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ON THE CLASSIFICATION OF BIORTHOGONAL SEQUENCES

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The work of various authors (e.g. Frink [**3**] and Markushevitch [**7**]) suggests the possibility of studying complete biorthogonal sequences in Banach spaces as a generalization of orthogonal families of continuous functions. But except for the case where the complete biorthogonal sequence is a Schauder basis such studies have not led to a very rich theory. The main reason for this is that an arbitrary complete biorthogonal sequence is not likely to have many helpful properties. For instance, in every separable Banach space X one can find a complete biorthogonal sequence $\{e_i, E_i\}$ which is not one-summable. (See Definition 1.1 (1) and the second paragraph of Section 5.) This means there is $x \in X$ such that x is not even in the closed linear span of

$$\left\{ \sum_{i=1}^{n} E_i(x) e_i : n = 1, 2, \ldots \right\}.$$

A second reason is that even rather mild conditions on a complete biorthogonal sequence results in the space having the approximation property or the metric approximation property.

This paper continues the work begun in [**8**] on the series summability of biorthogonal sequences. In Sections 1 and 2, eight formally distinct properties of a complete biorthogonal sequence are defined and equivalent characterizations given. All eight of these properties are independent of the order of the biorthogonal sequence, and they are all preserved by subsequences (Section 3). In Section 4 there are some criteria for the existence of various types of sequences.

1. Classification of biorthogonal sequences according to series summability. Throughout this paper $[e_i, E_i]$ will denote a complete biorthogonal sequence in a Banach space X. That is, $[e_i]$ the linear span of $\{e_i\}$ is dense in X, $\{E_i\}$ is total on X ($E_i(x) = 0$ for each i only when $x = 0$), and $E_i(e_j) = \delta_{ij}$. The symbol $E_i \otimes e_j$ denotes the one-dimensional linear mapping from X into X whose value at $x \in X$ is $E_i(x)e_j$. For A, a linearly independent subset of X, $\kappa(A)$ will denote the set of all non-negative finite linear combinations of vectors in A, i.e., the cone determined by A.

1.1 *Definitions.* Let $\{e_i, E_i\}$ be a complete biorthogonal sequence in a Banach space X. We shall say that $\{e_i, E_i\}$ is

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(1) *K-series summable* (for K a positive integer) if for each K-element subset of X the identity mapping from X into X lies in the closure of $[\{E_i \otimes e_i\}]$ with respect to the non-Hausdorff) topology of operators given by pointwise convergence on that set.

(2) *Finitely series summable* if the identity operators from X into X lies in the strong operator closure of $[\{E_i \otimes e_i\}]$.

(3) *Series summable* (cf. [**8**, Theorem 6.4]) if X has the approximation property and the identity mapping from X into X is in the w^*-closure of $[\{E_i \otimes e_i\}]$ in $L(X)$ (considered as a subspace of the dual space of $N(X)$ the nuclear mappings from X into X).

(4) *Strongly series summable* (cf. [**8**, Theorem 7.2]) if the identity mapping from X into X is the limit of a sequence in $[\{E_i \otimes e_i\}]$ with respect to the strong operator topology.

1.2 THEOREM. *Let $\{e_i, E_i\}$ be a complete biorthogonal sequence in a Banach space X.*

A. *The following statements are equivalent:*

(1) *$\{e_i, E_i\}$ is one-series summable.*

(2) *If J is any set of indices then*

$$\overline{[\{e_j : j \in J\}]} = \{x \in X : E_j(x) = 0, j \notin J\}.$$

(3) *For x in X and x' in X^*, $E_i(x)x'(e_i) = 0$ for each i implies $x'(x) = 0$.*

(4) *For each one-dimensional continuous linear mapping T from X into X, $E_i(Te_i) = 0$ for each i implies T has zero trace.*

(5) *For each x' in X^* there is a BK-space $S_{x'}$ and a continuous linear functional $E_{x'}$ on $S_{x'}$ such that $(x'(e_i)E_i(x)) \in S_{x'}$ and*

$$E_{x'}((x'(e_i)E_i(x)) = x'(x)$$

for each x in X.

(6) *For each x in X there is a sequence $\{T_n^{(x)}\}$ of finite-dimensional linear mappings which are diagonal with respect to $\{e_i, E_i\}$*

$$(i.e., T_n^{(x)}(y) = \textstyle\sum_{j=1}^{k_n} a_j^{(n)} E_j(y) e_j)$$

such that $\lim_n T_n^{(x)}x = x$.

B. *The following statements are equivalent for K a positive integer:*

(1) *$\{e_i, E_i\}$ is K-series summable.*

(2) *For $\{x_1, \ldots, x_M\} \subset X$ and $\{x_1', \ldots, x_M'\} \subset X^*$ with $M \leqq K$,*

$$\textstyle\sum_{n=1}^{M} x_n'(e_k)E_k(x_n) = 0$$

for $k = 1, 2, \ldots$ implies $\sum_{n=1}^{M} x_n'(x_n) = 0$.

(3) *For each finite-dimensional continuous linear mapping T from X into X of rank $\leqq K$, $E_i(Te_i) = 0$ for each i implies T has zero trace.*

(4) *For $\{x_1', \ldots, x_M'\} \subset X^*$ with $M \leqq K$, there is a BK-space S and a continuous linear functional E on S such that $(x_n'(e_i)E_i(x)) \in S$ for $n = 1, 2, \ldots, M$*

and each $x \in X$ *and*

$$E(x_n'(e_i)E_i(x)) = x_n'(x), \qquad n = 1, 2, \ldots M.$$

(5) *For each K-dimensional subspace F of X there is a sequence of finite dimensional diagonal linear mappings* $\{T_n^{(F)}\}$ *such that*

$$\lim_n T_n^{(F)} y = y$$

for y in F.

C. *The following statements are equivalent:*

(1) $\{e_i, E_i\}$ *is finitely series summable.*

(2) *For* $\{x_1, \ldots, x_M\} \subset X$ *and* $\{x_1', \ldots, X_M'\} \subset X^*$,

$$\sum_{n=1}^{M} x_n'(e_k)E_k(x_n) = 0$$

for $k = 1, 2, \ldots$ *implies* $\sum_{n=1}^{M} x_n'(x_n) = 0$.

(3) *For each finite dimensional continuous linear mapping T from X into X,* $E_i(Te_i) = 0$ *for each i implies T has trace zero.*

(4) *For every finite subset* $\{x_1', \ldots, x_M'\}$ *of* X^* *there is a BK-space S and a continuous linear functional E on S such that* $(x_n'(e_i)E_i(x)) \in S$ *for* $n = 1, 2, \ldots, M$ *and each* $x \in X$ *and*

$$E(x_n'(e_i)E_i(x)) = x_n'(x), \qquad n = 1, 2, \ldots, M.$$

(5) *The identity operator from X into X lies in the weak operators closure of* $[\{E_i \otimes e_i\}]$.

(6) *For any finite-dimensional subspace F of X there is a sequence* $\{T_n^{(F)}\}$ *of finite-dimensional diagonal linear mappings such that*

$$\lim_n T_n^{(F)} y = y$$

for $y \in F$.

D. *The following statements are equivalent:*

(1) $\{e_i, E_i\}$ *is series summable.*

(2) *X has the approximations property, and if T is a nuclear mapping from X into X for which* $E_i(Te_i) = 0$ *for each i, the trace of T is zero.*

(3) *X has the approximation property, and for each nuclear mapping T and each* $\epsilon > 0$ *there are numbers* $a_1, a_2, \ldots, a_n$ *such that*

$$\left| \mathrm{tr}(T) - \sum_{i=1}^{n} a_i E_i(Te_i) \right| < \epsilon.$$

(4) *There is a BK-space S containing all sequences of the form* $(x'(e_i)E_i(x))$ *with x in X and x′ in* X^* *and a continuous linear functional E on S for which*

$$E(x'(e_i)E_i(x)) = x'(x).$$

E. *The following statements are equivalent.*

(1) *$\{e_i, E_i\}$ is strongly series summable.*

(2) *The identity mapping from X into X is the limit of a sequence in $[\{E_i \otimes e_i\}]$ with respect to the weak operator topology.*

(3) *X has the approximation property and the identity mapping from X into X is the limit of a sequence in $[\{E_i \otimes e_i\}]$ in the w^*-topology on $L(X)$.*

(4) *X has the approximation property, and there is a row finite matrix $(a_{n,k})$ such that for each nuclear mapping T from X into X*

$$\operatorname{tr}(T) = \lim_n \sum_k a_{nk}E_k(Te_k).$$

(5) *There is a row finite matrix (a_{nk}) such that for each $x \in X$ and $x' \in X^*$*

$$x'(x) = \lim_n \sum_k a_{nk}E_k(x)x'(e_k).$$

(6) *There is a row finite matrix (a_{nk}) such that for each $x \in X$*

$$x = \lim_n \sum_k a_{nk}E_k(x)e_k.$$

Proof of A. (1) $\Leftrightarrow$ (6) follows since $L(X)$ with the topology of convergence on the set $\{x\}$ is first countable.

(6) $\Rightarrow$ (3). Given x in X and x' in X^* such that $E_i(x)x'(e_i) = 0$ for each i, let $\{T_n^{(x)}\}$ be a sequence of finite dimensional diagonal mappings for which $\lim_n T_n^{(x)}x = x$. Then $x'(x) = \lim_n x'(T_n^{(x)}x)$, but since $T_n^{(x)}$ is diagonal, $x'(T_n^{(x)}x) = 0$ for each n so that $x'(x) = 0$.

(3) $\Leftrightarrow$ (4) is clear.

(3) $\Rightarrow$ (2). It is not hard to see that the set on the left hand side of (2) is always contained in that on the right hand side.

Suppose there were a point x in X such that $E_j(x) = 0$ for each $j \notin J$ but such that $x \notin \overline{[\{e_j:j \in J\}]}$. There is then x' in X' such that $x'(e_j) = 0$ for $j \in J$ but $x'(x) = 1$. We should then have $E_i(X)x'(e_i) = 0$ for each i so that by (3), $x'(x) = 0$, a contradiction.

(2) $\Rightarrow$ (3). For $x \in X$ and $x' \in X'$ let $J = \{j:E_j(x) \neq 0\}$. Then $x \in \overline{[\{e_j:j \in J\}]}$, but by the hypothesis of (3) $x'(e_j) = 0$ for each $j \in I$, Hence $x'(x) = 0$.

(3) $\Rightarrow$ (5). For $x' \in X'$, let $S_{x'}$ consist of all sequences $(x'(e_i)E_i(x))$ as x ranges over X. Then $S_{x'}$ is a BK-space with norm

$$||(a_i)|| = \sup \{||x||:x'(e_i)E_i(x) = a_i \text{ for each } i\}.$$

Define $E_{x'}$ on $S_{x'}$ by $E_{x'}(x'(e_i)E_i(x)) = x'(x)$. By (3), $E_{x'}$ is well-defined; it is obviously continuous and linear.

(5) $\Rightarrow$ (3). If $x \in X$ and $x' \in X'$ are such that $E_i(x)x'(e_i) = 0$ for each i then $x'(x) = E_{x'}(0) = 0$.

(2) $\Rightarrow$ (6). For a given vector $x \in X$ let $J = \{j:E_j(x) \neq 0\}$. Then $x \in \overline{[\{e_j:j \in J\}]}$ so there is a row finite matrix (b_{nk}) such that $b_{nk} = 0$ if $k \notin J$

and

$$\lim_n \sum_k b_{nk} e_k = x.$$

Define a_{nk} to be $b_{nk}/E_k(x)$ if $E_k(x) \neq 0$ and 0 otherwise and define

$$T_n^{(x)}(y) = \sum_k a_{nk} E_k(y) e_k.$$

Then $\lim_n T_n^{(x)}(x) = x$.

We omit the proof of B which is like that of C.

Proof of C. (2) $\Leftrightarrow$ (3) is obvious.

(2) $\Rightarrow$ (4) Let S consist of all sequences of the form $(\sum_{n=1}^{M} x_n'(e_i) E_i^{(x)})_{i=1}^{\infty}$ as x ranges over X. Then S is a BK-space with the norm

$$||t|| = \inf \left\{ ||x|| : \sum_{n=1}^{M} x_n'(e_i) E_i(x) = t_i \text{ for each } i \right\}.$$

Define E on S by

$$E(t) = \sum_{n=1}^{M} x_n'(x), \qquad \sum_{n=1}^{M} x_n'(e_i) E_i(x) = t_i \text{ for each } i.$$

Then E is well defined by (2) and obviously continuous and linear. Moreover, for each $x \in X$

$$E(x_n'(e_i) E_i(x)) = x_n'(x) \quad \text{for } n = 1, 2, \dots, M.$$

(4) $\Rightarrow$ (2). For each $n = 1, 2, \dots, M$, let $s^{(n)} = (x_n'(e_k) E_k(x_n))_{k=1}^{\infty}$; then each $s^{(n)} \in S$ and by (3), $\sum_{n=1}^{M} s^{(n)} = 0$. Thus $E(\sum_{n=1}^{M} s^{(n)}) = \sum_{n=1}^{M} E(s^{(n)}) = \sum_{n=1}^{M} x_n'(x_n) = 0$.

(1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (5). This follows since the topological conjugate space of $L(X)$ with the strong and weak operator topologies is represented by the space of finite dimensional linear mappings from X into X by means of the bilinear form

$$\left\langle T, \sum_{n=1}^{M} x_n' \otimes x_n \right\rangle = \sum_{n=1}^{M} x_n'(Tx_n).$$

See Theorem VI. 1.4 of [**2**] and its proof.

(1) $\Leftrightarrow$ (6). This follows from the definition of strong operator topology.

Proof of D. (1) $\Leftrightarrow$ (2) If X has the approximation property then $L(X)$ is isometric to a subspace of the conjugate space of $N(X)$ the space of nuclear mappings in X.

(2) $\Leftrightarrow$ (3). This follows by definitions of w^*-closure

(1) $\Rightarrow$ (4). Let $S = \{(E_i(Te_i)) : T \text{ is a nuclear mapping from } X \text{ into } X\}$, and define E on S by

$$E(E_i(Te_i)) = \text{tr}(T).$$

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Then S and E have the required properties.

(4) $\Rightarrow$ (1). For each x' in X' define $T_{x'}$ from X into S by

$$T_{x'}x = x'(e_i)E_i(x).$$

Then since S is a BK-space, each $T_{x'}$ is continuous by the Closed Graph Theorem. The correspondence $x' \to T_{x'}$ is also continuous by the Closed Graph Theorem so there is $M > 0$ such that

$$(*) \quad ||T_{x'}x|| \leqq M||x||\,||x'||, \qquad x \in X, x' \in X^*.$$

Define F from $N(X)$ into S by

$$F(T) = \sum_k T_{y_k'}(y_k) = \left(\sum_k y_k'(e_i)E_i(y_k)\right)_i$$

when $T = \sum_k y_k'(x)y_k$ and $\sum_k ||y_k'||\,||y_k|| < \infty$.
The series converges for each T in $N(X)$ because of (*). Moreover, F is well defined because if

$$\sum_k y_k'(x)y_k = \sum_k z_k'(x)z_k$$

for each x in X we have

$$\sum_k y_k'(e_i)E_i(y_k) = \sum_k z_k'(e_i)E_i(z_k)$$

for each i. The mapping F is obviously linear; it is continuous because

$$||F(T)|| \leqq M||T||_N$$

by (*).

We can now define a continuous linear functional on $N(X)$ by $E(F(T))$. This functional coincides with the trace on finite dimensional operators so X has the approximation property by [**4**, Proposition 35]. Furthermore, if $E_i(Te_i) = 0$ for each i, $F(T) = 0$ so tr $(T) = E(F(T)) = 0$.

Most of the implications in E follow from [**8**, Theorem 7.2], and we omit its proof.

2. Classification of biorthogonal sequences according to positivity.

2.1 *Definitions.* Let $\{e_i, E_i\}$ be a complete biorthogonal sequence in a Banach space X. We shall say that $\{e_i, E_i\}$ is:

(1) *K-positive* (for K a positive integer) is for each K-element subset of X the identity mapping from X into X lies in the closure of $\kappa\{E_i \otimes e_i\}$ with respect to the (non-Hausdorff) topology of operators given by pointwise convergence on that set.

(2) *Finitely positive* if the identity operator from X into X lies in the strong operator closure of $\kappa\{E_i \otimes e_i\}$.

(3) *Positive* if X has the approximation property and the identity mapping from X into X in the w^* closure of $\kappa\{E_i \otimes e_i\}$ in $L(X)$.

(4) *Strongly positive* if the identity mapping from X into X is the limit of a sequence in $\kappa\{E_i \otimes e_i\}$ with respect to the strong operator topology.

The following theorem is analogous to Theorem 1.2, and we omit its proof. A positive continuous linear functional f on a BK-space S is one for which $f(a_i) \geqq 0$ whenever $a_i \geqq 0$ for each i.

2.2 THEOREM. *Let $\{e_i, E_i\}$ be a complete biothogonal sequence in a Banach space X.*

A. *The following statements are equivalent for K a positive integer:*

(1) *$\{e_i, E_i\}$ is K-positive.*

(2) *For $\{x_1, \ldots, x_M\} \subset X_n$ and $\{x_1', \ldots, x_M'\} \subset X^*$ with $M \leqq K$, $\sum_{n=1}^M x_n'(e_k)E(x_n) \geqq 0$ for $k = 1, 2, \ldots$ implies $\sum_{n=1}^M x_n'(x_n) \geqq 0$.*

(3) *For each finite dimensional continuous linear mapping T from X into X of* rank $\leqq K$, *$E_i(Te_i) \geqq 0$ for each i implies* $\mathrm{tr}(T) \geqq 0$.

(4) *For $\{x_1', \ldots, x_M'\} \subset X^*$ with $M \leqq K$, there is a BK-space S and a positive continuous linear functional E on S such that $(x_n'(e_i)E_i(x)) \in S$ for $n = 1, 2, \ldots, M$ and each $x \in X$ and*

$$E(x_n'(e_i)E_i(x)) = x_n'(x).$$

(5) *For each K-dimensional subspace F of X there is a sequence of finite-dimensional positive diagonal linear mappings*

$$\left\{T_n^{(F)} = \sum_{j=1}^{k_n} a_jE_j \otimes e_j, a_j \geqq 0 \text{ for each } j\right\}$$

such that

$$\lim_n T_n^{(F)}y = y$$

for y in F.

B. *The following statements are equivalent:*

(1) *$\{e_i, E_i\}$ is finitely positive.*

(2) *For $\{x_1, \ldots, x_M\} \subset X$ and $\{x_1', \ldots, x_M'\} \subset X^*$, $\sum_{n=1}^M x_n'(e_k)E_k(x_n) \geqq 0$ for $k = 1, 2, \ldots$ implies $\sum_{n=1}^M x_n'(x_n) \geqq 0$.*

(3) *For each finite-dimensional continuous linear mapping T from X into X, $E_i(Te_i) \geqq 0$ for each i implies* $\mathrm{tr}(T) \geqq 0$.

(4) *For every finite subset $\{x_1', \ldots, x_M'\}$ of X^* there is a BK-space S and a continuous positive linear functional E on S such that $(x_n'(e_i)E_i(x)) \in S$ for $n = 1, 2, \ldots, M$ and each $x \in X$ and*

$$E(x_n'(e_i)E_i(x)) = x_n'(x), \qquad n = 1, 2, \ldots, M.$$

(5) *The identity operator from X into X lies in the weak operator closure of $\kappa\{E_i \otimes e_i\}$.*

(6) *For any finite-dimensional subspace F of X there is a sequence $\{T_n^{(F)}\}$ of*

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finite-dimensional diagonal positive linear mappings $\{T_n^{(F)}\}$ *such that*

$$\lim_n T_n^{(F)} y = y$$

for $y \in F$.

C. *The following statements are equivalent:*

(1) $\{e_i, E_i\}$ *is positive.*

(2) X *has the approximation property, and if* T *is a nuclear mapping from* X *into* X *for which* $E_i(Te_i) \geqq 0$ *for each* i, *then* $\operatorname{tr}(T) \geqq 0$.

(3) X *has the approximation property and for each nuclear mapping* T *and each* $\epsilon > 0$ *there are numbers* $a_1, a_2, \ldots, a_n \geqq 0$ *such that*

$$\left| \operatorname{tr}(T) - \sum_{i=1}^{n} a_i E_i(Te_i) \right| < \epsilon.$$

(4) *There is a BK-space* S *containing all sequences of the form* $(x'(e_i)E_i(x))$ *with* x *in* X *and* x' *in* X^*, *and a continuous positive linear functional* E *on* S *for which*

$$E(x'(e_i)E_i(x)) = x'(x).$$

D. *The following statements are equivalent:*

(1) $\{e_i, E_i\}$ *is strongly positive.*

(2) *The identity mapping from* X *into* X *is the limit of a sequence in* $\kappa\{E_i \otimes e_i\}$ *with respect to the weak operator topology.*

(3) X *has the approximation property, and the identity mapping from* X *into* X *is in the* w^* *limit of a sequence in* $\kappa\{E_i \otimes e_i\}$ *in* $L(X)$.

(4) X *has the approximation property, and there is a row finite matrix* (a_{nk}) *of non-negative numbers such that for each nuclear mapping* T *from* X *into* X,

$$\operatorname{tr}(T) = \lim_n \sum_k a_{nk}(Te_k).$$

(5) *There is a row finite matrix* (a_{nk}) *of non-negative numbers such that for each* $x \in X$ *and* $x' \in X^*$,

$$x'(x) = \lim_n \sum_k a_{nk} E_k(x) x'(e_k).$$

(6) *There is a row finite matrix* (a_{nk}) *of non-negative numbers such that for each* $x \in X$

$$x = \lim_n \sum_k a_{nk} E_k(x) e_k.$$

3. Subsequences.

3.1 PROPOSITION. *Let* $\{e_i, E_i\}$ *be a complete biorthogonal sequence in a Banach space* X. *Let* $\{e_i, E_i : i \in J\}$ *be a subsequence of* $\{e_i, E_i\}$ *and let* Y *denote the closed linear span of* $\{e_i : i \in J\}$ *in* X. *If* $\{e_i, E_i\}$ *is of any one of the following types then* $\{e_i, E_i : i \in J\}$ *is also of that type in the space* Y: (a) *K-series sum-*

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mable (*b*) *finitely series summable*, (*c*) *series summable*, (*d*) *strongly series summable* (*e*) *K-positive*, (*f*) *finitely positive* (*g*) *positive*, (*h*) *strongly positive*.

Proof. (a) Suppose $\{x_1', \ldots, x_M'\} \subset Y^*$ with $M \leqq K$. For $n = 1, 2, \ldots, M$ let $\bar{x}_n'$ be an extension of x_n' to all of X. Using B(4) of Theorem 1.2 we obtain a BK-space S and a continuous linear functional E on S such that $(\bar{x}_n'(e_i)E_i(x)) \in S$ for each $x \in X$ and $n = 1, 2, \ldots, M$ and

$$E(\bar{x}_n'(e_i)E_i(x)) = \bar{x}_n'(x).$$

If $T = \{(a_i) \in S : a_i = 0 \text{ for } i \notin J\}$ then T is a closed subspace of S and thus a BK-space. Define F on T by $F(a_i) = E(a_i)$ for $(a_i) \in T$. Then F is a continuous linear functional on T. If $x \in Y$ then $E_i(x) = 0$ for $i \notin J$ so that $(x_n'(e_i)E_i(x)) = (\bar{x}_n(e_i)E_i(x)) \in T$ and

$$F(x_n'(e_i)E_i(x)) = E(\bar{x}_n(e_i)E_i(x)) = \bar{x}_n(x) = x_n(x)$$

for $n = 1, 2, \ldots, M$. Therefore $\{e_i, E_i : i \in J\}$ is K-series summable by B(4) of Theorem 1.2.

(b) If $\{e_i, E_i\}$ is finitely series summable then $\{e_i, E_i\}$ is K-series summable for each K. By (a), $\{e_i, E_i : i \in J\}$ is K-series summable for each K so $\{e_i, E_i : i \in J\}$ is series summable.

(c) The proof in this case follows the example of that for (a) using D(4) of Theorem 1.2 instead of B(4).

(d) Using E(6) of Theorem 1.2 we obtain a row finite matrix (a_{nk}) such that for each $x \in X$

$$x = \lim_n \sum_k a_{nk}E_k(x)e_k.$$

Let $(a_{nk})_{k \in J}$ be the row finite matrix obtained by deleting the columns of (a_{nk}) for $k \notin J$. Then for $x \in Y$, $E_k(x) = 0$ for $k \notin J$ so that

$$\begin{aligned} x &= \lim_n \sum_k a_{nk}E_k(x)e_k \\ &= \lim_n \sum_{k \in J} a_{nk}E_k(x)e_k. \end{aligned}$$

Therefore, $\{e_j, E_j : j \in J\}$ is strongly series summable.

We omit the proofs of (*e*), (*f*), (*g*) and (*h*) which are completely analogous to those of the series summable versions.

4. Existence. We recall that a family $\mathscr{P}$ of projections is called a *family of orthogonal projections* if for $P_1 \neq P_2$ in $\mathscr{P}$ we have $P_1P_2 = P_2P_1 = 0$.

4.1 PROPOSITION. (a) *A Banach space X admits a finitely series summable* (*respectively, finitely positive*) *complete biorthogonal sequence if and only if* (α) *X admits a countable family $\{P_n\}$ of orthogonal finite dimensional projections such that I is in the closure of $[P_n]$* (*respectively*, $\kappa\{P_n\}$) *in the strong operator topology*.

(b) *A Banach space X admits a series summable* (*respectively, positive*)

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complete biorthogonal sequence if and only if (β) *X has the approximation property and admits a countable family* $\{P_n\}$ *of orthogonal finite dimensional projections such that I is in the closure of* $[P_n]$ (*respectively,* $\kappa\{P_n\}$) *in the* w^**-topology on* $L(X)$ *considered as a subspace of* $N(X)^*$.

(c) *A Banach space X admits a strongly series summable* (*respectively, strongly positive*) *complete biorthogonal sequence if and only if* (γ) *X admits a countable family* $\{P_n\}$ *of orthogonal finite dimensional projections such that I is the limit of a sequence in* $[P_n]$ (*respectively, in* $\kappa(P_n\}$) *in the strong operator topology.*

Proof. The necessity of the conditions (α), (β) and (γ) follow from the fact that the set $\{E_i \otimes e_i\}$ is a family of one-dimensional orthogonal projections.

We demonstrate the sufficiency of the condition (α) in the finitely series summable case of (a). The reasoning is entirely analogous in all other cases.

Let $\{x_{i1}, x_{i2}, \ldots, x_{in_i}\}$ be a basis for the range of P_i and let $x_{i1}', x_{i2}', \ldots, x_{in_i}{}^i$ be the associated linear functionals. Let $E_{ik} = x_{ik}' \circ P_i$ and $e_{ik} = x_{ik}$. Then $\{e_{ik}, E_{ik}: k = 1, 2, \ldots, n_i : i = 1, 2, \ldots\}$ is a complete biorthogonal sequence in X. Since

$$P_i = \sum_{k=1}^{ni} E_{ik} \otimes e_{ik}$$

and I is in the closed linear span of $\{P_i\}$, the result follows.

4.2 THEOREM. *A Banach space X admits a positive complete biorthogonal sequence if* (*) *there exists in X a chain of subspaces* $\{M_\alpha : \alpha < \gamma\}$ *where* γ *is a countable limit ordinal such that*

(a) M_1 *is finite dimensional,*

(b) *for each* $\alpha < \gamma$, $M_{\alpha+1} \supset M_\alpha$ *and* $M_{\alpha+1}/M_\alpha$ *has finite dimension,*

(c) *there is a projection of norm one from* $M_{\alpha+1}$ *onto* M_α,

(d) $\overline{\bigcup_\alpha M_\alpha}$ *has finite codimension in* X,

(e) *for each limit ordinal* $\beta < \gamma$, $\bigcup_{\alpha<\beta} M_\alpha$ *is dense in* M_β.

Proof. We shall prove the theorem under the hypothesis that

$$\overline{\bigcup_\alpha M_\alpha} = X.$$

For each α let $P_{\alpha+1}$ be the projection from $M_{\alpha+1}$ onto M_α of norm one. Let $Z_1 = M_1$ and for each ordinal $\alpha < \gamma$ let $Z_{\alpha+1} = (I - P_{\alpha+1})M_{\alpha+1}$. Denote $[\bigcup_\alpha Z_{\alpha+1}]$ by S. If $x_{\beta_i} \in Z_{\beta_i}$ for $\beta_1 < \beta_2 < \ldots < \beta_k$ we have

$$||x_{\beta_1} + x_{\beta_2} + \ldots + x_{\beta_{k-1}}|| \leqq ||x_{\beta_1} + x_{\beta_2} + \ldots + x_{\beta_k}||$$

since $P_{\beta_k}(x_{\beta_1} + x_{\beta_2} + \ldots + x_{\beta_k}) = x_{\beta_1} + x_{\beta_2} + \ldots + x_{\beta_{k-1}}$. For each non-limit ordinal $\beta < \gamma$ and $y = x_{\beta_1} + x_{\beta_2} + \ldots + x_{\beta_k}$ with $x_{\beta_i} \in Z_{\beta_i}$ define

$$Q_\beta(y) = \sum_{\beta_i \leqq \beta} x_{\beta_i}.$$

Then Q_β is a projection fron S into S of norm one. Since S is dense in X we can extend each Q_β to all of X. For $y \in S$, $\lim_\beta Q_\beta(y) = y$ since $Q_\beta(y)$ is eventually equal to y. Thus by the Banach Steinhaus Theorem $\lim_\beta Q_\beta(x) = x$ for each $x \in X$. By the same sort of reasoning we can prove that for each $x \in X$ and each limit ordinal $\alpha < \gamma$ $\lim_{\beta<\alpha} Q_\beta(x)$ converges to a projection Q_α of norm one from X onto

$$\overline{[\bigcup_{\beta<\alpha} Z_\beta]}\,.$$

If for each non-limit ordinal $\beta < \gamma$, $R_\beta = Q_\beta - Q_{\beta-1}$, $\{R_\beta\}$ is on orthogonal family of projections from X into X such that I is in the closure of $\kappa\{R\}$ in the strong operator topology. Let $\{e_{\beta i}, E_{\beta i}: i = 1, 2, \ldots, n_\beta; \beta < \gamma\}$ (β is not a limit ordinal) be the system constructed as in Proposition 4.1.

Let T be the set of all indexed bounded families of numbers $(a_{\beta i}: i = 1, 2, \ldots, n_\beta, \beta < \gamma)$ such that the sum

$$\sum_{\beta<\gamma} \sum_{i=1}^{n_\beta} a_{\beta i} = E(a_{\beta i})$$

exists. Then T can be shown to be a BK-space and E a positive continuous linear functional on T. If $x \in X$ and $x_\alpha \in X^*$ $(E_{\beta'}(x)x'(e_{\beta i}): i = 1, 2, \ldots, n_\beta; \beta < \gamma)$ is in T and $E(E_{\beta i}(x)x'(e_{\beta i})) = x'(x)$ because

$$\sum_{\beta<\gamma} \sum_{i=1}^{n_\beta} E_{\beta i}(x)x'(e_{\beta i}) = \sum_{\beta<\gamma} x'\left(\sum_{i=1}^{n_\beta} E_{\beta i}(x)e_{\beta i}\right)$$
$$= \sum_{\beta<\gamma} x'(R_\beta x) = \lim_\alpha x'(Q_\alpha x) = x'(x).$$

Therefore, by C(4) of Theorem 2.2, $\{e_{\beta i}, E_{\beta i}\}$ is positive.

5. Observations, examples and problems. It is not hard to see that every permutation of a Schauder basis is a strongly positive complete biorthogonal sequence. Hence the following table of implications is valid for $\{e_i, E_i\}$ a complete biorthogonal sequence in a Banach space X:

Permutation of Schauder Basis
$\Downarrow$ $\Downarrow$
strongly positive $\Rightarrow$ strongly series summable
$\Downarrow$ $\Downarrow$
positive $\Rightarrow$ series summable
$\Downarrow$ $\Downarrow$
$k + 1$ positive $\Rightarrow$ $k + 1$ series summable
$\Downarrow$ $\Downarrow$
k positive $\Rightarrow$ k series summable
$\Downarrow$ $\Downarrow$
one positive $\Rightarrow$ one series summable.

The functions $\{1, \sin nx, \cos nx: n = 1, 2, \ldots\}$ and their biorthogonal functionals form a strongly positive complete biorthogonal sequence in $C[0, 2\pi]$, but no permutation of these functions is a basis. In [8, p. 524] is an example of a complete biorthogonal sequence in a Banach space which is not one-series summable. Professor W. B. Johnson has pointed out to the author that such a complete biorthogonal sequence can be constructed in every separable Banach space. Crone, Fleming and Jessup [1] have given an example of a series summable complete biorthogonal sequence which is not strongly series summable.

5.1 *Problem.* Are any of the implications above reversible?

5.2 *Problem.* Under what conditions on a Banach space X does it admit a complete biorthogonal sequence of a given type defined in 1.1 or 2.1?

In [6], Johnson showed that if X is a complex Banach space, and X^* has the λ-metric approximation property then X admits a strongly series complete biorthogonal sequence.

We can construct a system in an arbitrary locally convex space somewhat like a 1-series summable complete biorthogonal sequence but having a weaker biorthogonality property.

5.3 Proposition. (a) *In every locally convex space X there is a double family $\{x_\alpha, x_\alpha'\}$ with each $x_\alpha \in X$ and each $x_\alpha' \in X^*$ such that*

(1) $x_\alpha'(x_\alpha) = 1$ *for each* α;

(2) $x_\alpha'(x_\beta)x_\beta'(x_\alpha) = 0$ *for* $\alpha \neq \beta$;

(3) *if* $x \in X$ *and* $x' \in X^*$ *are such that* $x'(x_\alpha)x_\alpha'(x) = 0$ *for each* α *then* $x'(x) = 0$.

(b) *If X is a Banach space and X^* is separable then the system $\{x_\alpha, x_\alpha'\}$ given in* (a) *is countable.*

Proof. (a) Using a standard maximality argument we can construct a maximal system $\{x_\alpha, x_\alpha'\}$ having properties (1) and (2). Suppose $x'(x_\alpha)x_\alpha'(x) = 0$ for each α, and $x'(x) = a \neq 0$. Then $\{x_\alpha, x_\alpha'\} \cup \{x/a, x'\}$ has properties (1) and (2) and properly contains $\{x_\alpha, x_\alpha'\}$. This contradicts the maximality.

(b) If X is a Banach space and X^* is separable then $\mathscr{F}(X)$, the space of finite dimensional mappings from X into X with the greatest crossnorm $\|\ \|_n$ (nuclear norm) is separable. If $\{x_\alpha, x_\alpha'\}$ satisfies (1) and (2) then for $\alpha \neq \beta$,

$$\|x_\alpha' \otimes x_\alpha - x_\beta' \otimes x_\beta\|_n^2 \geqq |\mathrm{tr}([x_\alpha' \otimes x_\alpha - x_\beta' \otimes x_\beta] \circ [x_\alpha' \circ x_\alpha - x_\beta' \otimes x_\beta]) = 2.$$

Thus $\{x_\alpha' \otimes x_\alpha\}$ is discrete and consequently a countable set.

Let us call a system $\{x_\alpha, x_\alpha'\}$ in a locally convex space X which satisfies (1), (2), (3) of Proposition 5.3 a "maximal quasi-orthonormal" system. The shortcomings of sich a system are evident. For instance, "the expansion" with respect to such a system $\{x_\alpha, x_\alpha'\}$ for one of its vectors, say x_β, is not simply x_β but the formal series $\sum_\alpha x_\alpha'(x_\beta)x_\alpha$ which may not even converge. Even in a finite dimensional space a maximal quasi-orthonormal system need not be a biorthogonal system. For example: $\{(e_1, E_1), (e_2, E_1 + E_2), (e_2 - e_1, E_1)\}$ is maximal quasi-orthogonal in $\mathbf{R}^2$ where $e_1 = (1, 0)$, $e_2 = (0, 1)$, $E_1(x, y) = x$ and $E_2(x, y) = y$.

On the favorable side we have the following statement whose proof is like that of (3) $\Leftrightarrow$ (5) of Theorem 1.2-A.

5.4 PROPOSITION. *If* $\{x_\alpha, x_\alpha'\}$ *is a maximal quasi-orthonormal system in a Banach space* X *then for each* $x' \in X^*$ *there is a BK-space* S, *and a continuous linear functional* E *on* S_x, *such that* $(x_\alpha'(x)x'(x_\alpha) \in S_{x'}$ *for each* $x \in X$ *and*

$$E(x_\alpha'(x)x'(x_\alpha)) = x'(x).$$

Here $S_{x'}$ is a BK-space in the sense that it is a Banach space of functions on a set (not necessarily countable) such that the evaluation functionals are all continuous.

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