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SPECTRAL CONTINUITY FOR OPERATOR MATRICES

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Abstract. In this paper we prove that if $M_C = \begin{pmatrix} A & C \\ 0 & B \end{pmatrix}$ is a 2×2 upper triangular operator matrix on the Hilbert space $H \bigoplus K$ and if $\sigma(A) \cap \sigma(B) = \emptyset$, then σ is continuous at A and B if and only if σ is continuous at M_C, for every $C \in B(K, H)$.

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1. Introduction. Throughout this note let H and K be Hilbert spaces, let $B(H, K)$ denote the set of bounded linear operators from H to K, and abbreviate $B(H, H)$ to $B(H)$. If $T \in B(H)$ write $N(T)$ and $R(T)$ for the null space and range of T; $\alpha(T) = \dim N(T)$; $\beta(T) = \dim N(T^*)$; $\sigma(T)$ is the spectrum of T; $\sigma_a(T)$ is the approximate point spectrum of T; $\sigma_d(T)$ is the defect spectrum of T; $\pi_0(T)$ is the set of eigenvalues of T; $\pi_{00}(T)$ is the set of isolated points of $\sigma(T)$ that are eigenvalues of finite multiplicity. An operator $T \in B(H)$ is called *left semi-Fredholm* if it has closed range with finite dimensional null space and *right semi-Fredholm* if it has closed range with its range of finite co-dimension. If T is both left semi- and right semi-Fredholm, we call it *Fredholm*. The *index* of a left semi- and right semi-Fredholm operator $T \in B(H)$ is given by

$$i(T) = \alpha(T) - \beta(T).$$

The essential spectrum $\sigma_e(T)$, the Weyl spectrum $\omega(T)$ and the Browder spectrum $\sigma_b(T)$ of $T \in B(H)$ are defined in **[5]** and **[6]** as follows:

$$\sigma_e(T) = \{\lambda \in \mathbb{C} : T - \lambda I \ \text{ is not Fredholm}\};$$
$$\omega(T) = \{\lambda \in \mathbb{C} : T - \lambda I \ \text{ is not Weyl}\};$$
$$\sigma_b(T) = \{\lambda \in \mathbb{C} : T - \lambda I \ \text{ is not Browder}\}.$$

Evidently

$$\sigma_e(T) \subseteq \omega(T) \subseteq \sigma_b(T) = \sigma_e(T) \cup \operatorname{acc} \sigma(T),$$

where we write acc$G$ for the accumulation points of $G \subseteq \mathbb{C}$. We say that *Weyl's theorem holds for* $T \in B(H)$ if there is equality

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$$\sigma(T) \setminus \omega(T) = \pi_{00}(T), \tag{1.1}$$

and that *Browder's theorem holds for* $T \in B(H)$ if there is equality

$$\omega(T) = \sigma_b(T). \tag{1.2}$$

If (K_n) is a sequence of compact subsets of $\mathbb{C}$, then by the definition, its *limit inferior* is $\liminf K_n = \{\lambda \in \mathbb{C} : \text{there are } \lambda_n \in K_n \text{ with } \lambda_n \to \lambda\}$ and its *limit superior* is $\limsup K_n = \{\lambda \in \mathbb{C} : \text{ there are } \lambda_{n_k} \in K_{n_k} \text{ with } \lambda_{n_k} \to \lambda\}$. If $\liminf K_n = \limsup K_n$, then $\lim K_n$ is defined by this common limit. A mapping f, defined on $B(H)$, whose values are compact subsets of $\mathbb{C}$, is said to be *upper (lower) semi-continuous* at T, provided that if $T_n \to T$ (in the norm topology) then $\limsup f(T_n) \subset f(T)$ ($f(T) \subset \liminf f(T_n)$). If f is both upper and lower semi-continuous at T, then it is said to be *continuous at* T and in this case $\lim f(T_n) = f(T)$.

2. Main results. When $A, A_n \in B(H)$ and $B, B_n \in B(K)$ are given we denote by M_C and M_n the operators acting on $H \oplus K$ defined by

$$M_C = \begin{pmatrix} A & C \\ 0 & B \end{pmatrix}, \; M_n = \begin{pmatrix} A_n & C_n \\ 0 & B_n \end{pmatrix},$$

where $C, C_n \in B(K, H)$.

Consider the following example: let $U \in B(l_2)$ be the unilateral shift, $A_n = U$, $B_n = U^*$, and $C_n = \frac{1}{n}(I - UU^*)$. Then on $l_2 \oplus l_2$ we have

$$M_n = \begin{pmatrix} A_n & \frac{1}{n}(I - UU^*) \\ 0 & B_n \end{pmatrix} \to M = \begin{pmatrix} U & 0 \\ 0 & U^* \end{pmatrix},$$

as $n \longrightarrow \infty$. For operator matrices M_n and M we have $\sigma(M_n) = \{\lambda \in \mathbb{C} : |\lambda| = 1\}$ and $\sigma(M) = \{\lambda \in \mathbb{C} : |\lambda| \leq 1\}$. Therefore $\sigma(M_n) \not\to \sigma(M_n)$.

However, we have the following result.

THEOREM 2.1. *Let* $A \in B(H)$ *and* $B \in B(K)$ *be such that* $\sigma(A) \cap \sigma(B) = \emptyset$. *Then* σ *is continuous at* A *and* B *if and only if* σ *is continuous at* M_C, *for every* $C \in B(K, H)$.

Proof. Since $\sigma(A) \cap \sigma(B) = \emptyset$, there exists $\delta > 0$ such that $d(\sigma(A), \sigma(B)) > 3\delta$. Now, by the upper semi-continuity of the spectrum at A and B [**11**], for every sequence (A_n) in $B(H)$ and every sequence (B_n) in $B(K)$ such that $A_n \to A$ and $B_n \to B$ there exists a natural number n_0 such that $n \geq n_0 \Rightarrow \sigma(A_n) \subset (\sigma(A))_\delta$ and $\sigma(B_n) \subset (\sigma(B))_\delta$. Since $\sigma(A_n) \cap \sigma(B_n) = \emptyset$, for every $n \geq n_0$, we have that $\sigma(M_n) = \sigma(A_n) \cup \sigma(B_n)$.

($\Rightarrow$) Suppose that σ is continuous at A and B. Then

$$\sigma(M_c) = \sigma(A) \cup \sigma(B) \subset \liminf(\sigma(A_n) \cup \sigma(B_n)) = \liminf \sigma(M_n).$$

Therefore σ is lower semi-continuous at M_C, and hence σ is continuous at M_C for every $C \in B(K, H)$.

($\Leftarrow$) Suppose that σ is continuous at M_C for every $C \in B(K, H)$. We shall show that σ is continuous at A. Let $\lambda \in \sigma(A)$. Then $\lambda \notin \sigma(B)$ and

$$\lambda \in \sigma(A) \subset \sigma(M_C) \subset \liminf \sigma(M_n).$$

Therefore there exists a sequence (λ_n) such that $\lambda_n \in \sigma(M_n)$ and $\lambda_n \to \lambda$. But $\sigma(A_n) \cap \sigma(B_n) = \emptyset$, for every $n \geq n_0$; hence we have $\sigma(M_n) = \sigma(A_n) \cup \sigma(B_n)$. If there exists a subsequence (λ_{n_k}) of (λ_n) such that $\lambda_{n_k} \in \sigma(B_{n_k})$, then we have $\lambda \in \limsup \sigma(B_n) \subset \sigma(B)$. This is a contradiction. Therefore $\lambda_n \in \sigma(A_n)$, for every $n \geq n_0$. Thus $\lambda \in \liminf \sigma(A_n)$, and hence σ is continuous at A. Similarly, σ is continuous at B. □

If $A \in B(H)$ and $B \in B(K)$ such that $\omega(A) \cap \omega(B) = \emptyset$, then we have $\omega(M_C) = \omega(A) \cup \omega(B)$ [**10**, Theorem 4]. Now, we have the following theorem.

THEOREM 2.2. *Let $A \in B(H)$ and $B \in B(K)$ such that $\omega(A) \cap \omega(B) = \emptyset$. Then ω is continuous at A and B if and only if ω is continuous at M_C, for every $C \in B(K, H)$.*

Proof. Since ω is upper semi-continuous, the proof is similar to that of Theorem 2.1. □

α and β can be viewed as functions assigning $\alpha(T)$ and $\beta(T)$ to each $T \in B(H)$, respectively.

THEOREM 2.3. *Let $A \in B(H)$, $B \in B(K)$ and $C \in B(K, H)$ such that*
(1) *$\sigma(M_C) = \sigma_a(A) \cup \sigma_d(B) \cup \{\lambda \in \mathbb{C}; \alpha(B - \lambda) \neq \beta(A - \lambda)\}$;*
(2) *σ_a is continuous at A;*
(3) *σ_d is continuous at B.*
Then σ is continuous at M_C.

Proof. It is sufficient to show that σ is lower semi-continuous at M_C. Let $\lambda \in \sigma(M_C)$. We shall divide the proof into three cases.

Case 1. If $\lambda \in \sigma_a(A)$, then since σ_a is continuous at A there exists a natural number n_0 such that for every $n > n_0$ we have $\lambda \in \sigma_a(A_n) \subset \sigma(M_n)$.

Case 2. If $\lambda \in \sigma_d(B)$, then by continuity of σ_d at B there exists a natural number n_1 such that for every $n > n_1$ we have $\lambda \in \sigma_d(B_n) \subset \sigma(M_n)$.

Case 3. Suppose that $\lambda \in \sigma(M_C) \setminus (\sigma_a(A) \cup \sigma_d(B))$. Then we have $\alpha(B - \lambda) \neq \beta(A - \lambda)$, $\alpha(A - \lambda) = 0$, and $\beta(B - \lambda) = 0$. Therefore $i(A - \lambda) \neq i(B - \lambda)$, and hence it follows from the continuity of the index that there exists n_2 such that for $n > n_2$, $i(A_n - \lambda) \neq i(B_n - \lambda)$. Since functions α and β are continuous at A and B [**3**, Corollary 2.3], respectively, we have that $\alpha(B_n - \lambda) \neq \beta(A_n - \lambda)$. Therefore $\lambda \in \sigma(M_n)$ for every $n > n_2$. It follows that in all three cases σ is continuous at M_C. □

If M_C obeys Browder's theorem, then the Weyl spectrum, the Browder spectrum and the spectrum are continuous at M_C.

THEOREM 2.4. *Let $A \in B(H)$, $B \in B(K)$ and $C \in B(K, H)$ such that*
(1) *$\sigma(M_C)$ obeys Browder's theorem;*
(2) *σ_a is continuous at A;*

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(3) *σ_d is continuous at B.*
Then σ, ω, and σ_b are continuous at M_C, respectively.

Proof. Let $\lambda \in \sigma(M_C)$. If $\lambda \in \sigma_a(A) \cup \sigma_d(B) \cup \{\lambda \in \mathbb{C}; \alpha(B-\lambda) \neq \beta(A-\lambda)\}$, then it follows from Theorem 2.3 that $\lambda \in \liminf \sigma(M_n)$. Suppose now that

$$\lambda \in \sigma(M_C) \setminus [\sigma_a(A) \cup \sigma_d(B) \cup \{\lambda \in \mathbb{C}; \alpha(B-\lambda) \neq \beta(A-\lambda)\}].$$

Then $\alpha(A-\lambda) = \beta(B-\lambda) = 0$, $\alpha(B-\lambda) = \beta(A-\lambda)$, and so $i(A-\lambda) = -i(B-\lambda)$. By [**1**, Lemma 1.2], $i(M_C - \lambda) = 0$. Since M_C obeys Browder's theorem, $\lambda \notin \sigma_b(M_C)$. Therefore λ is an isolated point of $\sigma(M_C)$, and so $\lambda \in \liminf \sigma(M_n)$. Hence σ is continuous at M_C. It follows from [**2**, Theorem 2.2] that ω and σ_b are continuous at M_C. □

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