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The influence of stratification upon small-scale convectively-driven dynamos

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Abstract. In the quiet Sun, convective motions form a characteristic granular pattern, with broad upflows enclosed by a network of narrow downflows. Magnetic fields tend to accumulate in the intergranular lanes, forming localised flux concentrations. One of the most plausible explanations for the appearance of these quiet Sun magnetic features is that they are generated and maintained by dynamo action resulting from the local convective motions at the surface of the Sun. Motivated by this idea, we describe high resolution numerical simulations of nonlinear dynamo action in a (fully) compressible, non-rotating layer of electrically-conducting fluid. The dynamo properties depend crucially upon various aspects of the fluid. For example, the magnetic Reynolds number (Rm) determines the initial growth rate of the magnetic energy, as well as the final saturation level of the dynamo in the nonlinear regime. We focus particularly upon the ways in which the Rm-dependence of the dynamo is influenced by the level of stratification within the domain. Our results can be related, in a qualitative sense, to solar observations.

Keywords. Convection, magnetic fields, (magnetohydrodynamics:) MHD, Sun: magnetic fields, Sun: photosphere

1. Introduction

High resolution observations of the solar surface have greatly enhanced our theoretical understanding of the ways in which magnetic fields interact with turbulent convection in an electrically-conducting fluid. The time-dependent, near-surface, convective motions in the quiet Sun form a characteristic granular pattern at the solar photosphere (see, for example, Stix 2004). The dark intergranular lanes, which correspond to the convective downflows, contain mixed-polarity concentrations of vertical magnetic flux. These localised magnetic features show up as bright points in G-band images of the solar surface (see, for example, Sánchez Almeida *et al.* 2010). Peak magnetic field strengths in these regions are typically in excess of a kilogauss (see, for example, de Wijn *et al.* 2009, and references therein), which implies that the magnetic energy density of these features is about an order of magnitude larger than the mean kinetic energy density of the surrounding granular convection

It is plausible that the convective motions that are observed at the solar surface are themselves responsible for the generation of quiet Sun magnetic fields. A small-scale dynamo of this type would proceed independently of the large-scale dynamo processes that are responsible for driving the solar cycle. As demonstrated by Cattaneo (1999), standard Boussinesq convection in an electrically-conducting fluid (in the absence of rotation or shear) can drive a small-scale dynamo. More recent calculations have demonstrated that dynamo action is also possible in fully compressible convection (Abbett

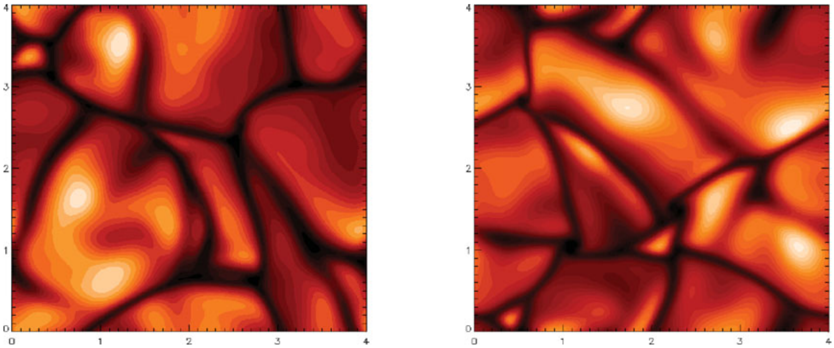

Figure 1. The temperature distribution for hydrodynamic convection in a horizontal plane just below the upper surface of the computational domain. Bright contours correspond to warmer fluid. Left: Moderately-stratified convection ($\theta = 3$). Right: Highly-stratified convection ($\theta = 10$).

2007; Vögler & Schüssler 2007; Käpylä, Korpi & Brandenburg 2008; Bushby, Proctor & Weiss 2010; Brummell, Tobias & Cattaneo 2010). However, there are still some aspects of convectively-driven dynamos that have not yet been addressed. The aim of this short paper is to investigate some of the ways in which the level of stratification within the fluid influences the efficiency of a convectively-driven dynamo.

2. An idealised model

We consider the behaviour of a compressible, electrically-conducting, monatomic gas, in the presence of a magnetic field. In this idealised model, we adopt constant values for the viscosity μ, the thermal conductivity K, the magnetic diffusivity η, and the magnetic permeability μ_0. The constant specific heat capacities (c_P and c_V) satisfy $c_P/c_V = 5/3$, whilst the gas constant is defined by $R_* = c_P - c_V$. Choosing a (non-rotating) Cartesian frame of reference in which the z-axis points vertically downwards, the gas occupies the region $0 \leqslant x \leqslant 4d$, $0 \leqslant y \leqslant 4d$, $0 \leqslant z \leqslant d$. The constant gravitational acceleration is given by $\mathbf{g} = g\hat{\mathbf{z}}$. The state of this system at position $\mathbf{x}$ and time t is defined by the density $\rho(\mathbf{x}, t)$, the temperature $T(\mathbf{x}, t)$, the velocity $\mathbf{u}(\mathbf{x}, t)$ and the magnetic field $\mathbf{B}(\mathbf{x}, t)$. The gas pressure, $P(\mathbf{x}, t)$, is given by $P = R_*\rho T$. The fluid variables satisfy periodic boundary conditions in each horizontal direction. The upper and lower bounding surfaces are held at fixed, uniform temperatures, with $T = T_0$ at $z = 0$ and $T = T_0 + \Delta T$ at $z = d$ (where a positive value of ΔT implies that the layer is heated from below). These boundaries are also assumed to be impermeable and stress-free in this idealised model. Vertical field boundary conditions are imposed on $\mathbf{B}$ at $z = 0$ and $z = d$. The evolution of this system is governed by the standard equations of (non-ideal) compressible magnetohydrodynamics (see, for example, Bushby *et al.* 2008).

In the absence of any magnetic fields, the governing equations have a hydrostatic solution, corresponding to a polytropic layer, in which $T = T_0\left(1 + \theta z/d\right)$ and $\rho = \rho_0\left(1 + \theta z/d\right)^m$. Here, the parameter $\theta = \Delta T/T_0$ is a measure of the thermal stratification of the layer, whilst $m = (gd/R_*\Delta T) - 1$ defines the polytropic index. For an unmagnetised monatomic gas, this polytropic equilibrium is unstable to convective perturbations provided that $m < 3/2$. The evolution of any convective perturbation depends crucially upon the other parameters in the system. We define the Prandtl number to be $\sigma = \mu c_P/K$, whilst the dimensionless thermal diffusivity is defined to be $\kappa = K/d\rho_0 c_p\left(R_* T_0\right)^{1/2}$. The parameter $\zeta_0 = \eta c_P \rho_0/K$ gives the ratio of the magnetic diffusivity to the thermal

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diffusivity at the upper surface of the domain. Two other relevant parameters for convection are the (mid-layer) Reynolds number,

$$Re = \frac{\rho_{mid} U_{rms} d}{\mu}, \tag{2.1}$$

(where U_{rms} is the rms velocity of the convection and ρ_{mid} is the mid-layer density of the unperturbed polytrope) whilst

$$Rm = \frac{U_{rms} d}{\eta}, \tag{2.2}$$

corresponds to the magnetic Reynolds number of the flow.

By carrying out three-dimensional numerical simulations (see, for example, Bushby *et al.* 2008 for numerical details), we investigate the dynamo properties of convection in two different polytropic layers, one moderately-stratified, the other highly-stratified. In both cases, $m = 1$ and $\sigma = 1$. In the moderately-stratified case, we set $\theta = 3$, which implies that the density and temperature both vary by a factor of 4 across the depth of the unperturbed layer. In the highly-stratified layer, we set $\theta = 10$, which means that the initial density and temperature profiles both vary by a factor of 11. Finally, $\kappa = 0.0245$ for the highly-stratified case and $\kappa = 0.00548$ for the moderately-stratified layer. These parameters have been chosen so that both calculations produce hydrodynamic convection with $Re \approx 150$. This is illustrated in Figure 1, which shows the temperature distribution (for each case) in a horizontal plane just below the upper surface of the computational domain. Note that the "granular" pattern is similar in both simulations, despite the differing levels of stratification. Some aspects of dynamo action in the highly-stratified case were described in a previous paper (Bushby, Proctor & Weiss 2010).

Once statistically-steady hydrodynamic convection has developed, we insert a weak vertical magnetic field into the flow. This has a simple cosine dependence upon x and y in order to ensure that there is no net flux across the computational domain. The fate of this field depends crucially upon the magnetic Reynolds number. If everything else is fixed, the value of Rm depends solely upon the value of ζ_0, with $Rm \propto (1/\zeta_0)$. Hence for a given hydrodynamic flow, different values of Rm can be investigated simply by changing the value of ζ_0. Since Re is fixed, varying Rm is equivalent to varying the magnetic Prandtl number (which is given by $Pm = Rm/Re$).

3. Numerical results

3.1. *The kinematic regime*

Provided that the energy of the initial magnetic field is very much smaller than the kinetic energy of the flow, the Lorentz force does not play a significant dynamical role during the early stages of the field evolution. During this kinematic phase, the magnetic energy fluctuates in time, but (on average) either grows or decays exponentially, depending upon the magnitude of the magnetic Reynolds number, Rm. Dynamo growth can only occur if Rm is large enough such that the inductive effects due to the fluid motions are strong enough to outweigh the dissipative effects of magnetic diffusion. Of course, exponential growth cannot proceed indefinitely. Eventually, the dynamo-generated magnetic field will become strong enough to exert a significant Lorentz force upon the flow, which forces the dynamo to saturate in the nonlinear regime.

Initially, however, we focus on the kinematic regime, indefinitely extending this phase of evolution by "switching off" the magnetic terms in the momentum and heat equations.

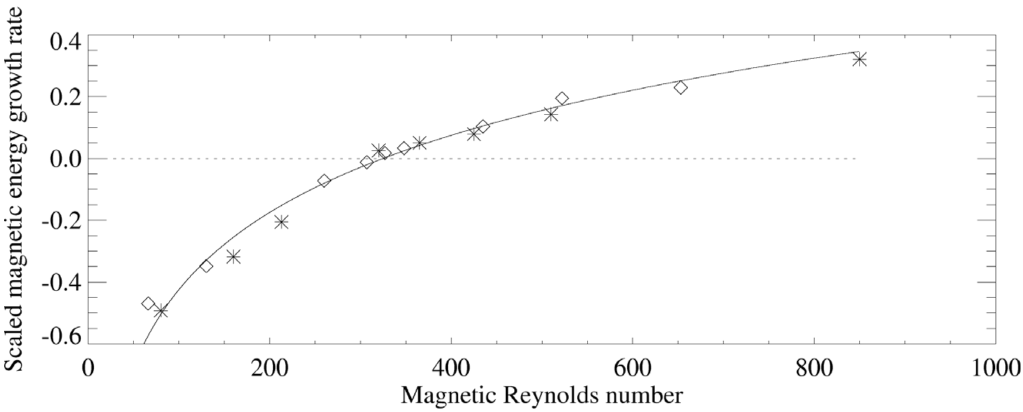

Figure 2. Scaled kinematic growth rates for the magnetic energy, as a function of the magnetic Reynolds number. Stars correspond to the moderately-stratified case, diamonds correspond to highly-stratified convection. The solid line is the curve $\Gamma = 0.36 \log_e(Rm/325)$.

A prolonged kinematic phase allows us to average growth rates over a long period of time, thus obtaining more accurate values for these quantities. In a previous paper (Bushby *et al.* 2010), it was shown that dynamo action is possible in the highly-stratified case provided that the magnetic Reynolds number exceeds a critical value of $Rm_{crit} \approx 325$. Furthermore, the results from that paper suggested that the kinematic growth rates (denoted here by Γ) had a logarithmic dependence upon Rm. More precisely, expressing the growth rates in terms an inverse (isothermal) acoustic travel time at the top of the computational domain, $(R_* T_0)^{1/2}/d$, the measured growth rates are a good fit to the following curve, $\Gamma = 0.21 \log_e(Rm/325)$. Although this logarithmic fit is rather empirical, a similar Rm-dependence has been found in a previous analytic study (Rogachevskii & Kleeorin 1997). In order to assess the influence that stratification has upon the kinematic regime, we repeated these calculations for our moderately-stratified convective layer. Again we found a critical magnetic Reynolds of $Rm_{crit} \approx 325$, whilst $\Gamma = 0.12 \log_e(Rm/325)$ gives a good fit to the Rm-dependence of the dynamo growth rates in moderately-stratified convection.

From these results, it is tempting to draw the conclusion that dynamo action is more efficient in highly stratified convection, with a higher kinematic growth rate for a given value of Rm. In fact, some care is needed here. The acoustic travel time that has been used to scale these growth rates corresponds to the time taken for a sound wave to cross a horizontal distance d across the upper surface of the domain. Clearly the surface surface sound speed is more representative of a "typical" sound speed in the moderately-stratified case than it is in the highly stratified simulation. So, in order to make a fairer comparison between the two cases, we scale the growth rates by the convective turnover time, d/U_{rms}. The results of this rescaling, for both stratifications, are shown in Figure 2. It is apparent from Figure 2 that all the data points lie close to the same best fit curve, $\Gamma = 0.36 \log_e(Rm/325)$. This suggests that the rescaled growth rates are effectively independent of the level of stratification in the domain. In other words, it is purely the convective turnover time that is responsible for setting the growth rate rather than any of the topological aspects of the flow that are associated with the compressibility.

3.2. *Nonlinear results*

The process of flux expulsion leads to the formation of vertical magnetic flux concentrations in the downflow regions at the edges of the granular convective cells. Whilst the field

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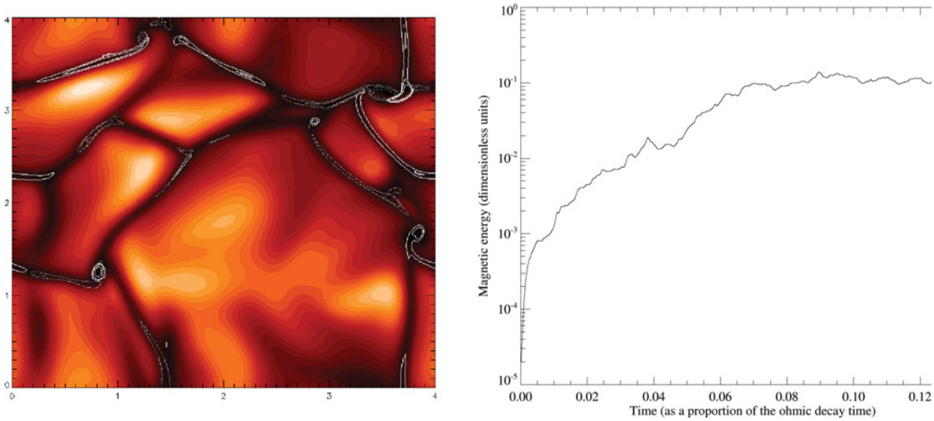

Figure 3. A nonlinear dynamo calculation, in moderately-stratified convection, for $Rm \approx 480$. Left: Contours of constant temperature, overlaid with contours of the vertical component of the magnetic field. Solid (dashed) contours correspond to positive (negative) values of B_z. Right: A log-linear plot showing the magnetic energy (in dimensionless units) as a function of time (expressed in terms of ohmic decay times).

is weak, this is a kinematic process, with no magnetic feedback upon the flow. In fully nonlinear dynamo simulations, the high magnetic pressure that is associated with these magnetic regions causes them to become partially evacuated, particularly in the upper layers of the domain. Amongst other things, this leads to an increase in the local Alfvén speed, as well as a reduction in the timescale that is associated with thermal diffusion. Both of these factors lead to a reduction in the time-step that must be taken in order to ensure stability of the (explicit) numerical scheme. Preliminary nonlinear results for highly-stratified convection were described in Bushby et al. (2010). Even for a relatively modest values of Rm ($Rm \approx 350$, which is less than 10% above the critical value for dynamo action), the partial evacuation that occurs is extremely significant. In such cases, the corresponding reduction in the time-step is so severe that it is not possible to carry out the calculation on a reasonable time-scale without (e.g.) imposing an artificial lower limit on the minimum density within the domain.

Partial evacuation is also a feature of nonlinear dynamo action in moderately-stratified convection. However, because the effects of compressibility are reduced (for example, the peak mach number at the surface is approximately 0.6, as opposed to 1.0 in the highly-stratified case), the effect is less dramatic. So, in this case, it is possible to carry out these simulations without having to deal with overly restrictive time-step constraints. A calculation of dynamo action in moderately-stratified convection, for $Rm \approx 480$, is illustrated in Figure 3. The field geometry is similar to that of the kinematic regime, with mixed polarity concentrations of vertical magnetic flux accumulating in the convective downflows. The minimum gas density within these magnetic regions is highly time-dependent, but a typical minimum value would be (approximately) 25% of the mean density of the non-magnetic surroundings. This dynamo has saturated in the nonlinear regime, reaching a state in which the total magnetic energy is approximately 5% of the total kinetic energy in the domain. The saturation level of such a dynamo clearly depends upon the magnetic Reynolds number, with larger values of Rm expected to produce more efficient dynamos. Early results for an ongoing calculation at $Rm \approx 800$ certainly support this idea. For a lower magnetic Reynolds number of $Rm \approx 360$, the magnetic energy saturates at approximately $3-4\%$ of the kinetic energy. For such a marginal dynamo ($Rm \approx 360$ is only 10% above Rm_{crit}), the magnetic energy exhibits significant fluctuations, so it is

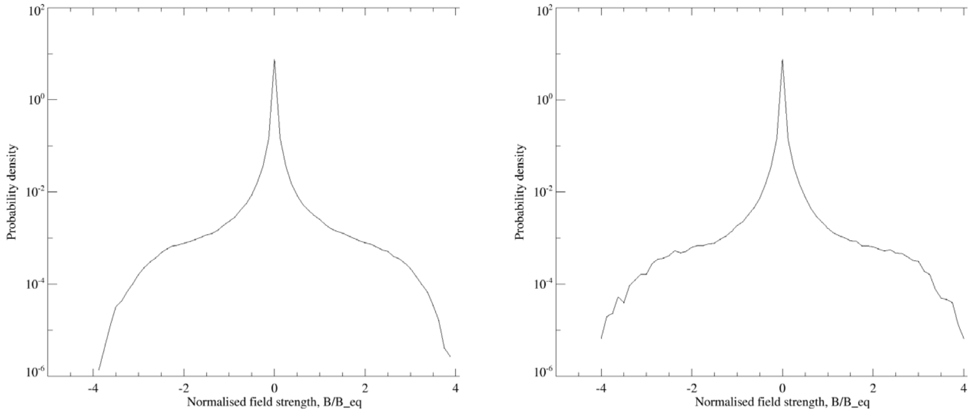

Figure 4. Probability density functions for the vertical component of the magnetic field (scaled by the equipartition value of B_z) in a nonlinear dynamo that is driven by moderately-stratified convection. Left: $Rm \approx 360$. Right: $Rm \approx 480$.

necessary to average over a significant fraction of an ohmic decay time in order to obtain a reliable estimate for the saturation level in this case. As described in Bushby *et al.* (2010), a highly-stratified calculation at $Rm \approx 350$ produces a dynamo in which the magnetic energy appears to be saturating at a similar level. However, due to fluctuations in the magnetic energy, it is unclear whether or not the dynamo has truly saturated in this highly-stratified case, so we will not attempt to draw any conclusions on the basis of this particular comparison.

Figure 4 shows probability density functions (PDFs) for the vertical magnetic field component at the upper surface, in moderately-stratified convection, at $Rm \approx 360$ and $Rm \approx 480$. In these PDFs, the strength of the magnetic field has been scaled by the value of B_z that would be in energy equipartition with the local convective motions at the surface of the domain. Both PDFs are consistent with a stretched exponential distribution, centred at $B_z = 0$. The peak fields that are obtained are roughly four times the equipartition value. This can be related to the quiet Sun, where the observed kilogauss-strength magnetic features exceed the equipartition field strength by a similar multiplicative factor. The processes which lead to the formation of such strong magnetic fields are discussed elsewhere (see, for example, Bushby *et al.* 2008), so will not be discussed in any detail here. The only thing that we note is that these super-equipartition magnetic features can only be in near-pressure balance with their non-magnetic surroundings if they are partially-evacuated.

4. Conclusions

These results clearly demonstrate that compressible convection in an electrically-conducting fluid can drive a small-scale dynamo, provided that the magnetic Reynolds number is large enough. For the two models of convection that are considered in this paper, which both have a Reynolds number of $Re \approx 150$, dynamo action is possible provided that Rm exceeds a critical value of $Rm_{crit} \approx 325$. In the kinematic regime, the growth rate of the magnetic energy appears to have a logarithmic dependence upon Rm. Furthermore, if the growth rates are scaled in terms of the convective turnover time, the result seems to be independent of the level of stratification, with all (scaled) growth rates appearing to lie close to the same curve, $\Gamma = 0.36 \log_e(Rm/325)$. Hence, although the flow structure will clearly be sensitive to the level of stratification within the domain, it appears to be solely the convective turnover time that is responsible for determining the growth rate of any dynamo. As in previous calculations (see, for example, Cattaneo 1999; Vögler & Schüssler 2007) the growth rate at large values of Rm is of the same order as the convective turnover time. There is no evidence from these kinematic results that high levels of stratification hinder the operation of convectively-driven dynamos. Therefore, the lack of dynamo action in the calculations of Stein, Bercik & Nordlund (2003) is probably not associated with the vertical asymmetry of the convective motions. In this context, it should be noted that the calculations of Stein *et al.* (2003) have an open (rather than an impermeable) lower boundary that allows magnetic flux to leave the domain. However, the penetrative dynamo calculations of Brummell *et al.* (2010) strongly suggest that an open lower boundary condition should not prohibit dynamo action. So, perhaps the lack of dynamo action in the calculations of Stein et al. (2003) was simply a consequence of Rm being too small.

In the nonlinear regime, the high magnetic pressure leads to the partial evacuation of the upper regions of the vertical magnetic flux concentrations. With our existing explicit code, this leads to numerical time-step constraints that rapidly become prohibitive in the case of highly-stratified convection. Only for a marginally excited dynamo, has it been possible to carry out a nonlinear dynamo calculation in the highly-stratified case, and even there it was necessary to impose an artificial lower bound on the minimum density within the computational domain. An implicit code would be needed to address this parameter regime in a more satisfactory manner. For moderately-stratified convection, on the other hand, it has been possible to carry out nonlinear dynamo simulations. At moderate values of Rm, the dynamo saturates in the nonlinear regime, reaching a state in which the total magnetic energy is a few percent of the total kinetic energy. At the surface of the computational domain, the partially-evacuated magnetic features achieve super-equipartition field strengths, as observed in the quiet Sun.

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Discussion

E. Zweibel: Convective collapse seems to be playing an important role in making the strongest fields (as a dynamo – yesterday's talk).

P. Bushby: Yes, we do observe an intensification process of this type, although we see more of a convective adjustment rather than a well-defined convective collapse instability.

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Certainly, without the associated reduction in the internal gas pressure, it would be impossible to generate the observed super-equipartition fields at the upper surface of the domain.

G. Vasil: What type of numerical method did you use to solve the equations?

P. Bushby: Horizontal derivatives are evaluated in Fourier space, whilst fourth-order finite differences are used to calculate the vertical derivatives. An explicit third-order Adams-Bashforth scheme is used to time-step the equations.

S. Tobias: You were very diplomatic about avoiding the argument as to whether putting more scale heights into a dynamo calculation "switches off" the dynamo. Do you have a comment on this?

P. Bushby: From these calculations, there is no evidence to suggest that an increased level of stratification will inhibit the dynamo.

D. Hughes: Would one expect very different levels of saturation for the compressible dynamo as compared to the Boussinesq model? What determines the peak fields in the compressible case?

P. Bushby: That is a difficult question to answer at the moment, as we have not yet reached high enough values of Rm to make a direct comparison with the Boussinesq calculation of Cattaneo (1999). Certainly it is feasible that the effects of magnetic pressure will have some influence upon the saturation level of the dynamo in the compressible case. Where significant partial evacuation occurs in the compressible case, the local magnetic pressure within a vertical flux concentration is comparable to the external gas pressure. Although the convective motions also play a role in the confinement of the flux concentrations, this total pressure balance sets an (approximate) upper limit on the peak magnetic fields that can be generated in the upper layers of the domain.

N. Brummell: Raising Rm having fixed other parameters implies changing Pm. Comments?

P. Bushby: Yes, that's right. For $Re \approx 150$, a marginal dynamo has a magnetic Prandtl number of $Pm \approx 2$. At fixed Re, $Pm \propto Rm$, so wherever dynamo action is found in these simulations, $Pm > 2$.

A. Brandenburg: For a small-scale dynamo one expects the Kazantsev scaling, where magnetic energy increases with wavenumber, k, like $k^{3/2}$, reaching a peak at the resistive scale k_η. The growth rate should then scale with the turnover rate at that scale, so it should scale with Rm like $Rm^{1/2}$ (see Haugen et. al 2004, PRE; Käpylä et. al 2008).

P. Bushby: Although the logarithmic fit to the growth rate curve is empirical, it seems to be fairly convincing, with a large number of data points spread over a significant range of values for Rm. The scalings that you refer to both seem to cover a small number of data points over a narrow range of magnetic Reynolds numbers. It's not clear to me that we should expect to see a Kazantsev-like scaling for dynamo action in compressible convection. Furthermore, I believe that you need $Pm \ll 1$ in order to justify this $Rm^{1/2}$ scaling. This is not the case here.