5}$$

From (3.5) it follows that

$$\left|\frac{f(z)+d_n}{f(1)+d_n}\right| \leqslant \exp\{-K\rho_n^{-1}\} \tag{3.6}$$

for all $n > N$. Therefore by (3.6) we obtain

$$\prod_{n=1}^{\infty} \left|\frac{f(z)+d_n}{f(1)+d_n}\right| \leqslant \prod_{i=1}^{N-1} \left|\frac{f(z)+d_n}{f(1)+d_n}\right| \cdot \exp\left\{\sum_{i=N}^{\infty} K\rho_i^{-1}\right\}. \tag{3.7}$$

https://doi.org/10.4153/CJM-1965-051-1 Published online by Cambridge University Press

Since

$$\sum_{i=N}^{\infty} \rho_n^{-1} = \infty$$

we conclude from (3.7) that

$$\prod_{n=1}^{\infty} \left[\frac{f(z)+d_n}{f(1)+d_n}\right] = 0.$$

This completes the proof.

THEOREM 3.2. *Suppose that*

$$\rho_n^{-1} = \infty, \qquad \lim_{n\to\infty} \theta_n = 0, \qquad \lim_{n\to\infty} \rho_n = \infty,$$

where the sum ranges over all positive ρ_n. *Then*

$$\lim_{n\to\infty} \sigma_n(z) = \infty$$

for all z *such that* $\mathrm{Re}\{f(z)\} > \mathrm{Re}\{f(1)\}$, *where* $\sigma_n(z)$ *and* $S_k(z)$ *are defined as in Theorem* 3.1.

Proof. Assume that z is given such that $\mathrm{Re}\{f(z)\} > \mathrm{Re}\{f(1)\}$. It is sufficient to show that

$$\prod_{n=1}^{\infty} \left[\frac{f(z)+d_n}{f(1)+d_n}\right] = \infty. \tag{3.8}$$

Since

$$\lim_{n\to\infty} \theta_n = 0 \quad \text{and} \quad \lim_{n\to\infty} \rho_n = \infty,$$

we have

$$\lim_{n\to\infty}\{|f(z)-d_n|^2 - |f(1)+d_n|^2\} = \infty.$$

Hence there exists an integer $N > 0$ such that

$$\left|\frac{f(z)+d_n}{f(1)+d_n}\right| > 1$$

when $n > N$. It follows that

$$\prod_{n=1}^{\infty} \left|\frac{f(z)+d_n}{f(1)+d_n}\right|^2 = \infty$$

if and only if

$$\sum_{n=1}^{\infty} \left[1 - \left|\frac{f(z)+d_n}{f(1)+d_n}\right|^2\right] = \infty. \tag{3.9}$$

We note that

$$1 - \left|\frac{f(z)+d_n}{f(1)+d_n}\right|^2 = \frac{2\rho_n(u-a)\cos\theta_n + 2\rho_n(v-b)\sin\theta_n + H}{|f(1)+d_n|^2},$$

https://doi.org/10.4153/CJM-1965-051-1 Published online by Cambridge University Press

where $H = u^2 + v^2 - a^2 - b^2$. Using the same procedure as in the proof of Theorem 3.1, we find that

$$\lim_{n\to\infty} \frac{2\rho_n^{\,2}[(u-a)\cos\theta_n + (v-b)\sin\theta_n] + \rho_n H}{|f(1)+d_n|^2} = 2(u-a).$$

Hence there exist a $K > 0$ and an integer $N > 0$ such that for all $n > N$ we have

$$\text{(3.10)}\qquad \frac{2\rho_n[(u-a)\cos\theta_n + (v-b)\sin\theta_n] + H}{|f(1)+d_n|^2} > K\rho_n^{-1}.$$

Since

$$\sum \rho_n^{-1} = \infty,$$

the relation (3.10) implies that

$$\sum_{n=1}^{\infty}\left[-1 + \left|\frac{f(z)+d_n}{f(1)+d_n}\right|^2\right] = \infty.$$

By (3.9) we have

$$\prod_{n=1}^{\infty}\left|\frac{f(z)+d_n}{f(1)+d_n}\right|^2 = \infty$$

so that

$$\text{(3.11)}\qquad \prod_{n=1}^{\infty}\left|\frac{f(z)+d_n}{f(1)+d_n}\right| = \infty.$$

The asserted result follows from (3.11).

THEOREM 3.3. *Suppose that z is given such that $|f(z)+\rho| < |f(1)+\rho|$ and that*

$$\lim_{n\to\infty}\rho_n = \rho \quad\text{and}\quad \lim_{n\to\infty}\theta_n = 0.$$

Then the (f, d_n)-method sums the geometric series (3.1) *to* $(1-z)^{-1}$.

Proof. We follow the same procedure as in Theorem 3.1. Thus we have

$$\text{(3.12)}\qquad \sigma_n(z) = \frac{1}{1-z} - \frac{z}{1-z}\prod_{i=1}^{n}\left[\frac{f(z)+d_i}{f(1)+d_i}\right].$$

Let $H = u^2 + v^2 - a^2 - b^2$. Since $1 + x \leqslant e^x$ for real x, we obtain

$$\left|\frac{f(z)+d_n}{f(1)+d_n}\right| \leqslant \exp\left\{\frac{2\rho_n[(u-a)\cos\theta_n + (v-b)\sin\theta_n] + H}{2|f(1)+d_n|^2}\right\}.$$

It follows from the hypothesis that

$$\lim_{n\to\infty}\frac{2\rho_n[(u-a)\cos\theta_n + (v-b)\sin\theta_n] + H}{2|f(1)+d_n|^2} = \frac{2\rho(u-a)+H}{2|f(1)+\rho|^{\,2}} < 0.$$

Hence there exist a $K > 0$ and an integer $N > 0$ such that

$$(3.13)\qquad \frac{2\rho_n[(u-a)\cos\theta_n + (v-b)\sin\theta_n] + H}{2|f(1)+d_n|^2} < -K$$

for $n > N$. Hence we obtain the relation

$$\prod_{i=1}^{n}\left|\frac{f(z)+d_i}{f(1)+d_i}\right| \leqslant \prod_{i=1}^{N-1}\left|\frac{f(z)+d_i}{f(1)+d_i}\right| \exp\left\{-\sum_{i=N}^{n} K\right\}$$

so that

$$\prod_{n=1}^{\infty}\left[\frac{f(z)+d_n}{f(1)+d_n}\right] = 0.$$

THEOREM 3.4. *Suppose that*

$$\lim_{n\to\infty} \rho_n = \rho, \qquad \lim_{n\to\infty} \theta_n = 0, \qquad \rho \neq -f(1),$$

and that z *is given such that* $|f(z)+\rho| > |f(1)+\rho|$. *Then*

$$\lim_{n\to\infty} \sigma_n(z) = \infty.$$

Proof. Since $|f(z)+\rho| < |f(1)+\rho|$, there exists an α such that $0 < \alpha < 8$ and

$$(3.14)\qquad |f(z)+\rho|^2 > |f(1)+\rho|^2\,(1+2\alpha).$$

Let $H = u^2+v^2-a^2-b^2$. It follows from (3.14) that

$$H \geqslant 2\rho(u-a) + 2\alpha\,|f(1)+\rho|^2$$

so that

$$(3.15)\qquad |f(z)+d_n|^2 - |f(1)+d_n|^2 \geqslant 2(u-a)(\rho_n\cos\theta_n - \rho) + 2\rho_n(v-b)\sin\theta_n + 2\alpha\,|f(1)+\rho|^2.$$

By hypothesis there exists an integer $N > 0$ such that for $n > N$

$$(3.16)\qquad |2(u-a)(\rho_n\cos\theta_n - \rho) + 2(v-b)\rho_n\sin\theta_n| < \alpha\,|f(1)+\rho|^2.$$

The relations (3.15) and (3.16) imply that

$$(3.17)\qquad |f(z)+d_n|^2 - |f(1)+d_n|^2 \geqslant \alpha\,|f(1)+\rho|^2.$$

Since $0 < \alpha < 8$, it follows from (3.17) that

$$\lim_{n\to\infty}\left|\frac{f(z)+d_n}{f(1)+d_n}\right| \geqslant 1+\alpha/4 > 1$$

so that we have

$$(3.18)\qquad \prod_{n=1}^{\infty}\left|\frac{f(z)+d_n}{f(1)+d_n}\right| = \infty.$$

It follows from (3.18) that

$$\lim_{n\to\infty}\sigma_n(z) = \infty$$

and the theorem is proved.

https://doi.org/10.4153/CJM-1965-051-1 Published online by Cambridge University Press

THEOREM 3.5. *If*

$$\lim_{n\to\infty} \rho_n = 0,$$

then the (f, d_n)*-method sums the geometric series* (3.1) *to* $(1 - z)^{-1}$ *for all* z *such that* $|f(z)| < |f(1)|$.

Proof. If $f(1) = 0$, the result follows immediately since there is no value of z for which $|f(z)| < |f(1)|$. So for the remainder of the proof we may suppose that $f(1) \neq 0$.

Since $1 + x \leqslant e^x$ for real x, we get

$$\left|\frac{f(z) + d_n}{f(1) + d_n}\right| \leqslant \exp\left\{\frac{H + 2\rho_n[(u - a)\cos\theta_n + (v - b)\sin\theta_n]}{2|f(1) + d_n|^2}\right\},$$

where $H = u^2 + v^2 - a^2 - b^2$. From the hypothesis we obtain

$$\lim_{n\to\infty} \frac{H + 2\rho_n[(u - a)\cos\theta_n + (v - b)\sin\theta_n]}{2|f(1) + d_n|^2} = \frac{H}{2|f(1)|^2}.$$

Hence there exist a $K > 0$ and an integer $N > 0$ such that for all $n > N$

$$-K > \frac{H + 2\rho_n[(u - a)\cos\theta_n + (v - b)\sin\theta_n]}{2|f(1) + d_n|^2}. \tag{3.19}$$

From (3.19) it follows that

$$\prod_{i=1}^{n} \left|\frac{f(z) + d_i}{f(1) + d_i}\right| \leqslant \prod_{i=1}^{N-1} \left|\frac{f(z) + d_i}{f(1) + d_i}\right| \exp\left\{-\sum_{i=N}^{n} K\right\},$$

which implies the theorem.

THEOREM 3.6. *Suppose that*

$$\lim_{n\to\infty} \rho_n = 0$$

and that z *is given such that* $|f(z)| > |f(1)|$. *Then*

$$\lim_{n\to\infty} \sigma_n(z) = \infty.$$

Proof. From the hypothesis it follows that

$$\lim_{n\to\infty} \{|f(z) + d_n|^2 - |f(1) + d_n|^2\} = |f(z)|^2 - |f(1)|^2 > 0.$$

Hence there exist an $\alpha > 0$ and an integer $N > 0$ such that

$$\left|\frac{f(z) + d_n}{f(1) + d_n}\right| > 1 + \alpha$$

for $n > N$. Therefore

$$\prod_{n=1}^{\infty} \left|\frac{f(z) + d_n}{f(1) + d_n}\right| = \infty,$$

from which the theorem follows.

In this section we have determined certain domains in the complex plane for which the (f, d_n)-method of summability sums the geometric series (3.1) to its analytic continuation $(1 - z)^{-1}$. There are several known results **(5)** which give information concerning the efficiency of a linear method of summability for summing a power series with positive radius of convergence to its analytic continuation. By using results of the type found in **(5)** and the theorems of this section one can determine a domain for which the (f, d_n)-method sums a power series with positive radius of convergence to its analytic continuation.

4. Special cases.

A. Let $f(z) = e^{u(z-1)}$, where u is real and $u \neq 0$, and let $d_n = n - 1$ for $n \geqslant 1$.

THEOREM 4.1. *The* $(e^{u[z-1]}, n - 1)$*-method is regular if and only if* $u > 0$.

Proof. If $u > 0$, then the $(e^{u[z-1]}, n - 1)$-method is regular by Theorem 2.3. Now suppose that $u < 0$. By substituting $e^{u(z-1)}$ for $f(z)$ and $n - 1$ for d_n in (1.1) and then letting $z = 0$, we obtain

$$a_{n0} = \prod_{k=1}^{n} \left[\frac{e^{-u} + k - 1}{k}\right].$$

Since $u < 0$, we have $a_{n0} > 1$ for each n. Hence the regularity condition (2.2) is not satisfied. Therefore if the $(e^{u[z-1]}, n - 1)$-method is regular, then $u > 0$.

Since the hypotheses of Theorem 3.1 are satisfied, it follows that the $(e^{u[z-1]}, n - 1)$-method sums the geometric series

$$\sum_{n=0}^{\infty} z^n \tag{4.1}$$

to $(1 - z)^{-1}$ for all z such that

$$\operatorname{Re}\{e^{u(z-1)}\} < 1;$$

that is, for all values of z which satisfy

$$e^{u(x-1)} \cos uy < 1. \tag{4.2}$$

The domain in which the $(e^{u[z-1]}, n - 1)$-method sums the geometric series is indicated in Figure 1.

THEOREM 4.2. *The* $(e^{u[z-1]}, n - 1)$*-method of summability provides a method of analytic continuation of the geometric series* (4.1) *to* $(1 - z)^{-1}$ *for all* $z \neq 1$.

Proof. The proof consists of showing that given any $z \neq 1$, a u can be chosen so that the $(e^{u[z-1]}, n - 1)$-sum of the geometric series (4.1) is $(1-z)^{-1}$. Assume that z is given.

Case 1. $y \neq 0$, x arbitrary. Choose u so that

$$\pi/2\,|y| < u < 3\pi/2\,|y|.$$

Hence $\pi/2 < |uy| < 3\pi/2$ and therefore $e^{u(x-1)} \cos uy < 1$.

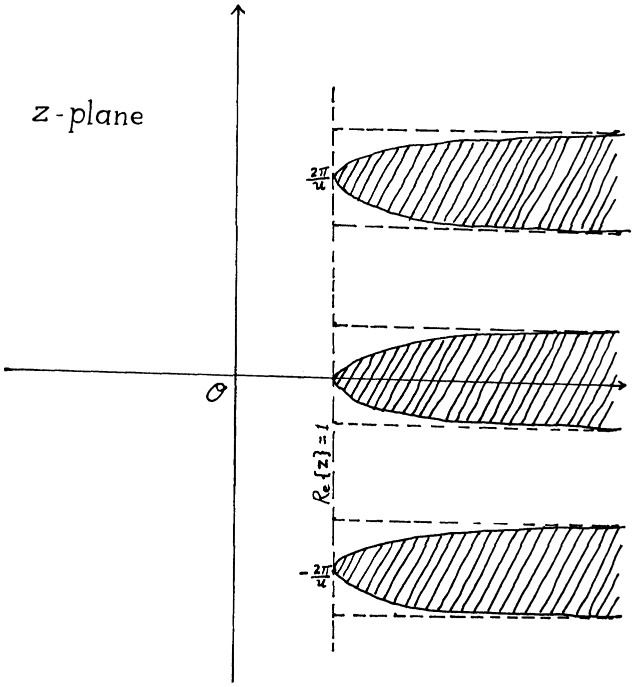

FIGURE 1
(The unshaded part of the plane is the domain of summability.)

Case 2. $y = 0, x < 1$. Choose any positive u. Then $e^{u(x-1)} \cos uy < 1$.

Case 3. $y = 0, x > 1$. Choose any negative u. Then $e^{u(x-1)} \cos uy < 1$.

B. Let $f(z) = e^{u(z-1)}$, where u is real and $u \neq 0$, and let $d_n = q \geqslant 0$ for all n.

THEOREM 4.3. *The* $(e^{u[z-1]}, q)$*-method is regular if and only if* $u > 0$.

The proof of Theorem 4.3 is analogous to that of Theorem 4.1. It follows from Theorem 3.3 that the $(e^{u[z-1]}, q)$-method sums the geometric series (4.1) to $(1 - z)^{-1}$ for all z such that

$$|e^{u(z-1)} + q| < 1 + q.$$

THEOREM 4.4. *If* $u > 0$, *the domain for which the* $(e^{u[z-1]}, q)$*-method sums the geometric series* (4.1) *to* $(1 - z)^{-1}$ *contains the half-plane* $\text{Re}\{z\} < 1$ *and is contained in the domain defined by* $\text{Re}\{e^{u(z-1)}\} < 1$.

Proof. Suppose that $\text{Re}\{z\} = x < 1$. Then

$$|e^{u(z-1)} + q| \leqslant e^{u(x-1)} + q < 1 + q,$$

which implies that the $(e^{u[z-1]}, q)$-sum of the geometric series (4.1) is $(1 - z)^{-1}$.

Suppose that $\operatorname{Re}\{z\} = x \geqslant 1$ and $e^{u(x-1)} \cos uy \geqslant 1$. Then the point $z = x + iy$ is not in the domain of summability since

$$e^{2u(x-1)} + 2qe^{u(x-1)} \cos uy \geqslant 1 + 2q. \tag{4.3}$$

The relation (4.3) implies that

$$|e^{u(z-1)} + q|^2 \geqslant |1 + q|^2$$

so that we obtain

$$|e^{u(z-1)} + q| \geqslant 1 + q. \tag{4.4}$$

If $x > 1$, then each of the relations (4.3) and (4.4) can be replaced by strict inequality and the desired result follows from Theorem 3.4. If $x = 1$, then $\cos uy = 1$ so that neither the $(e^{u[z-1]}, q)$-method nor the $(e^{u[z-1]}, n - 1)$-method sums the geometric series (4.1) to $(1 - z)^{-1}$ for such a z.

THEOREM 4.5. *If $p > q$, the domain of summability in which the $(e^{u[z-1]}, p)$-method sums the geometric series* (4.1) *includes the corresponding domain of summability of the $(e^{u[z-1]}, q)$-method.*

Proof. When $x < 1$, both methods under consideration sum the geometric series (4.1) to $(1 - z)^{-1}$. Neither method sums the geometric series to $(1 - z)^{-1}$ for a value of z for which $e^{u(x-1)} \cos uy \geqslant 1$. So assume that $e^{u(x-1)} \cos uy < 1$. The domains of summability corresponding to p and q are defined by the inequalities

$$e^{2u(x-1)} + 2p[e^{u(x-1)} \cos uy - 1] - 1 < 0 \tag{4.5}$$

and

$$e^{2u(x-1)} + 2q[e^{u(x-1)} \cos uy - 1] - 1 < 0 \tag{4.6}$$

respectively. But since $p >$ q and $e^{u(x-1)} \cos uy - 1 < 0$,

$$2q[e^{u(x-1)} \cos uy] > 2p[e^{u(x-1)} \cos uy].$$

Therefore if z is a point such that (4.6) is satisfied, then (4.5) is satisfied. Hence the theorem follows.

C. Let $f(z) = az^m$, $a > 0$, m a positive integer.

THEOREM 4.6. *Let α be given such that $0 < \alpha < \pi/2$. Suppose there exist an $\epsilon > 0$ and an integer $N > 0$ such that $\theta_n > \alpha$ and $\rho_n > \epsilon$ for all $n > N$. Then the (az^m, d_n)-method is not regular.*

Proof. Suppose that the (az^m, d_n)-method is regular. Define $\lambda_n = \rho_n \exp(i\beta_n)$, where $\beta_n = \theta_n - \alpha$ and define b_{nk} by the relations

$$\begin{gathered} b_{00} = 1, \\ b_{0k} = 0 \qquad (k \neq 0), \\ \prod_{i=1}^{n} \left| \frac{az^m + \lambda_i}{a + \lambda_i} \right| = \sum_{k=0}^{\infty} b_{nk} z^k \qquad (n \geqslant 1). \end{gathered} \tag{4.7}$$

https://doi.org/10.4153/CJM-1965-051-1 Published online by Cambridge University Press

The elements a_{nk} of (1.1) can be written in the form

$$a_{nk} = \frac{1}{2\pi i}\int_C \prod_{i=1}^{n} \left|\frac{at^m + d_i}{a + d_i}\right| \frac{dt}{t^{k+1}},$$

where $f(z)$ is replaced by az^m and C is any circle with the origin as centre. Integrating we get

$$a_{nk} = \frac{a^{k/m}}{\prod_{i=1}^{n}(a + d_i)}\left[\sum d_1^{s_1} d_2^{s_2} \ldots d_n^{s_n}\right], \tag{4.8}$$

where $s_i = 0$ or 1 and the sum is taken over all s_i's for which

$$s_1 + \ldots + s_n = (mn - k)/m.$$

Similarly,

$$b_{nk} = \frac{a^{k/m}}{\prod_{i=1}^{n}(a + \lambda_i)}\left[\sum d_1^{s_1} d_2^{s_2} \ldots d_n^{s_n}\right] e^{-i\alpha(mn-k)/m}.$$

This implies that

$$a^{k/m}\left|\sum d_1^{s_1} \ldots d_n^{s_n}\right| = |b_{nk}| \prod_{i=1}^{n} |a + \lambda_i|. \tag{4.9}$$

Hence by (4.9) and (4.8) it follows that

$$\sum_{k=0}^{mn} |a_{nk}| = \prod_{i=1}^{n} \left|\frac{a + \lambda_i}{a + d_i}\right| \sum_{k=0}^{mn} |b_{nk}|$$

so that

$$\sum_{k=0}^{mn} |a_{nk}| \geqslant \prod_{i=1}^{n} \left|\frac{a + \lambda_i}{a + d_i}\right| \tag{4.10}$$

since

$$\sum_{k=0}^{mn} |b_{nk}| \geqslant \left|\sum_{k=0}^{mn} b_{nk}\right| = 1.$$

The assumption that (az^m, d_n) is regular and relation (4.10) imply that

$$\prod_{i=1}^{n} \left|\frac{a + \lambda_i}{a + d_i}\right|$$

is a bounded function of n. By hypothesis and since $-\pi \leqslant \theta_n \leqslant \pi$, there exists a positive integer N such that $\alpha < \theta_n \leqslant \pi$ for all $n \geqslant N$, which implies that $\theta_n > \beta_n > 0$. Hence

$$|a + \lambda_n|^2 > |a + d_n|^2$$

so that

$$\prod_{i=1}^{n} \left| \frac{a + \lambda_i}{a + d_i} \right|$$

is a monotone function of n for all $n \geqslant N$. Hence

$$\prod_{i=1}^{n} \left| \frac{a + \lambda_i}{a + d_i} \right|$$

is bounded if and only if

$$\prod_{i=1}^{\infty} \left| \frac{a + \lambda_i}{a + d_i} \right|^2 \tag{4.11}$$

converges. But (4.11) converges if and only if

$$\sum_{i=1}^{\infty} \left[-1 + \left| \frac{a + \lambda_i}{a + d_i} \right|^2 \right]$$

is convergent. If $n \geqslant N$, then $\alpha < \beta_n + \theta_n < 2\pi - \alpha$ and so

$$\sin[(\beta_n + \theta_n)/2] > \sin(\alpha/2).$$

It follows that

$$-1 + \left| \frac{a + \lambda_n}{a + d_n} \right|^2 = \frac{2a\rho_n (\cos \beta_n - \cos \theta_n)}{a^2 + 2a\rho_n \cos \theta_n + \rho_n^2} \tag{4.12}$$

$$\geqslant \frac{4a\rho_n \sin^2(\alpha/2)}{(a + \rho_n)^2} \geqslant 4 \sin^2(\alpha/2) \frac{\rho_n / a}{(1 + \rho_n/a)^2}.$$

By supposition (az^m, d_n) is regular, which implies that

$$\sum_{n=1}^{\infty} \rho_n^{-1} = \infty.$$

Since, by hypothesis, ρ_n is bounded away from zero, it follows that

$$\sum_{n=1}^{\infty} \frac{\rho_n/a}{(1 + \rho_n/a)^2} = \infty. \tag{4.13}$$

It now follows from (4.12) and (4.13) that

$$\sum_{i=1}^{\infty} \left[-1 + \left| \frac{a + \lambda_i}{a + d_i} \right|^2 \right] = \infty.$$

Hence by (4.10) and (4.11) we find that

$$\sum_{k=0}^{mn} |a_{nk}|$$

is not uniformly bounded for all n, which proves the theorem.

The following example shows that in Theorem 4.6 the restriction that ρ_n be bounded away from zero cannot be removed.

EXAMLPE 4.1. *Let*

$$d_n = n^{-2} e^{i\pi/4} \tag{4.14}$$

for all positive integers n and let $f(z) = z$. *Hence*

$$\sum_{n=1}^{\infty} \frac{1}{|1 + d_n|} \geqslant \sum_{n=1}^{\infty} \frac{1}{1 + n^{-2}} = \infty$$

and

$$\sum_{n=1}^{\infty} \rho_n \theta_n^2 = (\pi^2/16) \sum_{n=1}^{\infty} n^{-2} = \pi^4/96.$$

Thus it follows from Corollary 2.8 *that the corresponding* (z, d_n)*-method is regular.*

Example 4.1 furnishes us with a counterexample to two statements made by Cowling and Miracle (**3**, Theorems 2.2 and 2.4). By replacing π by $-\pi$ in (4.14), we get a counterexample to (**3**, Theorem 2.3).

Even if ρ_n is bounded away from zero, the (az^m, d_n)-method may be regular when

$$\lim_{n\to\infty} \theta_n \neq 0$$

as the following example shows.

EXAMPLE 4.2. *Let* $f(z) = z$ *and let*

$$d_n = \begin{cases} n \text{ if } n \text{ is not the square of a positive integer,} \\ n \cdot \exp\{(-1)^{n_i}\} \text{ if } n \text{ is the square of a positive integer.} \end{cases}$$

Hence

$$\sum_{n=1}^{\infty} \frac{1}{|f(1) + d_n|} \geqslant \sum_{n=1}^{\infty} \frac{1}{1 + n} = \infty$$

and

$$\sum_{n=1}^{\infty} \frac{\theta_n^2}{\rho_n}$$

converges. It follows from Corollary 2.7 *that the corresponding* (z, d_n)*-method is regular.*

Example 4.2 answers the open problem in (**3**, p. 424): to find a sequence $\{d_n\}$ of type 2 such that

$$\lim_{n\to\infty} \arg (d_n) \neq 0$$

and such that the (z, d_n)-matrix is regular; or to show that no such sequence exists.

THEOREM 4.7. *Let* α *be given such that* $-\pi/2 < \alpha < 0$. *Suppose there exist an* $\epsilon > 0$ *and an integer* $N > 0$ *such that* $\theta_n < \alpha$ *and* $\rho_n > \epsilon$ *for all* $n > N$. *Then the* (az^m, d_n)*-method is not regular.*

THEOREM 4.8. *Suppose that* $\theta_n = \alpha$ *for all* n *and that there exist an* $\epsilon > 0$ *and an integer* $N > 0$ *such that* $\rho_n > \epsilon$ *for all* $n > N$. *Then if the* (az^m, d_n)-*method is regular,* $\alpha = 0$.

D. Let $f(z) = z^m$, where m is a positive integer, and let $d_n = n - 1$ for all positive integers n.

PROPERTY 4.1. *The* (z^m, n)-*method is regular for each* m.

PROPERTY 4.2. *The* (z^m, n)-*method sums the geometric series to* $(1 - z)^{-1}$ *for all* z *which satisfy* $\mathrm{Re}\{z^m\} < 1$.

We notice that the (z^m, n)-method of summability sums the geometric series to $(1 - z)^{-1}$ in the generalized Borel polygon. When $m = 1$, we get the Lototsky method **(2)**.

REFERENCES

1. R. P. Agnew, *Euler transformations*, Amer. J. Math., *66* (1944), 318–338.
2. ——— *The Lototsky method for the evaluation of series*, Michigan Math. J., *4* (1957), 105–128.
3. V. F. Cowling and C. L. Miracle, *Some results for the generalized Lototsky transform*, Can. J. Math., *14* (1962), 418–435.
4. A. Jakimovski, *A generalization of the Lototsky method*, Michigan Math. J., *6* (1959), 270–290.
5. Y. Okada, *Ueber die Annäherung analytischer Funktionen*, Math. Z., *23* (1925), 62–71.
6. V. Vuckovic, *The mutual inclusion of Karamata–Stirling methods of summation*, Michigan Math. J., *6* (1959), 291–297.

Hattiesburg, Mississippi