

Well-posedness for nonlinear SPDEs with strongly continuous perturbation

Guy Vallet

LMAP UMR CNRS 5142, IPRA BP 1155, 64013 Pau Cedex, France
(guy.vallet@univ-pau.fr)

Aleksandra Zimmermann

Faculty of Mathematics, Thea-Leymann-Str. 9, 45127 Essen, Germany
(aleksandra.zimmermann@uni-due.de)

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We consider the well-posedness of a stochastic evolution problem in a bounded Lipschitz domain $D \subset \mathbb{R}^d$ with homogeneous Dirichlet boundary conditions and an initial condition in $L^2(D)$. The main technical difficulties in proving the result of existence and uniqueness of a solution arise from the nonlinear diffusion-convection operator in divergence form which is given by the sum of a Carathéodory function satisfying p -type growth associated with coercivity assumptions and a Lipschitz continuous perturbation. In particular, we consider the case $1 < p < 2$ with an appropriate lower bound on p determined by the space dimension. Another difficulty arises from the fact that the additive stochastic perturbation with values in $L^2(D)$ on the right-hand side of the equation does not inherit the Sobolev spatial regularity from the solution as in the multiplicative noise case.

Keywords: Nonlinear SPDE; strongly continuous perturbation; additive stochastic forcing; pseudomonotone SPDE

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1. Introduction

1.1. Statement of the problem

Let (Ω, \mathcal{F}, P) be a complete, countably generated probability space (e.g. the classical Wiener space), $D \subset \mathbb{R}^d$ be a bounded Lipschitz domain, $T > 0$, $Q_T := (0, T) \times D$. We are interested in a result of existence and uniqueness of the solution to

$$\begin{aligned} du - \operatorname{div}(a(x, u, \nabla u) + F(u)) dt &= \Phi dW(t) && \text{in } \Omega \times Q_T, \\ u &= 0 && \text{on } \Omega \times (0, T) \times \partial, \\ u(0, \cdot) &= u_0 && \in L^2(D). \end{aligned} \quad (1.1)$$

The nonlinear diffusion-convection operator of Leray–Lions’ type is defined as the sum of a Carathéodory function $a : D \times \mathbb{R}^{d+1} \rightarrow \mathbb{R}^d$ satisfying appropriate growth and coercivity assumptions which will be given below and $F : \mathbb{R} \rightarrow \mathbb{R}^d$ Lipschitz continuous with Lipschitz constant $L > 0$, such that $F(0) = 0$. On the right-hand side,

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$\Phi \in L^2(\Omega; \mathcal{C}([0, T]; HS(L^2(D))))$ is progressively measurable, where $HS(L^2(D))$ is the space of Hilbert–Schmidt operators from $L^2(D)$ to $L^2(D)$. W is a cylindrical Wiener process in $L^2(D)$ with respect to a filtration $(\mathcal{F}_t)_{t \in [0, T]}$, denoted (\mathcal{F}_t) in the sequel, satisfying the usual assumptions. More precisely, W is defined in the following sense (see, e.g. [3]): for a fixed orthonormal basis $(e_n)_{n \in \mathbb{N}^*}$ of $L^2(D)$ and a sequence of independent, real-valued (\mathcal{F}_t) -Brownian motions $(\beta_n)_{n \in \mathbb{N}^*}$, we define for any $t \in [0, T]$ and $u \in L^2(D)$, $\langle W(t), u \rangle := \sum_{n=1}^{\infty} (e_n, u)_{L^2(D)} \beta_n(t)$. One can check (see, e.g. [3, p. 85]) that for any $u \in L^2(D)$, $\langle W, u \rangle$ is a real-valued, (\mathcal{F}_t) -Wiener process such that $\mathbb{E}[\langle W(t), u \rangle \langle W(s), v \rangle] = \min(t, s) (u, v)_{L^2(D)}$ for any $u, v \in L^2(D)$, for all $s, t \in [0, T]$. W can be represented in the following way: for any sequence $(a_n) \in l^1(\mathbb{R}^+)$ it is easy to check that $W(t) := \sum_{n=1}^{\infty} \sqrt{a_n} (e_n / \sqrt{a_n}) \beta_n(t)$ is a Q -Wiener process with positive definite, symmetric and nuclear covariance operator $Q = \text{diag}(a_n)$ in the (bigger) Hilbert space

$$U := Q^{-1/2}(L^2(D)) \tag{1.2}$$

which is obtained as the completion of $L^2(D)$ with respect to the norm $\|\cdot\|_U$ induced by the scalar product $(u, v)_U := (Q^{1/2}u, Q^{1/2}v)_{L^2(D)}$. However, the stochastic integral $\int_0^t \Phi \, dW(s)$, $t \in [0, T]$, can be defined independently of the representation of W and the choice of Q by $\Phi \, dW(t) := \sum_{n=1}^{\infty} \Phi(e_n) \, d\beta_n(t)$.

1.2. Motivation and former results

The technical novelties of this contribution arise from the stochastic forcing of the nonlinear diffusion-convection operator $u \mapsto -\text{div}(a(x, u, \nabla u) + F(u))$. The diffusion part $u \mapsto -\text{div} a(x, u, \nabla u)$ is a monotone operator with p -growth and coercivity conditions for $2d/(d+1) < p < \infty$ where $d \in \mathbb{N}^*$ is the space dimension (see § 1.3 for more details). In particular, the assumptions include a class of p -Laplacian operators with $1 < p < 2$, p appropriately bounded away from 1. In contrast, the convection part $u \mapsto -\text{div} F(u)$ is, in general, not monotone but strongly continuous. It is well-known that a semigroup representation of the solution in the sense of [3] is not available in this nonlinear case. Moreover, classical well-posedness theory for monotone SPDEs (see, e.g. [11]) and for locally monotone SPDEs (see [10]) does not apply for a general perturbation of the type $-\text{div} F(u)$ for Lipschitz continuous $F : \mathbb{R} \rightarrow \mathbb{R}^d$. Therefore, our aim is to show well-posedness results by using a semi-implicit Euler–Maruyama time discretization (i.e. implicit in the operator part and explicit in the noise one). From the *a priori* estimates, we get weak convergences of approximate solutions with respect to (ω, t, x) , but compactness arguments can be applied uniquely with respect to the variables (t, x) . It is therefore not possible to identify the limit of the convection part of the operator. However, it is possible to show the existence of a martingale solution by adapting argumentations based on convergence in law and Skorokhod’s representation theorem. Then, since a pathwise uniqueness result can be obtained by using a L^1 -contraction principle, the existence of a strong solution follows by the argument of convergence in probability of Gyöngy and Krylov [8]. Those techniques are well-known for evolution problems with a multiplicative stochastic perturbation and have been applied in, e.g. [4, 6] and by many other authors in the last few decades.

In our case, this method allows to pass to the limit in the nonlinear convection operator. Taking advantage of the additive character of the stochastic perturbation, we can recover the stochastic integral by using a result detailed by Debussche *et al.* in [4]. Since the stochastic integral takes values in $L^2(D)$, it does not inherit spatial Sobolev regularity of the solution. Therefore, the compactness arguments are more subtle than in the case of a multiplicative stochastic perturbation as considered in [17] for the operator $u \mapsto -\operatorname{div}(|\nabla u|^{p-2}\nabla u + F(u))$ with $p > 2$.

1.3. Assumptions on the nonlinear operator

The vector field $a : D \times \mathbb{R} \times \mathbb{R}^d \rightarrow \mathbb{R}^d$, $(x, \lambda, \xi) \mapsto a(x, \lambda, \xi)$ is a Carathéodory function in the sense that the mapping $(\lambda, \xi) \mapsto a(x, \lambda, \xi)$ is continuous for almost every $x \in D$ and $x \mapsto a(\cdot, \lambda, \xi)$ is measurable for every $(\lambda, \xi) \in \mathbb{R} \times \mathbb{R}^d$. In the following, let $2d/(d + 1) < p < \infty$ and $p' = p/(p - 1)$. We impose the following conditions:

(A1) a is monotone with respect to its last variable only,

$$[a(x, \lambda, \xi) - a(x, \lambda, \eta)] \cdot (\xi - \eta) \geq 0$$

for all $\lambda \in \mathbb{R}$, $\xi, \eta \in \mathbb{R}^d$ and almost every $x \in D$.

(A2) There exists $\kappa \in L^1(D)$, some constants $C_a^1 > 0$, $C_a^2 \geq 0$, $C_a^3 \geq 0$ and a nonnegative function $g \in L^{p'}(D)$ such that

$$a(x, \lambda, \xi) \cdot \xi \geq \kappa(x) + C_a^1|\xi|^p, \tag{1.3}$$

$$|a(x, \lambda, \xi)| \leq C_a^2|\xi|^{p-1} + C_a^3|\lambda|^{p-1} + g(x) \tag{1.4}$$

for all $\lambda \in \mathbb{R}$, $\xi \in \mathbb{R}^d$ and almost every $x \in D$.

(A3) There exists a constant $C_a^4 \geq 0$ and a nonnegative function $h \in L^{p'}(D)$ such that

$$|a(x, \lambda_1, \xi) - a(x, \lambda_2, \xi)| \leq [C_a^4|\xi|^{p-1} + h(x)]|\lambda_1 - \lambda_2|$$

for all $\lambda_1, \lambda_2 \in \mathbb{R}$, for all $\xi \in \mathbb{R}^d$ and almost all $x \in D$.

From assumptions (A1)–(A2) it follows that the nonlinear operator

$$\mathcal{A} : W_0^{1,p}(D) \rightarrow W^{-1,p'}(D), \quad u \mapsto \mathcal{A}(u) := -\operatorname{div} a(x, u, \nabla u)$$

is well-defined, hemicontinuous and pseudomonotone (see [12, lemma 2.32]). From assumption (1.3) of (A2), it follows that \mathcal{A} is coercive.

Since $p > 2d/(d + 1) > 2d/(d + 2)$, this implies that $p^* := dp/(d - p) > 2$ when $p < d$ and $W_0^{1,p}(D)$ is compactly embedded into $L^2(D)$. In addition, the stricter condition $p > 2d/(d + 1)$ implies that $p' < p^*$. Therefore, $L^{p^*}(D)$ is continuously embedded in $L^{p'}(D)$ and the operator $u \in W_0^{1,p}(D) \mapsto -\operatorname{div}(a(x, u, \nabla u) + F(u))$ is well defined. An example of an operator satisfying (A1)–(A3) is given by $\mathcal{A}(u) = -\operatorname{div}(\varphi(x, u)|\nabla u|^{p-2}\nabla u)$ for a strictly positive, Lipschitz continuous (with respect to the second variable) and bounded function φ .

1.4. Strong solutions

Let us recall the notion of strong solution to (1.1):

DEFINITION 1.1. A (strong) solution to (1.1) is a predictable process $u \in L^2(\Omega; \mathcal{C}([0, T]; L^2(D))) \cap L^p(\Omega; L^p(0, T; W_0^{1,p}(D)))$ such that $u(0, \cdot) = u_0$ in $L^2(D)$ and

$$u(t) - u_0 - \int_0^t \operatorname{div}(a(x, u, \nabla u) + F(u)) \, ds = \int_0^t \Phi \, dW(s),$$

in $L^2(D)$ for all $t \in [0, T]$, a.s. in Ω .

1.5. Main result and outline

THEOREM 1.1. For any $u_0 \in L^2(D)$ and any progressively measurable $\Phi \in L^2(\Omega; \mathcal{C}([0, T]; HS(L^2(D))))$ there exists a strong solution to (1.1).

The proof of theorem 1.1 is contained in the following section. It is based on an approximation procedure by a time discretization of (1.1) introduced in §2.1. Since there is a lack of compactness with respect to $\omega \in \Omega$, we use the theorems of Prokhorov and Skorokhod to obtain a.s. the convergence of the sequence of approximate solutions [2].

2. Proof of theorem 1.1

2.1. Time discretization

For $N \in \mathbb{N}^*$, let $0 = t_0 < t_1 < \dots < t_N = T$ be an equidistant subdivision of the interval $[0, T]$ with $\tau := T/N = t_{k+1} - t_k$ for all $k = 0, \dots, N - 1$. For $u_0 \in L^2(D)$, $u^0 = u_0^\tau$ given by lemma A.1 and $\Phi_k := \Phi(t_k)$ for $k = 0, \dots, N - 1$, we introduce the semi-implicit Euler–Maruyama scheme

$$u^{k+1} - u^k - \tau \operatorname{div}(a(x, u^{k+1}, \nabla u^{k+1}) + F(u^{k+1})) = \Phi_k \Delta_{k+1} W, \tag{2.1}$$

where $\Phi_k \Delta_{k+1} W := \Phi(t_k)(W(t_{k+1}) - W(t_k))$ for $k = 0, \dots, N - 1$.

LEMMA 2.1. For any $\tau > 0$ and any given \mathcal{F} -measurable $u^k \in L^2(D)$, there exists a unique \mathcal{F} -measurable function $u^{k+1} : \Omega \rightarrow W_0^{1,p}(D)$ satisfying (2.1) in $L^2(D)$, for a.e. $\omega \in \Omega$. If moreover u^k is \mathcal{F}_{t_k} -measurable, then u^{k+1} is $\mathcal{F}_{t_{k+1}}$ -measurable.

Proof. The operator $A_\tau : W_0^{1,p}(D) \rightarrow W^{-1,p'}(D)$ defined by

$$\langle A_\tau(u), v \rangle_{W^{-1,p'}, W_0^{1,p}} := (u, v)_2 + \tau \int_D (a(x, u, \nabla u) + F(u)) \cdot \nabla v \, dx$$

for $u, v \in W_0^{1,p}(D)$ is a pseudomonotone operator, since it is the sum of a pseudomonotone operator and a strongly continuous perturbation (see, e.g. [12, corollary 2.12, §2.4.3]). Thanks to (A2), condition (1.3) and Gauss–Green’s

theorem, A_τ is coercive, i.e.

$$\lim_{\|u\|_{W_0^{1,p}(D)} \rightarrow \infty} \frac{\langle A_\tau(u), u \rangle_{W^{-1,p'}(D), W_0^{1,p}(D)}}{\|u\|_{W_0^{1,p}(D)}} = +\infty.$$

Let $V \subset W_0^{1,p}(D)$ be bounded, i.e. $\sup_{u \in V} \|u\|_{W_0^{1,p}(D)} \leq C_V$ for some constant $C_V \geq 0$. In the following, $C_i \geq 0$, for $i = 1, 2, 3 \dots$ will be constants. For any $u \in V$ and any $v \in W_0^{1,p}(D)$ such that $\|v\|_{W_0^{1,p}(D)} \leq 1$ we have

$$|\langle A_\tau(u), v \rangle_{W^{-1,p'}, W_0^{1,p}}| \leq I_1 + I_2 + I_3. \tag{2.2}$$

Using Cauchy–Schwarz inequality and the continuous embedding of $W_0^{1,p}(D)$ into $L^2(D)$ for $p \geq 2d/(d + 1)$, we have

$$I_1 = |(u, v)_2| \leq \|u\|_2 \|v\|_2 \leq C_1 \|u\|_{W_0^{1,p}(D)} \|v\|_{W_0^{1,p}(D)} \leq C_1 C_V. \tag{2.3}$$

From (1.4) of (A2), Young’s and Poincaré’s inequalities it follows that

$$\begin{aligned} I_2 &= \tau \int_D |a(x, u, \nabla u) \cdot \nabla v| \, dx \\ &\leq \frac{\tau}{p'} C_2 \int_D |\nabla u|^p + |u|^p + g(x)^{p'} \, dx + \frac{\tau}{p} \|v\|_{W_0^{1,p}(D)}^p \\ &\leq \frac{\tau}{p'} C_2 (\|u\|_{W_0^{1,p}(D)}^p + \|g\|_{p'}^{p'}) + \frac{\tau}{p} \leq C_3 (C_V^p + \|g\|_{p'}^{p'} + 1). \end{aligned} \tag{2.4}$$

We recall that thanks to the condition $p > 2d/(d + 1)$, it follows that $L^{p^*}(D)$ is continuously embedded into $L^{p'}(D)$ and therefore we can estimate

$$\begin{aligned} I_3 &= \tau \int_D |F(u)| |\nabla v| \, dx \leq \frac{\tau}{p'} \int_D |F(u)|^{p'} \, dx + \frac{\tau}{p} \|v\|_{W_0^{1,p}(D)}^p \\ &\leq \frac{\tau}{p'} L^{p'} \|u\|_{p'}^{p'} + \frac{\tau}{p} \leq C \|u\|_{p^*}^{p'} + \frac{\tau}{p} \leq C_4 (\|u\|_{W_0^{1,p}(D)}^p + 1) \leq C_4 (C_V^p + 1), \end{aligned} \tag{2.5}$$

where $L > 0$ is the Lipschitz constant of F . From equations (2.2)–(2.5) it follows that A_τ is bounded and, by Brezis’ theorem (see, e.g. [12, theorem 2.6 on p. 33]), A_τ is onto $W^{-1,p'}(D)$. In order to show that A_τ is injective, we fix $f \in W^{-1,p'}(D)$ and assume that there exist $u_1, u_2 \in W_0^{1,p}(D)$ such that $A_\tau(u_1) = A_\tau(u_2) = f$ in $W^{-1,p'}(D)$. Then, for the non-decreasing Lipschitz continuous approximation of the sign function $(\text{sign}_\delta)_{\delta \geq 0}$ defined for all $r \in \mathbb{R}$ by $\text{sign}_\delta(r) = \max(-1, \min(1, r/\delta))$, the chain rule for Sobolev functions yields $J_1 + J_2 + J_3 = 0$, where

$$\begin{aligned} J_1 &= \int_D (u_1 - u_2) \text{sign}_\delta(u_1 - u_2) \, dx, \\ J_2 &= \int_D \text{sign}'_\delta(u_1 - u_2) (a(x, u_1, \nabla u_1) - a(x, u_2, \nabla u_2)) \cdot \nabla(u_1 - u_2) \, dx, \\ J_3 &= \int_D \text{sign}'_\delta(u_1 - u_2) (F(u_1) - F(u_2)) \cdot \nabla(u_1 - u_2) \, dx. \end{aligned}$$

It follows immediately that $\lim_{\delta \rightarrow 0^+} J_1 = \int_D |u_1 - u_2| dx$. Now we write $J_2 = J_{2,1} + J_{2,2}$, where, by (A1),

$$J_{2,1} = \int_D \text{sign}'_{\delta}(u_1 - u_2)(a(x, u_1, \nabla u_1) - a(x, u_1, \nabla u_2)) \cdot \nabla(u_1 - u_2) dx \geq 0$$

and by (A3)

$$\begin{aligned} |J_{2,2}| &\leq \int_D \text{sign}'_{\delta}(u_1 - u_2) |a(x, u_1, \nabla u_2) - a(x, u_2, \nabla u_2)| |\nabla(u_1 - u_2)| dx \\ &\leq \frac{1}{\delta} \int_{\{|u_1 - u_2| < \delta\}} (C_a^4 |\nabla u_2|^{p-1} + h(x)) |u_1 - u_2| |\nabla(u_1 - u_2)| dx \\ &\leq C \left(\int_D |\nabla u_2|^p + h(x)^{p'} dx \right)^{1/p'} \left(\int_{\{|u_1 - u_2| < \delta\}} |\nabla(u_1 - u_2)|^p dx \right)^{1/p}. \end{aligned}$$

Since $\nabla(u_1 - u_2) = 0$ a.e. in $\{u_1 = u_2\}$ for Sobolev functions, $\lim_{\delta \rightarrow 0^+} |J_{2,2}| = 0$ and $\limsup_{\delta \rightarrow 0^+} J_2 \geq 0$. Since F is Lipschitz continuous with Lipschitz constant $L > 0$, we have

$$|J_3| \leq \frac{1}{\delta} \int_{\{|u_1 - u_2| < \delta\}} L |u_1 - u_2| |\nabla(u_1 - u_2)| dx,$$

thus $\lim_{\delta \rightarrow 0^+} J_3 = 0$. Combining the above results, we obtain $u_1 = u_2$.

It is left to show that $A_{\tau}^{-1} : W^{-1,p'}(D) \rightarrow W_0^{1,p}(D)$ is demi-continuous. For $f \in W^{-1,p'}(D)$ and u such that $A_{\tau}(u) = f$, using Gauss–Green’s theorem on the convection term, we get $\langle f, u \rangle_{W^{-1,p'}(D), W_0^{1,p}(D)} = \|u\|_2^2 + \tau \int_D a(x, u, \nabla u) \cdot \nabla u dx$.

By (A2), $\int_D a(x, u, \nabla u) \cdot \nabla u dx \geq -\|\kappa\|_1 + C_a^1 \|\nabla u\|_p^p$ and therefore, using Young’s inequality, for any $\delta > 0$ we get

$$\begin{aligned} \|u\|_2^2 - \tau \|\kappa\|_1 + \tau C_a^1 \|\nabla u\|_p^p &\leq \langle f, u \rangle_{W^{-1,p'}(D), W_0^{1,p}(D)} \\ &\leq \frac{C_{\tau,\delta}}{p'} \|f\|_{W^{-1,p'}(D)}^{p'} + \frac{\tau \delta}{p} \|\nabla u\|_p^p \end{aligned} \tag{2.6}$$

with a constant $C_{\tau,\delta} \geq 0$. Let $(f_n) \subset W^{-1,p'}(D)$ be a sequence converging to f in $W^{-1,p'}(D)$. For all $n \in \mathbb{N}^*$, we define

$$u_n := A_{\tau}^{-1}(f_n). \tag{2.7}$$

From (2.6) it follows that there exists a not relabelled subsequence of (u_n) , $u \in W_0^{1,p}(D)$ and B in $L^{p'}(D)^d$ such that $u_n \rightharpoonup u$ in $W_0^{1,p}(D)$, $u_n \rightarrow u$ in $L^2(D)$ thanks to $p^* > 2$ and $a(x, u_n, \nabla u_n) \rightharpoonup B$ in $L^{p'}(D)^d$ for $n \rightarrow \infty$. Using these convergence

results and (2.7), we get

$$\begin{aligned} & \|u\|_2^2 + \tau \limsup_{n \rightarrow \infty} \int_D a(x, u_n, \nabla u_n) \cdot \nabla u_n \, dx \\ &= \langle f, u \rangle_{W^{-1,p'}(D), W_0^{1,p}(D)} = \|u\|_2^2 + \tau \int_D B \cdot \nabla u \, dx. \end{aligned} \tag{2.8}$$

Thus, from (2.8) it follows that

$$\begin{aligned} & \limsup_{n \rightarrow \infty} \langle A_\tau(u_n), u_n - u \rangle_{W^{-1,p'}(D), W_0^{1,p}(D)} \\ &= \limsup_{n \rightarrow \infty} \int_D a(x, u_n, \nabla u_n) \cdot \nabla u_n \, dx - \int_D B \cdot \nabla u \, dx = 0 \end{aligned} \tag{2.9}$$

and since A_τ is pseudomonotone, (2.9) implies $A_\tau u = f$.

Consequently, $u_n = A_\tau^{-1}(f_n) \rightharpoonup A_\tau^{-1}(f) = u$ in $W_0^{1,p}(D)$, a priori for a subsequence. Since u is unique, it follows that the whole sequence (u_n) converges to u weakly in $W_0^{1,p}(D)$ for $n \rightarrow \infty$ and A_τ^{-1} is demi-continuous.

By assumption, $\Phi_k \Delta_{k+1} W + u^k$ is \mathcal{F} (resp. $\mathcal{F}_{t_{k+1}}$) measurable, thus

$$u_{k+1} = A_\tau^{-1}(\Phi_k \Delta_{k+1} W + u^k)$$

is weakly \mathcal{F} (resp. $\mathcal{F}_{t_{k+1}}$) measurable, thus measurable by the theorem of Pettis (see [18, V. 4, p. 131]) since $W_0^{1,p}(D)$ is separable. The lemma is proved. \square

2.2. Estimates

LEMMA 2.2. For $u_0 \in L^2(D)$, and $k = 0, \dots, N - 1$, let u^{k+1} be the solution to (2.1). Then,

$$\begin{aligned} & \frac{1}{2} \mathbb{E} (\|u^{k+1}\|_2^2 - \|u^k\|_2^2) + \frac{1}{4} \mathbb{E} \|u^{k+1} - u^k\|_2^2 + \tau C_a^1 \mathbb{E} \|\nabla u^{k+1}\|_p^p \\ & \leq \tau \|\kappa\|_1 + \tau \mathbb{E} \|\Phi_k\|_{HS(L^2(D))}^2. \end{aligned} \tag{2.10}$$

Proof. Taking u^{k+1} as a test function in (2.1), we get

$$\begin{aligned} & (u^{k+1} - u^k, u^{k+1})_2 - \tau \langle \operatorname{div}(a(x, u^{k+1}, \nabla u^{k+1}) + F(u^{k+1})), u^{k+1} \rangle_{W^{-1,p'}(D), W_0^{1,p}(D)} \\ &= (\Phi_k \Delta_{k+1} W, u^{k+1})_2 \Leftrightarrow I_1 + I_2 + I_3 = I_4 \end{aligned} \tag{2.11}$$

where $I_1 := (u^{k+1} - u^k, u^{k+1})_2 = \frac{1}{2} (\|u^{k+1}\|_2^2 - \|u^k\|_2^2 + \|u^{k+1} - u^k\|_2^2)$, and, using (A2), $I_2 := \tau \int_D a(x, u^{k+1}, \nabla u^{k+1}) \cdot \nabla u \, dx \geq -\tau \|\kappa\|_1 + \tau C_a^1 \|\nabla u\|_p^p$.

From Gauss–Green’s theorem it follows that

$$I_3 = \tau \int_D F(u^{k+1}) \cdot \nabla u^{k+1} \, dx = 0,$$

and we still have to estimate $I_4 = (\Phi_k \Delta_{k+1} W, u^{k+1} - u^k)_2 + (\Phi_k \Delta_{k+1} W, u^k)_2$. Combining (2.11) with the above estimates and taking expectation we arrive at

$$\begin{aligned} & \frac{1}{2} \mathbb{E} (\|u^{k+1}\|_2^2 - \|u^k\|_2^2 + \|u^{k+1} - u^k\|_2^2) + \tau C_a^1 \mathbb{E} \|\nabla u\|_p^p \\ & \leq \tau \|\kappa\|_1 + \mathbb{E}(\Phi_k \Delta_{k+1} W, u^{k+1} - u^k)_2 + \mathbb{E}(\Phi_k \Delta_{k+1} W, u^k)_2. \end{aligned}$$

Since u^k is \mathcal{F}_{t_k} -measurable and $W(t_{k+1}) - W(t_k)$ is \mathcal{F}_{t_k} -independent, we have

$$\mathbb{E}(\Phi_k \Delta_{k+1} W, u^k)_2 = \mathbb{E}(u^k, \mathbb{E}[\Phi_k(W(t_{k+1}) - W(t_k)) | \mathcal{F}_{t_k}])_2 = 0.$$

Using Hölder’s and Young’s inequalities it follows that for any $\alpha > 0$

$$\mathbb{E}(\Phi_k \Delta_{k+1} W, u^{k+1} - u^k)_2 \leq \frac{1}{2} \left(\frac{1}{\alpha} \mathbb{E} \left\| \int_{t_k}^{t_{k+1}} \Phi_k \, dW(t) \right\|_2^2 + \alpha \mathbb{E} \|u^{k+1} - u^k\|_2^2 \right). \tag{2.12}$$

By Itô’s isometry, setting $\alpha = \frac{1}{2}$ in (2.12) yields

$$\begin{aligned} & \mathbb{E}(\Phi_k \Delta_{k+1} W, u^{k+1} - u^k) \\ & \leq \mathbb{E} \int_{t_k}^{t_{k+1}} \|\Phi_k\|_{HS(L^2(D))}^2 \, dt + \frac{1}{4} \mathbb{E} \|u^{k+1} - u^k\|_2^2 \\ & = \tau \mathbb{E} \|\Phi_k\|_{HS(L^2(D))}^2 + \frac{1}{4} \mathbb{E} \|u^{k+1} - u^k\|_2^2, \end{aligned}$$

and therefore (2.10) holds. □

DEFINITION 2.1. For $N \in \mathbb{N}^*$, $\tau > 0$, we introduce the right-continuous step function

$$u_N^r(t) = \sum_{k=0}^{N-1} u^{k+1} \chi_{[t_k, t_{k+1})}(t), \quad t \in [0, T),$$

the left-continuous, (\mathcal{F}_t) -adapted step function

$$\Phi_N(t) = \sum_{k=0}^{N-1} \Phi_k \chi_{(t_k, t_{k+1}]}(t), \quad t \in (0, T],$$

the continuous, square-integrable (\mathcal{F}_t) -martingale

$$M_N(t) = \int_0^t \Phi_N \, dW(s), \quad t \in [0, T]$$

and the piecewise affine functions

$$\widehat{u}_N(t) := \sum_{k=0}^{N-1} \left(\frac{u^{k+1} - u^k}{\tau} (t - t_k) + u^k \right) \chi_{[t_k, t_{k+1})}(t), \quad t \in [0, T), \quad \widehat{u}_N(T) = u^N,$$

$$\widehat{M}_N(t) = \sum_{k=0}^{N-1} \left(\frac{M_N(t_{k+1}) - M_N(t_k)}{\tau} (t - t_k) + M_N(t_k) \right) \chi_{[t_k, t_{k+1})}(t), \quad t \in [0, T].$$

LEMMA 2.3. *There exist a generic constants $K_i \geq 0$, $i = 1, \dots, 6$ not depending on the discretization parameters such that*

$$\sum_{k=0}^{N-1} \mathbb{E} \|u^{k+1} - u^k\|_2^2 \leq K_1, \tag{2.13}$$

$$\max_{n=0, \dots, N} \mathbb{E} \|u^n\|_2^2 \leq K_2, \tag{2.14}$$

$$\mathbb{E} \int_0^T \|u_N^r\|_2^2 dt \leq K_3 \quad \text{and} \quad \mathbb{E} \int_0^T \|\widehat{u}_N\|_2^2 dt \leq K_4, \tag{2.15}$$

$$\mathbb{E} \int_0^T \|\nabla u_N^r\|_p^p dt \leq K_5 \quad \text{and} \quad \mathbb{E} \int_0^T \|\nabla \widehat{u}_N\|_p^p dt \leq K_6. \tag{2.16}$$

Proof. We fix $n \in \{1, \dots, N\}$, take the sum over $0, \dots, n - 1$ in (2.10) to get

$$\begin{aligned} & \frac{1}{2} \mathbb{E} \|u^n\|_2^2 - \frac{1}{2} \mathbb{E} \|u^0\|_2^2 + \frac{1}{4} \sum_{k=0}^{n-1} \mathbb{E} \|u^{k+1} - u^k\|_2^2 + C_a^1 \sum_{k=0}^{n-1} \tau \mathbb{E} \|\nabla u^{k+1}\|_p^p \\ & \leq \sum_{k=0}^{n-1} \tau \mathbb{E} \|\Phi_k\|_{HS(L^2(D))}^2 + T \|\kappa\|_1. \end{aligned} \tag{2.17}$$

From (2.17) and lemma A.1 (see appendix) it follows that

$$\begin{aligned} & \frac{1}{2} \mathbb{E} \|u^n\|_2^2 + \frac{1}{4} \sum_{k=0}^{n-1} \mathbb{E} \|u^{k+1} - u^k\|_2^2 + C_a^1 \int_0^T \mathbb{E} \|\nabla u_N^r\|_p^p dt \\ & \leq \frac{1}{2} \mathbb{E} \|u_0\|_2^2 + T \mathbb{E} \|\Phi\|_{\mathcal{C}([0, T]; HS(L^2(D)))}^2 + T \|\kappa\|_1 \end{aligned}$$

and we get (2.13), (2.14) and the first inequality of (2.16). Now, from (2.14) it follows that $\mathbb{E} \int_0^T \|u_N^r\|_2^2 dt \leq T \max_{k=1, \dots, N} \mathbb{E} \|u^k\|_2^2 \leq K_3$. Since there exists a constant $C \geq 0$ such that $\mathbb{E} \int_0^T \|\widehat{u}_N\|_2^2 dt \leq C\tau \sum_{k=0}^N \mathbb{E} \|u^k\|_2^2$, we have shown both

inequalities of (2.15). Thanks to lemma A.1 it follows that

$$\begin{aligned} \mathbb{E} \int_0^T \|\nabla \hat{u}_N\|_p^p dt &= \mathbb{E} \sum_{k=0}^{N-1} \int_{t_k}^{t_{k+1}} \left\| \frac{\nabla u^{k+1} - \nabla u^k}{\tau} (t - t_k) + \nabla u^k \right\|_p^p dt \\ &\leq \mathbb{E} \sum_{k=0}^{N-1} \left(\int_{t_k}^{t_{k+1}} \frac{t - t_k}{\tau} \|\nabla u^{k+1}\|_p^p dt + \int_{t_k}^{t_{k+1}} \frac{t_{k+1} - t}{\tau} \|\nabla u^k\|_p^p dt \right) \\ &= \frac{\tau}{2} \mathbb{E} \sum_{k=0}^{N-1} (\|\nabla u^{k+1}\|_p^p + \|\nabla u^k\|_p^p) \leq C + \frac{\tau}{2} \|\nabla u^0\|_p^p \leq C \end{aligned}$$

and this yields the second inequality of (2.16). □

LEMMA 2.4. *There exists a constant $K \geq 0$ such that*

$$\mathbb{E} \max_{n=0, \dots, N} \|u^n\|_2^2 \leq K. \tag{2.18}$$

In particular, from (2.18) it follows that there exists a possibly different, but not re-labelled constant $K \geq 0$ such that $\mathbb{E} \sup_{t \in [0, T]} \|\hat{u}_N\|_2^2 < K$ and $\mathbb{E} \sup_{t \in [0, T]} \|u_N^r\|_2^2 < K$ for all $N \in \mathbb{N}^$.*

Proof. Taking u^{k+1} as a test function in (2.1), using (A2), Gauss–Green’s theorem on the convection term and Hölder’s and Young’s inequalities we get

$$\|u^{k+1}\|_2^2 - \|u^k\|_2^2 \leq 2\tau \|\kappa\|_1 + \|\Phi_k \Delta_{k+1} W\|_2^2 + 2(\Phi_k \Delta_{k+1} W, u^k)_2. \tag{2.19}$$

Summing over $k = 0, \dots, n - 1$ with $n \in \{1, \dots, N\}$ in (2.19) and applying we get

$$\|u^n\|_2^2 - \|u^0\|_2^2 \leq 2T \|\kappa\|_1 + \sum_{k=0}^{n-1} \|\Phi_k \Delta_{k+1} W\|_2^2 + 2 \sum_{k=0}^{n-1} (\Phi_k \Delta_{k+1} W, u^k)_2$$

and therefore, applying lemma A.1, taking the maximum over $0, \dots, n$ and then taking expectation we arrive at

$$\begin{aligned} \mathbb{E} \max_{n=0, \dots, N} \|u^n\|_2^2 &\leq \|u_0\|_2^2 + 2T \|\kappa\|_1 + \sum_{k=0}^{N-1} \mathbb{E} \|\Phi_k \Delta_{k+1} W\|_2^2 \\ &\quad + 2\mathbb{E} \left(\max_{n=1, \dots, N} \sum_{k=0}^{n-1} (\Phi_k \Delta_{k+1} W, u^k)_2 \right) \\ &= \|u_0\|_2^2 + 2T \|\kappa\|_1 + I_1 + I_2, \end{aligned} \tag{2.20}$$

where, by Itô’s isometry, $I_1 = \sum_{k=0}^{N-1} \mathbb{E} \|\int_{t_k}^{t_{k+1}} \Phi_k dW(t)\|_2^2 \leq T \mathbb{E} \|\Phi\|_{C([0, T]; HS(L^2(D)))}^2$. We have $\sum_{k=0}^{n-1} (\Phi_k \Delta_{k+1} W, u^k)_2 = \int_0^{t_n} (\Phi_N, u_N^l)_2 dW(t)$, where u_N^l is the piecewise

constant, left-continuous function defined by $u_N^l(t) = u^k$ for $t \in (t_k, t_{k+1}]$, $k = 0, \dots, N - 1$. Thus,

$$I_2 = 2\mathbb{E} \max_{n=1, \dots, N} \int_0^{t_n} (\Phi_N, u_N^l)_2 \, dW(t) \leq 2\mathbb{E} \max_{n=1, \dots, N} \left| \int_0^{t_n} (\Phi_N, u_N^l)_2 \, dW(t) \right|. \tag{2.21}$$

Now, by Burkholder’s and Hölder’s inequalities it follows from (2.21)

$$\begin{aligned} I_2 &\leq 6\mathbb{E} \left(\int_0^T \|(\Phi_N, u_N^l)_2\|_{HS(L^2(D); \mathbb{R})}^2 \, dt \right)^{1/2} \\ &\leq 6\mathbb{E} \left[\max_{n=0, \dots, N} \|u^n\|_2 \sqrt{T} \|\Phi\|_{C([0, T]; HS(L^2(D)))} \right]. \end{aligned} \tag{2.22}$$

Using Young’s inequality with $\alpha > 0$ from (2.22), it follows that

$$\begin{aligned} I_2 &\leq 3\alpha \mathbb{E} \max_{n=0, \dots, N} \|u^n\|_2^2 + \frac{3T}{\alpha} \mathbb{E} \|\Phi\|_{C([0, T]; HS(L^2(D)))}^2 \\ &\leq 3\alpha \mathbb{E} \max_{n=0, \dots, N} \|u^n\|_2^2 + 3\alpha \|u_0\|_2^2 + \frac{3T}{\alpha} \mathbb{E} \|\Phi\|_{C([0, T]; HS(L^2(D)))}^2. \end{aligned}$$

Plugging the estimates for I_1 and I_2 into (2.20), choosing $\alpha > 0$ such that $1 - 3\alpha > 0$, the assertion follows. \square

LEMMA 2.5. *There exists $K \geq 0$ not depending on $N \in \mathbb{N}^*$ such that*

$$\mathbb{E} \int_0^T \left\| \frac{d}{dt} (\widehat{u}_N - \widehat{M}_N) \right\|_{W^{-1, p'}(D)}^{p'} \, dt \leq K. \tag{2.23}$$

Proof. We fix $k = 0, \dots, N - 1$. For all $t \in (t_k, t_{k+1})$,

$$\frac{d}{dt} (\widehat{u}_N - \widehat{M}_N) = \operatorname{div}(a(x, u^{k+1}, \nabla u^{k+1}) + F(u^{k+1})). \tag{2.24}$$

For any $v \in W_0^{1, p}(D)$, using (A2) we get

$$\begin{aligned} &\int_D |(a(x, u^{k+1}, \nabla u^{k+1}) + F(u^{k+1})) \cdot \nabla v| \, dx \\ &\leq \int_D (C_a^2 |\nabla u^{k+1}|^{p-1} + C_a^3 |u^{k+1}|^{p-1} + g(x) + L|u^{k+1}|) |\nabla v| \, dx = I_1 + I_2, \end{aligned} \tag{2.25}$$

where $L > 0$ is the Lipschitz constant of F . Thanks to Young’s and Poincaré’s inequalities, a positive constant C_1 exists such that

$$\begin{aligned} I_1 &= \int_D (C_a^2 |\nabla u^{k+1}|^{p-1} + C_a^3 |u^{k+1}|^{p-1} + g(x)) |\nabla v| \, dx \\ &\leq \frac{C_1}{p'} (\|\nabla u^{k+1}\|_p^p + \|g\|_{p'}^{p'}) + \frac{1}{p} \|v\|_{W_0^{1, p}(D)}^p. \end{aligned} \tag{2.26}$$

Since $p^* > p'$, $L^{p^*}(D) \subset L^{p'}(D)$ with continuous injection. Therefore, using Hölder's, Sobolev's and Young's inequalities, we get

$$I_2 = L \int_D |u^{k+1}| |\nabla v| \, dx \leq \frac{C_2}{p} \|\nabla u^{k+1}\|_p^p + \frac{1}{p'} \|v\|_{W_0^{1,p}(D)}^{p'} \tag{2.27}$$

for a constant $C_2 \geq 0$. Using (2.24), (2.25), (2.26) and (2.27) and taking expectation it follows that

$$\begin{aligned} \mathbb{E} \int_0^T \left\| \frac{d}{dt} (\widehat{u}_N - \widehat{M}_N) \right\|_{W^{-1,p'}(D)}^{p'} dt &\leq \frac{TC_3}{p'} \|g\|_{p'} + T + C_3 \mathbb{E} \tau \sum_{k=0}^{N-1} \|\nabla u^{k+1}\|_p^p \\ &= \frac{TC}{p'} \|g\|_{p'} + T + C \mathbb{E} \int_0^T \|\nabla u_N^r\|_p^p dt \end{aligned} \tag{2.28}$$

for a constant $C_3 \geq 0$. Now, the assertion follows from (2.16) of lemma 2.3. \square

LEMMA 2.6. *There exists a constant $K \geq 0$ such that*

$$\mathbb{E} \int_0^T \|u_N^r - \widehat{u}_N\|_2^2 dt \leq K\tau. \tag{2.29}$$

Proof. We have $\mathbb{E} \int_0^T \|u_N^r - \widehat{u}_N\|_2^2 dt = \tau/3 \sum_{k=0}^{N-1} \mathbb{E} \|u^{k+1} - u^k\|_2^2$ and the assertion follows from (2.13) of lemma 2.3. \square

LEMMA 2.7. *There exists $\gamma > 0$ and $K \geq 0$ not depending on $N \in \mathbb{N}^*$ such that*

$$\mathbb{E} \sup_{k \in \{0, \dots, N-1\}} \sup_{s \in [t_k, t_{k+1}]} \left\| \int_{t_k}^s \Phi_N \, dW(t) \right\|_{L^2(D)}^2 \leq K\tau^\gamma. \tag{2.30}$$

Proof. We fix $k \in \{0, \dots, N-1\}$ and $s \in [t_k, t_{k+1}]$. Then we have

$$\left\| \int_{t_k}^s \Phi_N \, dW(t) \right\|_{L^2(D)}^2 \leq C \|\Phi_k\|_{L^2(D)}^2 \|W(s) - W(t_k)\|_U^2, \tag{2.31}$$

where $C \geq 0$ is a constant that may change from line to line. Now, from [7], [14, example 2.4.1] for any $q \geq 1$ and $\alpha > 1/q$, it follows that

$$\begin{aligned} \|W(s) - W(t_k)\|_U &\leq C\tau^{\alpha-1/q} \left(\int_0^T \int_0^T \frac{\|W(t) - W(r)\|_U^q}{|t-r|^{\alpha q+1}} dt dr \right)^{1/q} \\ &= C\tau^{\alpha-1/q} X^{1/q} \end{aligned}$$

where U is defined in (1.2) and $X := \int_0^T \int_0^T (\|W(t) - W(r)\|_U^q) / |t-r|^{\alpha q+1} dt dr$ is a real-valued random variable. Consequently,

$$\|W(s) - W(t_k)\|_U^2 \leq C\tau^{2(\alpha-1/q)} X^{2/q}. \tag{2.32}$$

Plugging (2.32) into (2.31), we get

$$\begin{aligned} & \sup_{k \in \{0, \dots, N-1\}} \sup_{s \in [t_k, t_{k+1}]} \left\| \int_{t_k}^s \Phi_N \, dW(t) \right\|_{L^2(D)}^2 \\ & \leq C \left(\sup_{k \in \{0, \dots, N-1\}} \|\Phi_k\|_{HS(L^2(D))}^2 \right) \sup_{k \in \{0, \dots, N-1\}} \sup_{s \in [t_k, t_{k+1}]} \|W(s) - W(t_k)\|_U^2 \\ & \leq C\tau^\gamma X^{2/q} \sup_{t \in [0, T]} \|\Phi(t)\|_{L^2(D)}^2 \end{aligned} \tag{2.33}$$

for any $q \geq 1$, $\alpha > 1/q$, $\gamma := 2(\alpha - 1/q) > 0$. For $q > 2$ with Jensen’s inequality, it follows that

$$\mathbb{E}(X^{2/q}) \leq (\mathbb{E}(X))^{2/q} = \left(\int_0^T \int_0^T \frac{\mathbb{E}\|W(t) - W(r)\|_U^q}{|t - r|^{\alpha q + 1}} \, dt \, dr \right)^{2/q}. \tag{2.34}$$

Since $\mathbb{E}\|W(t) - W(r)\|_U^q \leq C|t - r|^{q/2}$, for some $c \geq 0$ one gets, for $q > 2$ and $\alpha \in (1/q, \frac{1}{2})$

$$\mathbb{E}(X) \leq C \int_0^T \int_0^T |t - r|^{q/2 - \alpha q - 1} \, dt \, dr < +\infty. \tag{2.35}$$

Taking expectation in (2.33) and using (2.34) and (2.35), (2.30) holds true. \square

As a consequence, we can estimate the difference between the piecewise affine and the time-continuous approximations of the stochastic integral:

LEMMA 2.8. *There exist constants $\gamma > 0$, $K \geq 0$ such that, for all $N \in \mathbb{N}^*$,*

$$\mathbb{E} \sup_{t \in [0, T]} \|M_N(t) - \widehat{M}_N(t)\|_2^2 \leq K\tau^\gamma. \tag{2.36}$$

Proof. We fix $N \in \mathbb{N}^*$. For $k \in \{0, \dots, N - 1\}$ and $t \in [t_k, t_{k+1})$

$$\begin{aligned} \|M_N(t) - \widehat{M}_N(t)\|_2^2 &= \left\| \int_{t_k}^t \Phi_N \, dW(s) - \frac{t - t_k}{\tau} \int_{t_k}^{t_{k+1}} \Phi_N \, dW(s) \right\|_2^2 \\ &\leq C \left\| \int_{t_k}^t \Phi_N \, dW(s) \right\|_2^2 + \left\| \int_{t_k}^{t_{k+1}} \Phi_N \, dW(s) \right\|_2^2 \end{aligned}$$

and therefore

$$\sup_{t \in [0, T]} \|M_N(t) - \widehat{M}_N(t)\|_2^2 \leq 2 \sup_{k=0, \dots, N-1} \sup_{t \in [t_k, t_{k+1})} \left\| \int_{t_k}^t \Phi_N \, dW(s) \right\|_2^2$$

and the assertion follows from lemma 2.7. \square

2.3. Convergence of approximate solutions

2.3.1. *Tightness* In the multiplicative noise case, the stochastic integral inherits the spatial Sobolev regularity from the solution. In the additive noise case, this is not possible and the compactness argument for processes with values in $L^2(D)$ is more subtle.

LEMMA 2.9. $\lim_{N \rightarrow \infty} M_N = \lim_{N \rightarrow \infty} \widehat{M}_N = \int_0^\cdot \Phi \, dW(t)$ in $L^2(\Omega; \mathcal{C}([0, T]; L^2(D)))$.

Proof. From Burkholder–Davies–Gundy’s inequality it follows that

$$\mathbb{E} \sup_{[0, T]} \left\| \int_0^t \Phi_N - \Phi \, dW(s) \right\|_2^2 \leq \mathbb{E} \int_0^T \|\Phi_N - \Phi\|_2^2 \, ds. \tag{2.37}$$

Since $\Phi_N \rightarrow \Phi$ for $N \rightarrow \infty$ in $L^2(\Omega; L^2(0, T; HS(L^2(D))))$, the right-hand side of (2.37) converges to 0 when $N \rightarrow \infty$. Now, the convergence result for (\widehat{M}_N) is a direct consequence of lemma 2.8. \square

LEMMA 2.10. For $N \in \mathbb{N}^*$, let μ_N be the law of

$$X_N := (\widehat{u}_N, \widehat{u}_N - \widehat{M}_N, M_N, \widehat{M}_N, \Phi_N, W).$$

Then, $(\mu_N)_{N \in \mathbb{N}^*}$ is a uniformly tight sequence.

Proof. By lemma 2.9, \widehat{M}_N converges to $\int_0^\cdot \Phi \, dW(s)$ in $L^2(\Omega; \mathcal{C}([0, T]; L^2(D)))$, thus also in law. Since $\mathcal{C}([0, T]; L^2(D))$ is a Polish space, from the theorem of Prokhorov, it follows that for any $\epsilon > 0$, there exists a compact set $K_\epsilon^1 \subset \mathcal{C}([0, T], L^2(D))$ such that $P[\widehat{M}_N \notin K_\epsilon^1] < \epsilon$, for all $N \in \mathbb{N}^*$. For the same reason, $P[M_N \notin K_\epsilon^1] < \epsilon$ for all $N \in \mathbb{N}^*$, for the same compact set. Since $\Phi_N \rightarrow \Phi$ in $L^2(\Omega; L^2(0, T; HS(L^2(D))))$ for $N \rightarrow \infty$, with similar arguments, it follows that for any $\epsilon > 0$, there exists a compact set $K_\epsilon^2 \subset L^2(0, T; HS(L^2(D)))$ such that $P[\Phi_N \notin K_\epsilon^2] < \epsilon$ for all $N \in \mathbb{N}^*$. Moreover, for any $\epsilon > 0$, there exists a compact set $K_\epsilon^3 \subset \mathcal{C}([0, T]; U)$, where U is defined in (1.2), such that $P[W \notin K_\epsilon^3] < \epsilon$. From lemmas 2.4 and 2.5, it follows that $\widehat{u}_N - \widehat{M}_N$ is bounded in $L^2(\Omega; \mathcal{C}([0, T]; L^2(D)))$ and $(d/dt)(\widehat{u}_N - \widehat{M}_N)$ is bounded in $L^{p'}(\Omega \times (0, T); W^{-1, p'}(D))$, thus $(\widehat{u}_N - \widehat{M}_N)$ is bounded in $L^{\min\{2, p'\}}(\Omega; \mathcal{W})$, where

$$\mathcal{W} := \left\{ u \in L^2(0, T; L^2(D)) \mid \frac{d}{dt} u \in L^{p'}(0, T; W^{-1, p'}(D)) \right\}$$

and this space is compactly embedded into $\mathcal{C}([0, T]; W^{-1, p'}(D))$ by [13, corollary 1]. Therefore, from Markov inequality, it follows that for any $\epsilon > 0$, there exists $M_\epsilon > 0$ and $K_\epsilon^4 = B_{\mathcal{W}}(0, M_\epsilon)$ such that $P[\widehat{u}_N - \widehat{M}_N \notin K_\epsilon^4] < \epsilon$, for all N , with the additional information that K_ϵ^4 is a relatively compact set in $\mathcal{C}([0, T]; W^{-1, p'}(D))$. Thanks to lemma 2.3, (\widehat{u}_N) is bounded in $L^p(\Omega, L^p(0, T; W_0^{1, p}(D)))$, thus again, using Markov inequality, we obtain: for any $\epsilon > 0$, there exists $M > 0$ such that $P[\|\widehat{u}_N\|_{L^p(0, T; W_0^{1, p}(D))} > M] < \epsilon$ for all $N \in \mathbb{N}^*$. Then, for $\epsilon > 0$, we can define

$K_\epsilon := \bar{B}(0, M)_{L^p(0, T; W_0^{1, p}(D))} \times K_\epsilon^4 \times K_\epsilon^1 \times K_\epsilon^1 \times K_\epsilon^2 \times K_\epsilon^3$ and

$$\tilde{K}_\epsilon := \left(\bar{B}(0, M)_{L^p(0, T; W_0^{1, p}(D))} \cap [K_\epsilon^4 + K_\epsilon^1] \right) \times K_\epsilon^4 \times K_\epsilon^1 \times K_\epsilon^1 \times K_\epsilon^2 \times K_\epsilon^3.$$

Consequently, if μ_N is the law of $X_N = (\hat{u}_N, \hat{u}_N - \widehat{M}_N, M_N, \widehat{M}_N, \Phi_N, W)$, on one hand we have

$$\begin{aligned} \mu_N(K_\epsilon) &= P\left[X_N^{-1}(K_\epsilon)\right] \\ &= P\left[\{\hat{u}_N \in \bar{B}(0, M)_{L^p(0, T; W_0^{1, p}(D))}\} \cap \{\hat{u}_N - \widehat{M}_N \in K_\epsilon^4\} \cap \{M_N \in K_\epsilon^1\} \right. \\ &\quad \left. \cap \{\widehat{M}_N \in K_\epsilon^1\} \cap \{\Phi_N \in K_\epsilon^2\} \cap \{W \in K_\epsilon^3\}\right] \\ &= 1 - P\left[\{\hat{u}_N \notin \bar{B}(0, M)_{L^p(0, T; W_0^{1, p}(D))}\} \cup \{\hat{u}_N - \widehat{M}_N \notin K_\epsilon^4\} \right. \\ &\quad \left. \cup \{M_N \notin K_\epsilon^1\} \cup \{\widehat{M}_N \notin K_\epsilon^1\} \cup \{\Phi_N \notin K_\epsilon^2\} \cup \{W \notin K_\epsilon^3\}\right] \\ &\geq 1 - 6\epsilon \end{aligned}$$

for all $N \in \mathbb{N}^*$. On the other hand, since $\hat{u}_N = \hat{u}_N - \widehat{M}_N + \widehat{M}_N$,

$$\begin{aligned} \mu_N(K_\epsilon) &= P\left[X_N^{-1}(K_\epsilon)\right] \\ &= P\left[\{\hat{u}_N - \widehat{M}_N + \widehat{M}_N \in \bar{B}(0, M)_{L^p(0, T; W_0^{1, p}(D))}\} \cap \{\hat{u}_N - \widehat{M}_N \in K_\epsilon^4\} \right. \\ &\quad \left. \cap \{M_N \in K_\epsilon^1\} \cap \{\widehat{M}_N \in K_\epsilon^1\} \cap \{\Phi_N \in K_\epsilon^2\} \cap \{W \in K_\epsilon^3\}\right] \\ &= P\left[\{\hat{u}_N \in \bar{B}(0, M)_{L^p(0, T; W_0^{1, p}(D))} \cap (K_\epsilon^4 + K_\epsilon^1)\} \cap \{\hat{u}_N - \widehat{M}_N \in K_\epsilon^4\} \right. \\ &\quad \left. \cap \{M_N \in K_\epsilon^1\} \cap \{\widehat{M}_N \in K_\epsilon^1\} \cap \{\Phi_N \in K_\epsilon^2\} \cap \{W \in K_\epsilon^3\}\right] \\ &= P\left(X_N^{-1}(\tilde{K}_\epsilon)\right) = \mu_N(\tilde{K}_\epsilon). \end{aligned}$$

Note that by [13, lemma 9], $\bar{B}(0, M)_{L^p(0, T; W_0^{1, p}(D))} \cap (K_\epsilon^4 + K_\epsilon^1)$ is relatively compact in $L^2(0, T, L^2(D))$ since $K_\epsilon^4 + K_\epsilon^1$ is relatively compact in $L^2(0, T; W^{-1, p'}(D))$ and $W_0^{1, p}(D)$ is compactly embedded into $L^2(D)$. Thus, (μ_N) is a uniformly tight sequence. \square

PROPOSITION 2.11. *The sequence (ν_N) of laws induced by*

$$Y_N = (u_N^r - \hat{u}_N, \hat{u}_N, \hat{u}_N - \widehat{M}_N, M_N, \widehat{M}_N, \Phi_N, W)$$

on the product space

$$\begin{aligned} \mathcal{X} &:= L^2(Q_T) \times L^2(Q_T) \times \mathcal{C}([0, T]; W^{-1, p'}(D)) \times \mathcal{C}([0, T]; L^2(D)) \\ &\quad \times \mathcal{C}([0, T]; L^2(D)) \times L^2(0, T; HS(L^2(D))) \times \mathcal{C}([0, T]; U), \end{aligned}$$

where U is defined in (1.2), is tight and therefore, passing to a not relabelled subsequence if necessary, there exist probability measures ν_1 on $L^2(Q_T)$, ν_2 on

$\mathcal{C}([0, T]; W^{-1,p'}(D))$ and a probability measure

$$\nu := \left(0, \nu_1, \nu_2, \mathcal{L} \left(\int_0^\cdot \Phi \, dW(s) \right), \mathcal{L} \left(\int_0^\cdot \Phi \, dW(s) \right), \mathcal{L}(\Phi), \mathcal{L}(W) \right)$$

on \mathcal{X} such that $\nu_N \Rightarrow \nu$, i.e., $\lim_{N \rightarrow \infty} \int_{\mathcal{X}} f \, d\nu_N = \int_{\mathcal{X}} f \, d\nu$ for all $f \in C_b(\mathcal{X})$.

Proof. Note that for all $N \in \mathbb{N}^*$, $\nu_N = (\mathcal{L}(u_N^r - \widehat{u}_N), \mu_N)$ and the tightness of (μ_N) follows from lemma 2.10. The tightness of $(\mathcal{L}(u_N^r - \widehat{u}_N))$ is a direct consequence of lemma 2.6 and the converse part of Prokhorov’s theorem. \square

2.3.2. *Compactness* Now, we are in position to apply the theorem of Skorokhod: by Appendix A.3, there exists a new probability space $(\Omega', \mathcal{F}', P')$ ¹ such that, passing to a subsequence if necessary, (Y_N) converges almost surely in \mathcal{X} . Without changing the notation of random variables with the same law in order not to overload the presentation, the semi-implicit Euler scheme (2.1) is satisfied on Ω' .² Moreover, remark A.1 and in particular the *a priori* estimates developed in lemmas 2.2 to 2.6 hold true on $(\Omega', \mathcal{F}', P')$. Thus, on $(\Omega', \mathcal{F}', P')$,

- there exists a $L^2(Q_T)$ -valued random variable u_∞ such that $\mathcal{L}(u_\infty) = \mathcal{L}(\nu_1)$ and $\lim_{N \rightarrow \infty} \widehat{u}_N = u_\infty$ in $L^2(Q_T)$ a.s. in Ω' . Then, lemma 2.4 and Vitali’s theorem yield the convergence of (\widehat{u}_N) to u_∞ in $L^\ell(\Omega'; L^2(Q_T))$ for all $1 \leq \ell < 2$.
- $(u_N^r - \widehat{u}_N)$ converges a.s. to 0 in $L^2(Q_T)$ and one proves similarly, or by using (2.29), that $\lim_{N \rightarrow \infty} u_N^r = u_\infty$ in $L^2(Q_T)$ a.s. in Ω' and in $L^\ell(\Omega'; L^2(Q_T))$ for all $1 \leq \ell < 2$.
- In particular, the above convergence holds in $L^\ell(\Omega' \times (0, T); L^2(D))$ and, up to a subsequence if necessary, $(u_N^r(\omega, t))$ converges to $u_\infty(\omega, t)$ in $L^2(D)$, a.e. in $\Omega' \times (0, T)$.
- There exists a $\mathcal{C}([0, T]; L^2(D))$ -valued random variable M_∞ such that $\mathcal{L}(M_\infty) = \mathcal{L}(\int_0^\cdot \Phi \, dW(s))$ and $\lim_{N \rightarrow \infty} \widehat{M}_N = \lim_{N \rightarrow \infty} M_N = M_\infty$ in $\mathcal{C}([0, T]; L^2(D))$ a.s. in Ω' . Using remark A.1, lemma 2.9 and the lemma of Brezis–Lieb (see appendix), we also obtain the convergence in $L^2(\Omega'; \mathcal{C}([0, T]; L^2(D)))$.
- There exists a $\mathcal{C}([0, T]; HS(L^2(D)))$ -valued random variable Φ_∞ such that $\mathcal{L}(\Phi_\infty) = \mathcal{L}(\Phi)$ and $\lim_{N \rightarrow \infty} \Phi_N = \Phi_\infty$ in $L^2(0, T; HS(L^2(D)))$ a.s. in Ω' . Again, remark A.1 and Brezis–Lieb’s lemma yield the convergence in $L^2(\Omega'; L^2(0, T; HS(L^2(D))))$.
- Further, there exists a $\mathcal{C}([0, T]; W^{-1,p'}(D))$ -valued random variable B_∞ on Ω' such that $\mathcal{L}(B_\infty) = \nu_2$ and $\lim_{N \rightarrow \infty} (\widehat{u}_N - \widehat{M}_N) = B_\infty$ in $\mathcal{C}([0, T]; W^{-1,p'}(D))$ a.s. in Ω' and in $L^\ell(\Omega, \mathcal{C}([0, T]; W^{-1,p'}(D)))$ for any $1 \leq \tilde{\ell} < \min(2, p')$ by Vitali’s theorem. Thanks to the previous convergence results, $(\widehat{u}_N) = (\widehat{u}_N - \widehat{M}_N + \widehat{M}_N)$ converges a.s. in $\mathcal{C}([0, T]; W^{-1,p'}(D))$. Thus, $B_\infty = u_\infty - M_\infty$,

¹This can be $(0, 1)$ with the Lebesgue σ -field and measure and expectation denoted \mathbb{E}' .

²Note that, because of Skorokhod’s theorem, W_N needs to be indexed by N in Ω' .

$u_\infty \in \mathcal{C}([0, T]; W^{-1,p'}(D))$ and lemma 2.4 with Vitali's theorem yield the convergence of \widehat{u}_N to u_∞ in $L^\ell(\Omega'; \mathcal{C}([0, T]; W^{-1,p'}(D)))$ for all $1 \leq \ell < 2$.

- Finally, there exists a random variable W_∞ with values in $\mathcal{C}([0, T]; U)$, such that $\mathcal{L}(W_\infty) = \mathcal{L}(W)$, $\lim_{N \rightarrow \infty} W_N = W_\infty$ in $\mathcal{C}([0, T]; U)$ a.s. in Ω' and in $L^2(\Omega'; \mathcal{C}([0, T]; U))$ thanks to remark A.1 and Brezis–Lieb's lemma.

2.3.3. *First identifications at the limits* Note that W_N is (\mathcal{F}_t^N) -measurable, where (\mathcal{F}_t^N) is the filtration generated by W_N and, for any $h \in U$, $t \in [0, T]$, $0 \leq s \leq t$ and all $\psi \in \mathcal{C}_b(\mathcal{C}([0, s]; U))$

$$\mathbb{E}'[(W_N(t) - W_N(s), h)_U \psi(W_N)] = \mathbb{E}[(W(t) - W(s), h)_U \psi(W)] = 0,$$

since W is a (\mathcal{F}_t) -martingale. Thus, W_N is a (\mathcal{F}_t^N) -martingale satisfying $W_N(0) = 0$ for all $N \in \mathbb{N}^*$. With similar arguments, one can show that $\ll W_N \gg_{t=0} = tQ$ for all $t \in [0, T]$ and all $N \in \mathbb{N}^*$, where Q is the covariance operator of W . From Levy's theorem (see [3, proposition 3.11, p. 75]) it follows that W_N is a cylindrical Wiener process with values in $L^2(D)$ with respect to (\mathcal{F}_t^N) . In particular $(\Omega', \mathcal{F}', (\mathcal{F}_t^N), \mathbb{P}', W_N)$ is a stochastic basis in the sense of [4, lemma 2.1, p. 1126]. We recall that we recovered the time discretization scheme in Ω' and therefore the following approximate equations hold true: for all $N \in \mathbb{N}^*$, we have

$$(\widehat{M}_N - \widehat{u}_N)(t) - u_0 - \int_0^t \operatorname{div}(a(x, u_N^r, \nabla u_N^r) + F(u_N^r)) \, ds = 0 \tag{2.38}$$

in $W^{-1,p'}(D)$ for all $t \in [0, T]$, a.s. in Ω' . Thanks to lemma 2.3, we have the following weak convergences:

LEMMA 2.12. *Up to a subsequence, $u_N^r \rightharpoonup u_\infty$ weakly in $L^p(\Omega'; L^p(0, T; W_0^{1,p}(D)))$ for $N \rightarrow \infty$. Moreover, there exists $G \in L^{p'}(\Omega' \times Q_T)$ such that $a(x, u_N^r, \nabla u_N^r) \rightharpoonup G$ weakly in $L^{p'}(\Omega' \times Q_T)$.*

Now we are ready to pass to the limit in the approximate equations:

PROPOSITION 2.13.

$$u_\infty(t) = u_0 + \int_0^t \operatorname{div}(G + F(u_\infty)) \, ds + M_\infty(t) \tag{2.39}$$

holds in $L^2(D)$ a.s. in Ω' for all $t \in [0, T]$.

REMARK 2.1. Using lemma 2.4, one gets that (\widehat{u}_N) is bounded in $L^2(\Omega'; L^\infty(0, T; L^2(D)))$, thus in $L_w^2(\Omega'; L^\infty(0, T; L^2(D)))$ where w stands for the weak-* measurability. Since [5, theorem 8.20.3, p. 606]

$$L_w^2(\Omega'; L^\infty(0, T; L^2(D))) \simeq (L^2(\Omega'; L^1(0, T; L^2(D))))^*,$$

the theorem of Banach–Alaoglu yields $u_\infty \in L_w^2(\Omega'; L^\infty(0, T; L^2(D)))$. On the other hand, since \widehat{u}_N converges to u_∞ in $L^\ell(\Omega'; \mathcal{C}([0, T]; W^{-1,p'}(D)))$, it follows that u_∞ is a random variable with values in the space of weakly continuous functions

$C_w([0, T]; L^2(D))$ [15, lemma 1.4, p. 263]. In particular, $u_\infty(t) \in L^2(D)$ a.s. in Ω' for all $t \in [0, T]$. Therefore, (2.39) holds in $L^2(D)$, a.s. in Ω' , for all $t \in [0, T]$.

Proof of proposition 2.13. We fix $\psi \in W_0^{1,p}(D)$, $\xi \in \mathcal{D}(0, T)$, $A \in \mathcal{F}'$ and use $\psi\xi\chi_A$ as a test function in (2.38). Integrating over Ω' we arrive at

$$\begin{aligned} & \int_A \int_0^T \left\langle \frac{d}{dt} (\widehat{u}_N - \widehat{M}_N)(t), \psi \right\rangle_{W^{-1,p'}(D), W_0^{1,p}(D)} \xi \, dt \, dP' \\ & + \int_A \int_0^T \int_D (a(x, u_N^r, \nabla u_N^r) + F(u_N^r)) \cdot \nabla \psi \xi \, dx \, dt \, dP' = 0. \end{aligned} \tag{2.40}$$

Let us write (2.40) as $I_1 + I_2 + I_3 = 0$, where

$$\begin{aligned} I_1 &= \int_A \int_0^T \left\langle \frac{d}{dt} (\widehat{u}_N - \widehat{M}_N)(t), \psi \right\rangle_{W^{-1,p'}(D), W_0^{1,p}(D)} \xi \, dt \, dP' \\ &= \int_A \int_0^T \frac{d}{dt} \langle (\widehat{u}_N - \widehat{M}_N)(t), \psi \rangle_{W^{-1,p'}(D), W_0^{1,p}(D)} \xi \, dt \, dP' \\ &= - \int_A \int_0^T \langle (\widehat{u}_N - \widehat{M}_N)(t), \psi \rangle_{W^{-1,p'}(D), W_0^{1,p}(D)} \xi_t \, dt \, dP', \\ I_2 &= \int_A \int_0^T \int_D a(x, u_N^r, \nabla u_N^r) \cdot \nabla \psi \xi \, dx \, dt \, dP', \\ I_3 &= \int_A \int_0^T \int_D F(u_N^r) \cdot \nabla \psi \xi \, dx \, dt \, dP'. \end{aligned}$$

Since $\widehat{u}_N - \widehat{M}_N$ converges to $u_\infty - M_\infty$ in $L^{\tilde{\ell}}(\Omega'; \mathcal{C}([0, T]; W^{-1,p'}(D)))$, it follows that

$$\lim_{N \rightarrow \infty} I_1 = \int_A \int_0^T \langle (\widehat{u}_\infty - \widehat{M}_\infty)(t), \psi \rangle_{W^{-1,p'}(D), W_0^{1,p}(D)} \xi_t \, dx \, dt \, dP'.$$

We recall that, by remark 2.1, $(u_\infty - M_\infty)(t) \in L^2(D)$ for all $t \in [0, T]$, a.s. in Ω' . Thus

$$\lim_{N \rightarrow \infty} I_1 = \int_A \int_0^T \int_D (u_\infty - M_\infty) \partial_t \xi \psi \, dx \, dt \, dP'. \tag{2.41}$$

According to lemma 2.12 $a(x, u_N^r, \nabla u_N^r) \rightharpoonup G$ weakly in $L^{p'}(\widehat{\Omega} \times Q_T)^d$ and therefore

$$\lim_{N \rightarrow \infty} I_2 = \int_A \int_0^T \int_D G \cdot \nabla \psi \xi \, dx \, dt \, dP'. \tag{2.42}$$

$L \geq 0$ being the Lipschitz constant of F , we have, thanks to (2.16)

$$\mathbb{E}' \int_0^T \|F(u_N^r)\|_{W_0^{1,p}(D)}^p \, dt \leq L \mathbb{E}' \int_0^T \|\nabla u_N^r\|_p^p \, dt \leq C$$

where $C \geq 0$ is a constant not depending on $N \in \mathbb{N}^*$.

Consequently, there exists $H \in L^p(\Omega'; L^p(0, T; (W_0^{1,p}(D))^d))$ such that $F(u_N^r) \rightharpoonup H$ weakly in $L^p(\Omega'; L^p(0, T; W_0^{1,p}(D))^d)$. Since $u_N^r \rightarrow u_\infty$ in $L^1(\Omega' \times Q_T; L^2(D)^d)$, $F(u_n^r) \rightarrow F(u_\infty)$ holds in $L^1(\Omega' \times Q_T; L^2(D))$ and $H = F(u_\infty)$. Now, by Sobolev embedding, it follows that $F(u_N^r) \rightharpoonup F(u_\infty)$ weakly in $L^p(\Omega'; L^p(0, T; L^{p^*}(D)^d))$. Since $p' < p^*$, we conclude that $F(u_N^r) \rightharpoonup F(u_\infty)$ weakly in $L^p(\Omega'; L^p(0, T; L^{p'}(D)^d))$. Therefore, it follows that

$$\lim_{N \rightarrow \infty} I_3 = \int_A \int_0^T \int_D F(u_\infty) \cdot \nabla \psi \xi \, dx \, dt \, dP'. \tag{2.43}$$

Now, from (2.41)–(2.43) it follows that

$$- \int_A \int_0^T \int_D (u_\infty - M_\infty) \xi_t \psi + (G + F(u_\infty)) \cdot \nabla \psi \xi \, dx \, dt \, dP' = 0 \tag{2.44}$$

for all $A \in \hat{\mathcal{F}}$, $\xi \in \mathcal{D}(0, T)$ and all $\psi \in W_0^{1,p}(D)$.

From (2.44) it follows that $(d/dt)(u_\infty - M_\infty) - \operatorname{div}(G + F(u_\infty)) = 0$ in $L^{p'}(0, T; W^{-1,p'}(D))$, a.s. in Ω' . We recall that $u_\infty \in \mathcal{C}([0, T]; W^{-1,p'}(D))$ a.s. in Ω' and since, a.s. in Ω' , $M_\infty \in \mathcal{C}([0, T]; L^2(D))$, we have

$$u_\infty(t) - u_\infty(0) - \int_0^t \operatorname{div}(G + F(u_\infty)) \, ds = M_\infty(t) \tag{2.45}$$

for all $t \in [0, T]$ in $L^2(D)$ a.s. in Ω' . □

REMARK 2.2. Note that for any $t \in [0, T]$,

$$\delta_t : L^2(\Omega'; \mathcal{C}([0, T]; W^{-1,p'}(D))) \rightarrow L^2(\Omega'; W^{-1,p'}(D)), \quad u \mapsto u(t)$$

is continuous and therefore $u_\infty(0) = \lim_{N \rightarrow \infty} \hat{u}_N(0) = u_0$ in $L^2(\Omega'; W^{-1,p'}(D))$.

2.4. Martingale identification argument

We denote the augmentation of the filtration $\sigma(W_\infty(s), \Phi_\infty(s), u_\infty(s))_{0 \leq s \leq t}$, $t \in [0, T]$ by (\mathcal{F}_t^∞) . In the following two lemmas, we will show that W_∞ is a cylindrical Wiener process with values in $L^2(D)$ with respect to (\mathcal{F}_t^∞) . To this end, we first show that W_∞ is a (\mathcal{F}_t^∞) -martingale. Since u_∞ is in $\mathcal{C}([0, T]; W^{-1,p'}(D))$ a.s. in Ω' , it is a stochastic process and therefore (\mathcal{F}_t^∞) is well defined.

LEMMA 2.14. W_∞ is a (\mathcal{F}_t^∞) -martingale.

Proof. By definition of (\mathcal{F}_t^∞) , W_∞ is adapted to (\mathcal{F}_t^∞) . For all $t \in [0, T]$, $0 \leq s \leq t$, $\psi \in \mathcal{C}_b(\mathcal{C}([0, s]; U) \times L^2(0, s; HS(L^2(D))) \times L^2((0, s) \times D))$ and $h \in U$

$$\begin{aligned} & \mathbb{E}'[(W_\infty(t) - W_\infty(s), h)_U \psi(W_\infty, \Phi_\infty, u_\infty)] \\ &= \lim_{N \rightarrow \infty} \mathbb{E}'[(W_N(t) - W_N(s), h)_U \psi(W_N, \Phi_N, u_N^r)] \\ &= \lim_{N \rightarrow \infty} \mathbb{E}[(W(t) - W(s), h)_U \psi(W, \Phi_N, u_N^r)] = 0 \end{aligned}$$

since Φ_N and u_N^r are (\mathcal{F}_t) -adapted processes for all $N \in \mathbb{N}^*$ and W is a (\mathcal{F}_t) -martingale. □

LEMMA 2.15. W_∞ is a cylindrical Wiener process with values in $L^2(D)$ with respect to (\mathcal{F}_t^∞) .

Proof. Since we already know that W_∞ is a (\mathcal{F}_t^∞) -martingale with $W_\infty(0) = 0$, according to [3, theorem 4.4, p. 89], it is left to show that

$$\ll W_\infty \gg_t = tQ \quad \text{for all } t \in [0, T], \tag{2.46}$$

where Q is the covariance operator of W . Recall that $\ll W_N \gg_t = tQ$ for all $t \in [0, T]$ and all $N \in \mathbb{N}^*$. Let U be the Hilbert space defined in (1.2) and (g_l) be an orthonormal basis of U . For all $t \in [0, T]$, $0 \leq s \leq t$, $\psi \in \mathcal{C}_b(\mathcal{C}([0, s]; U) \times L^2(0, s; HS(L^2(D))) \times L^2((0, s) \times D))$ and $n, m \in \mathbb{N}^*$, combining the convergence results from the previous section with the dominated convergence theorem of Lebesgue, it follows that

$$\begin{aligned} & \mathbb{E}' [((W_\infty, g_n, g_m)(t) - (W_\infty, g_n, g_m)(s) - ((t - s)Q(g_n), g_m)_U)\psi(W_\infty, \Phi_\infty, u_\infty)] \\ &= \lim_{N \rightarrow \infty} \mathbb{E}' [((W_N, g_n, g_m)(t) - (W_N, g_n, g_m)(s) \\ &\quad - ((t - s)Q(g_n), g_m)_U)\psi(W_N, \Phi_N, u_N^r)] \\ &= \lim_{N \rightarrow \infty} \mathbb{E} [((W, g_n, g_m)(t) - (W, g_n, g_m)(s) \\ &\quad - ((t - s)Q(g_n), g_m)_U)\psi(W, \Phi_N, u_N^r)] = 0, \end{aligned}$$

where $(W, g_n, g_m)(r) := (W(r), g_n)_U (W(r), g_m)_U$ for $W(r) \in U$, $r \in [0, T]$, thus (2.46) holds true. In particular, $(\Omega', \mathcal{F}', (\mathcal{F}_t^\infty), \mathbb{P}', W_\infty)$ is a stochastic basis in the sense of [4, lemma 2.1, p. 1126]. □

From § A.3 of the appendix it follows that Φ_N is a (\mathcal{F}_t^N) -predictable process with values in $HS(L^2(D))$ and that $M_N(t) = \int_0^t \Phi_N dW_N(s)$. In order to pass to the limit, we recall the following lemma:

LEMMA 2.16. [4, lemma 2.1, p. 1126] Let (Ω, \mathcal{F}, P) be a fixed probability space, W a cylindrical Wiener process on (Ω, \mathcal{F}, P) with values in $L^2(D)$ with respect to a given filtration (\mathcal{F}_t) and X a separable Hilbert space. Consider a sequence of stochastic bases $S_n = (\Omega, \mathcal{F}, (\mathcal{F}_t^n), P, W_n)$, that is a sequence so that each W_n is a cylindrical Wiener process with values in $L^2(D)$ with respect to (\mathcal{F}_t^n) . Assume that (G_n) is a collection of $HS(L^2(D); X)$ -valued, (\mathcal{F}_t^n) -predictable processes such that $G_n \in L^2(0, T; HS(L^2(D); X))$ a.s. in Ω and $G \in L^2(0, T; HS(L^2(D); X))$ is (\mathcal{F}_t) -predictable. If $W_n \rightarrow W$ in probability in $\mathcal{C}([0, T]; U)$, where U is given in (1.2) and $G_n \rightarrow G$ in probability in $L^2(0, T; HS(L^2(D); X))$, then

$$\lim_{n \rightarrow \infty} \int_0^\cdot G_n dW_n(s) = \int_0^\cdot G dW(s)$$

in probability in $L^2(0, T; X)$.

Consequently, we have $M_\infty(t) = \int_0^t \Phi_\infty dW_\infty(s)$ for all $t \in [0, T]$, a.s. in Ω' and therefore

$$u_\infty(t) = u_0 + \int_0^t \operatorname{div}(G + F(u_\infty)) ds + \int_0^t \Phi_\infty dW_\infty(s) \tag{2.47}$$

in $W^{-1,p'}(D)$, for all $t \in [0, T]$, a.s. in Ω' . By lemma 2.4 and Itô's formula (see, e.g. [10, theorem 4.2.5]), it follows that u_∞ is a square-integrable, (\mathcal{F}_t^∞) -adapted, continuous process with values in $L^2(D)$, and (2.47) holds in $L^2(D)$ a.s. in Ω' .

2.5. Monotonicity argument

LEMMA 2.17. $G = a(x, u_\infty, \nabla u_\infty)$ in $L^{p'}(\Omega' \times Q_T)^d$.

Proof. Taking u^{k+1} as a test function in the discretized equation (2.1) in Ω' , using Gauss–Green's theorem for the convection term and taking expectation we get

$$\begin{aligned} & \frac{1}{2} \mathbb{E}' (\|u^{k+1}\|_2^2 - \|u^k\|_2^2 + \|u^{k+1} - u^k\|_2^2) + \tau \mathbb{E}' \int_D a(x, u^{k+1}, \nabla u^{k+1}) \cdot \nabla u^{k+1} dx \\ & = \mathbb{E}' (M_N(t_{k+1}) - M_N(t_k), u^{k+1})_2. \end{aligned} \tag{2.48}$$

Since $\mathbb{E}' (M_N(t_{k+1}) - M_N(t_k), u^k)_2 = 0$, using Young's inequality in (2.48) we get

$$\begin{aligned} & \mathbb{E}' (\|u^{k+1}\|_2^2 - \|u^k\|_2^2) + 2\tau \mathbb{E}' \int_D a(x, u^{k+1}, \nabla u^{k+1}) \cdot \nabla u^{k+1} dx \\ & \leq \mathbb{E}' \|M_N(t_{k+1}) - M_N(t_k)\|_2^2. \end{aligned} \tag{2.49}$$

Summing over $k = 0, \dots, N - 1$ in (2.49) it follows that

$$\begin{aligned} & \mathbb{E}' \left[\|\hat{u}_N(T)\|_2^2 - \|u^0\|_2^2 + 2 \int_0^T \int_D a(x, u_N^r, \nabla u_N^r) \cdot \nabla u_N^r dx dt \right] \\ & - \sum_{k=0}^{N-1} \mathbb{E}' \left\| \int_{t_k}^{t_{k+1}} \Phi_N dW_N(t) \right\|_2^2 \leq 0 \end{aligned} \tag{2.50}$$

where, by Itô's isometry,

$$\begin{aligned} \sum_{k=0}^{N-1} \mathbb{E}' \left\| \int_{t_k}^{t_{k+1}} \Phi_N dW_N(t) \right\|_2^2 & = \sum_{k=0}^{N-1} \mathbb{E}' \int_{t_k}^{t_{k+1}} \|\Phi_N\|_{HS(L^2(D))}^2 dt \\ & = \mathbb{E}' \int_0^T \|\Phi_N\|_{HS(L^2(D))}^2 dt. \end{aligned} \tag{2.51}$$

By equality of laws, from (2.50) and (2.51) it follows that

$$\begin{aligned} & \mathbb{E}' \left[\|\hat{u}_N(T)\|_2^2 - \|u^0\|_2^2 + 2 \int_0^T \int_D a(x, u_N^r, \nabla u_N^r) \cdot \nabla u_N^r dx dt \right] \\ & - \mathbb{E}' \int_0^T \|\Phi_N\|_{HS(L^2(D))}^2 dt \leq 0. \end{aligned} \tag{2.52}$$

On the other hand, Itô's formula and (2.47) yield

$$\begin{aligned} & \|u_\infty(T)\|_2^2 - \|u_0\|_2^2 + 2 \int_0^T \int_D G \cdot \nabla u_\infty \, dx \, dt \\ &= \int_0^T \|\Phi_\infty\|_{HS(L^2(D))}^2 \, dt + 2 \int_0^T (u_\infty, \Phi_\infty \, dW_\infty(t))_2, \end{aligned} \tag{2.53}$$

a.s. in Ω' , therefore

$$\mathbb{E}' \left[\|u_\infty(T)\|_2^2 - \|u_0\|_2^2 + 2 \int_0^T \int_D G \cdot \nabla u_\infty \, dx \, dt \right] - \mathbb{E}' \int_0^T \|\Phi_\infty\|_{HS(L^2(D))}^2 \, dt = 0.$$

From the above equation and (2.52), it follows that ($u^0 = u_0^r$ is given by lemma A.1)

$$\begin{aligned} & \mathbb{E}' \|u_\infty(T)\|_2^2 + 2\mathbb{E}' \int_0^T \int_D G \cdot \nabla u_\infty \, dx \, dt - \mathbb{E}' \int_0^T \|\Phi_\infty\|_{HS(L^2(D))}^2 \, dt \\ & \geq \mathbb{E}' \|\hat{u}_N(T)\|_2^2 + 2\mathbb{E}' \int_0^T \int_D a(x, u_N^r, \nabla u_N^r) \cdot \nabla u_N^r \, dx \, dt \\ & \quad - \mathbb{E}' \int_0^T \|\Phi_N\|_{HS(L^2(D))}^2 \, dt + \|u_0\|_2^2 - \|u_0^r\|_2^2, \end{aligned} \tag{2.54}$$

hence

$$\begin{aligned} & 2\mathbb{E}' \int_0^T \int_D G \cdot \nabla u_\infty \, dx \, dt \\ & \geq \mathbb{E}' (\|\hat{u}_N(T)\|_2^2 - \|u_\infty(T)\|_2^2) + \mathbb{E}' \int_0^T \|\Phi_N\|_{HS(L^2(D))}^2 - \|\Phi_\infty\|_{HS(L^2(D))}^2 \, dt \\ & \quad + 2\mathbb{E}' \int_0^T \int_D a(x, u_N^r, \nabla u_N^r) \cdot \nabla u_N^r \, dx \, dt + \|u_0\|_2^2 - \|u_0^r\|_2^2. \end{aligned} \tag{2.55}$$

Since \hat{u}_N converges to u_∞ in $L^\ell(\Omega'; \mathcal{C}([0, T]; W^{-1,p'}(D)))$ for some $1 \leq \ell < 2$, for any t , $\hat{u}_N(t)$ converges to $u_\infty(t)$ in $L^\ell(\Omega'; W^{-1,p'}(D))$. On the other hand, the sequence $\hat{u}_N(t)$ is bounded in $L^2(\Omega' \times D)$ and one is able to conclude that $\hat{u}_N(t)$ converges weakly to $u_\infty(t)$ in $L^2(\Omega' \times D)$. Therefore, by lower semi-continuity, $\liminf_{N \rightarrow \infty} \mathbb{E}' \|\hat{u}_N(T)\|_2^2 \geq \mathbb{E}' \|u_\infty(T)\|_2^2$ and since

$$\mathbb{E}' \int_0^T \|\Phi_N\|_{HS(L^2(D))}^2 \, dt \rightarrow \mathbb{E}' \int_0^T \|\Phi_\infty\|_{HS(L^2(D))}^2 \, dt \tag{2.56}$$

for $N \rightarrow \infty$, it follows that

$$\begin{aligned} \mathbb{E}' \int_0^T \int_D G \cdot \nabla u_\infty \, dx \, dt & \geq \limsup_{N \rightarrow \infty} \mathbb{E}' \int_0^T \int_D a(x, u_N^r, \nabla u_N^r) \cdot \nabla u_N^r \, dx \, dt \\ & \quad + \liminf_{N \rightarrow \infty} \mathbb{E}' \|\hat{u}_N(T)\|_2^2 - \mathbb{E}' \|u_\infty(T)\|_2^2 \\ & \geq \limsup_{N \rightarrow \infty} \mathbb{E}' \int_0^T \int_D a(x, u_N^r, \nabla u_N^r) \cdot \nabla u_N^r \, dx \, dt. \end{aligned} \tag{2.57}$$

The next step is to prove that

$$\mathbb{E}' \int_0^T \int_D G \cdot \nabla u_\infty \, dx \, dt \leq \liminf_{N \rightarrow \infty} \mathbb{E}' \int_0^T \int_D a(x, u_N^r, \nabla u_N^r) \cdot \nabla u_N^r \, dx \, dt. \tag{2.58}$$

Following the proof of [12, lemma 8.8, p. 208], for any $N \in \mathbb{N}^*$ and almost every $(\omega, t) \in \Omega' \times (0, T)$, we define

$$\xi_N(\omega, t) := \int_D a(x, u_N^r, \nabla u_N^r) \cdot \nabla (u_N^r - u_\infty) \, dx$$

and observe that by (A2) and Young’s inequality, there exist $\delta, C > 0$ such that

$$\xi_N(\omega, t) \geq \delta \int_D |\nabla u_N^r|^p \, dx - C \left(\int_D |\nabla u_\infty|^p \, dx + 1 \right) \tag{2.59}$$

for any $N \in \mathbb{N}^*$, a.s. in $\Omega' \times (0, T)$.

As a direct consequence of (2.59), we get that $\liminf_{N \rightarrow \infty} \xi_N(\omega, t) > -\infty$ a.s. in $\Omega' \times (0, T)$. If we can show that $\liminf_{N \rightarrow \infty} \xi_N(\omega, t) \geq 0$, a.s. in $\Omega' \times (0, T)$, then (2.58) would follow from Fatou’s lemma. Assume that for a given (ω, t) , $\liminf_{N \rightarrow \infty} \xi_N(\omega, t) = a < 0$. This yields the existence of a subsequence N' (depending on ω and t) such that $\lim_{N' \rightarrow \infty} \xi_{N'}(\omega, t) = a < 0$, thus $\xi_{N'}(\omega, t) < a/2 < 0$ after a certain time, considered in the next few lines as the starting point of the sequence. Then, (2.59) ensures the boundedness of $u_{N'}^r(\omega, t)$ in $W_0^{1,p}(D)$. Since $u_{N'}^r(\omega, t)$ converges to $u_\infty(\omega, t)$ in $L^2(D)$, one is able to conclude that $u_{N'}^r(\omega, t)$ converges also weakly in $W_0^{1,p}(D)$. Since the nonlinear operator

$$\mathcal{A} : W_0^{1,p}(D) \rightarrow W^{-1,p'}(D), \quad u \mapsto \mathcal{A}(u) := -\operatorname{div} a(x, u, \nabla u)$$

is pseudomonotone, $a = \liminf_{N' \rightarrow \infty} \xi_{N'}(\omega, t) \geq 0$. This is a contradiction and one concludes that $\liminf_{N \rightarrow \infty} \xi_N(\omega, t) \geq 0$, for almost any (ω, t) in $\Omega' \times (0, T)$ and therefore (2.58) is true. As a consequence of (2.57) and (2.58),

$$\lim_{N \rightarrow \infty} \mathbb{E}' \int_0^T \int_D a(x, u_N^r, \nabla u_N^r) \cdot \nabla u_N^r \, dx \, dt = \mathbb{E}' \int_0^T \int_D G \cdot \nabla u_\infty \, dx \, dt \tag{2.60}$$

and, for any $\vec{\xi} \in L^p(\Omega' \times Q_T)^d$,

$$\lim_{N \rightarrow \infty} \mathbb{E}' \int_0^T \int_D a(x, u_N^r, \nabla u_N^r) \cdot (\nabla u_N^r - \vec{\xi}) \, dx \, dt = \mathbb{E}' \int_0^T \int_D G \cdot (\nabla u_\infty - \vec{\xi}) \, dx \, dt. \tag{2.61}$$

Since for almost any (ω, t) in $\Omega' \times (0, T)$, $\liminf_{N \rightarrow \infty} \xi_N(\omega, t) \geq 0$, one gets that $\lim_{N \rightarrow \infty} \xi_N^-(\omega, t) = 0$ a.s. in Ω' , where $-r^- := \min(0, r)$ for $r \in \mathbb{R}$. As by (2.59),

$$0 \geq -\xi_N^- \geq -C \left(\int_D |\nabla u_\infty|^p \, dx + 1 \right) \in L^1(\Omega' \times (0, T)),$$

one concludes that ξ_N^- converges to 0 in $L^1(\Omega' \times (0, T))$. From (2.61) in particular we get $\lim_{N \rightarrow \infty} \mathbb{E}' \int_0^T \xi_N \, dt = 0$, and therefore (ξ_N) converges to 0 in $L^1(\Omega' \times (0, T))$.

Up to a subsequence denoted by N' , one can consider that $\lim_{N' \rightarrow \infty} \xi_{N'} = 0$ a.s. in $\Omega' \times (0, T)$ and that there exists $\Theta \in L^1(\Omega' \times (0, T))$ such that $|\xi_{N'}| \leq \Theta$ for all $N' \in \mathbb{N}^*$ a.e. in $\Omega' \times (0, T)$.

Since $u_N^r \rightarrow u_\infty$ for $N \rightarrow \infty$ in $L^2(D)$ a.e. in $\Omega' \times (0, T)$, (2.59) with the above majoration by Θ yield the boundedness of $u_{N'}^r(\omega, t)$ in $W_0^{1,p}(D)$, thus $u_{N'}^r(\omega, t) \rightharpoonup u_\infty(\omega, t)$ in $W_0^{1,p}(D)$ and $u_{N'}^r(\omega, t) \rightarrow u_\infty(\omega, t)$ in $L^p(D)$ for almost every $(\omega, t) \in \Omega' \times (0, T)$. By assumption (A2), for any $\vec{\xi} \in L^p(\Omega' \times Q_T)^d$, the properties of Nemytskii's operators yield $a(x, u_{N'}^r(\omega, t), \vec{\xi}) \rightarrow a(x, u_\infty(\omega, t), \vec{\xi})$ in $L^{p'}(D)$ for almost every $(\omega, t) \in \Omega' \times (0, T)$ and for $N' \rightarrow \infty$, so that

$$\lim_{N' \rightarrow \infty} \int_D a(x, u_{N'}^r, \vec{\xi}) \cdot (\nabla u_{N'}^r - \vec{\xi}) \, dx = \int_D a(x, u_\infty, \vec{\xi}) \cdot (\nabla u_\infty - \vec{\xi}) \, dx \tag{2.62}$$

a.e. in $\Omega' \times (0, T)$. The same assumption ensures, for almost every $(\omega, t) \in \Omega' \times (0, T)$, the boundedness of the sequence $(a(x, u_{N'}^r(\omega, t), \nabla u_{N'}^r(\omega, t)))$ in $L^{p'}(D)$ and, up to a subsequence denoted N'' (depending *a priori* on (ω, t)), the weak convergence of $a(x, u_{N''}^r(\omega, t), \nabla u_{N''}^r(\omega, t))$ to some $\theta(\omega, t)$ in $L^{p'}(D)$. By assumption (A1),

$$0 \leq \int_D (a(x, u_{N''}^r, \nabla u_{N''}^r) - a(x, u_{N''}^r, \vec{\xi})) \cdot (\nabla u_{N''}^r - \vec{\xi}) \, dx.$$

Then, thanks to the above convergence, (2.62) and $\lim_{N' \rightarrow \infty} \xi_{N'} = 0$, one gets that $0 \leq \int_D (\theta - a(x, u_\infty, \vec{\xi})) \cdot (\nabla u_\infty - \vec{\xi}) \, dx$ and Minty's argument yields

$$\theta(\omega, t) = a(x, u_\infty(\omega, t), \nabla u_\infty(\omega, t)). \tag{2.63}$$

This implies $a(x, u_{N'}^r(\omega, t), \nabla u_{N'}^r(\omega, t)) \rightharpoonup a(x, u_\infty(\omega, t), \nabla u_\infty(\omega, t))$ weakly in $L^{p'}(D)$, where N' can be chosen independently of $(\omega, t) \in \Omega' \times (0, T)$. Since $\lim_{N' \rightarrow \infty} \xi_{N'}(\omega, t) = 0$, from (2.63) it follows that for any given $\vec{\xi} \in L^p(\Omega \times Q_T)^d$, for almost every $(\omega, t) \in \Omega' \times (0, T)$,

$$\begin{aligned} & \lim_{N' \rightarrow \infty} \int_D a(x, u_{N'}^r(\omega, t), \nabla u_{N'}^r(\omega, t)) \cdot (\nabla u_{N'}^r(\omega, t) - \vec{\xi}(\omega, t)) \, dx \\ &= \int_D a(x, u_\infty(\omega, t), \nabla u_\infty(\omega, t)) \cdot (\nabla u_\infty(\omega, t) - \vec{\xi}(\omega, t)) \, dx. \end{aligned} \tag{2.64}$$

Similarly to (2.59), for $\vec{\xi} \in L^p(\Omega \times Q_T)^d$,

$$\int_D a(x, u_N^r, \nabla u_N^r) \cdot (\nabla u_N^r - \vec{\xi}) \, dx \geq -C \left(\int_D |\vec{\xi}|^p \, dx + 1 \right) \tag{2.65}$$

a.e. in $\Omega' \times (0, T)$, for all $N \in \mathbb{N}^*$ and, by Fatou's lemma, from (2.64) and (2.65) it follows that

$$\begin{aligned} & \liminf_{N' \rightarrow \infty} \mathbb{E}' \int_0^T \int_D a(x, u_{N'}^r, \nabla u_{N'}^r) \cdot (\nabla u_{N'}^r - \vec{\xi}) \, dx \, dt \\ & \geq \mathbb{E}' \int_0^T \int_D a(x, u_\infty, \nabla u_\infty) \cdot (\nabla u_\infty - \vec{\xi}) \, dx \, dt. \end{aligned} \tag{2.66}$$

Back to (2.61), (2.66) yields

$$\mathbb{E}' \int_0^T \int_D G \cdot (\nabla u_\infty - \vec{\xi}) \, dx \, dt \geq \mathbb{E}' \int_0^T \int_D a(x, u_\infty, \nabla u_\infty) \cdot (\nabla u_\infty - \vec{\xi}) \, dx \, dt$$

for any $\vec{\xi} \in L^p(\Omega \times Q_T)^d$, and therefore it follows that $G = a(\cdot, u_\infty, \nabla u_\infty)$. \square

3. Uniqueness and existence of strong solutions

In this section, we show the existence of strong solutions adapting the argument of Gyöngy and Krylov (see [8]). First, we show a pathwise L^1 -contraction principle:

PROPOSITION 3.1. *Assume that for a given stochastic basis $(\Omega, \mathcal{F}, (\mathcal{F}_t), P)$ and for an arbitrary $\Phi \in L^2(\Omega \times (0, T); HS(L^2(D)))$ there exist solutions u_1, u_2 to (1.1) with respect to the initial values u_{01} and u_{02} in $L^2(D)$ respectively. Then,*

$$\int_D |u_1(t) - u_2(t)| \, dx \leq \int_D |u_{01} - u_{02}| \, dx \tag{3.1}$$

for all $t \in [0, T]$, a.s. in Ω .

Proof. For $\delta > 0$, consider η_δ , the Lipschitz approximation of the sign function defined by $\eta_\delta(r) = \max(-1, \min(r/\delta, 1))$. Now, for $\delta > 0$ and $r \in \mathbb{R}$ we define $N_\delta(r) := \int_0^r \eta_\delta(s) \, ds$. Subtracting the equation for u_2 from the equation for u_1 , the noise terms are eliminated and one gets pathwise the PDE

$$\partial_t(u_1 - u_2) - \operatorname{div}[a(\cdot, u_1, \nabla u_1) - a(\cdot, u_2, \nabla u_2) + F(u_1) - F(u_2)] = 0.$$

Testing the equation with $\eta_\delta(u_1(t) - u_2(t))$, we get $I_1 = I_2 + I_3$ for all $t \in [0, T]$ a.s. in Ω , where

$$\begin{aligned} I_1 &= \int_D N_\delta(u_1 - u_2)(t) \, dx - \int_D N_\delta(u_{01} - u_{02}) \, dx, \\ I_2 &= - \int_0^t \int_D (a(x, u_1, \nabla u_1) - a(x, u_2, \nabla u_2)) \cdot \nabla(u_1 - u_2) \eta'_\delta(u_1 - u_2) \, dx \, ds, \\ I_3 &= - \int_0^t \int_D (F(u_1) - F(u_2)) \cdot \nabla(u_1 - u_2) \eta'_\delta(u_1 - u_2) \, dx \, ds, \end{aligned} \tag{3.2}$$

and $\eta'_\delta(u_1 - u_2) = 1/\delta \chi_{\{|u_1 - u_2| < \delta\}} \geq 0$ for any $\delta > 0$. Now we write $I_2 = I_{2,1} + I_{2,2}$ where, by (A1)

$$I_{2,1} = - \int_0^t \int_D (a(x, u_1, \nabla u_1) - a(x, u_1, \nabla u_2)) \cdot \nabla(u_1 - u_2) \eta'_\delta(u_1 - u_2) \, dx \, ds \leq 0.$$

Using (A3) and Hölder’s inequality we get

$$\begin{aligned}
 |I_{2,2}| &= \left| \int_0^t \int_D (a(x, u_1, \nabla u_2) - a(x, u_2, \nabla u_2)) \cdot \nabla(u_1 - u_2) \eta'_\delta(u_1 - u_2) \, dx \, ds \right| \\
 &\leq \int_0^t \int_D (C_a^4 |\nabla u_2|^{p-1} + h(x)) |u_1 - u_2| |\nabla(u_1 - u_2)| \eta'_\delta(u_1 - u_2) \, dx \, ds \\
 &\leq C \left(\int_{\{|u_1 - u_2| < \delta\}} |\nabla(u_1 - u_2)|^p \, dx \, ds \right)^{1/p},
 \end{aligned}$$

where $C \geq 0$ is a constant not depending on $\delta > 0$. Therefore, $\lim_{\delta \rightarrow 0^+} I_{2,2} = 0$ and consequently it follows that $\limsup_{\delta \rightarrow 0^+} I_2 \leq 0$ for all $t \in [0, T]$, a.s. in Ω . Since $N_\delta(u_1 - u_2)(t)$ converges to $|(u_1 - u_2)(t)|$ for $\delta \rightarrow 0^+$, and $|N_\delta(u_1 - u_2)(t)| \leq |(u_1 - u_2)(t)|$ for all $\delta > 0$, it follows that

$$\lim_{\delta \rightarrow 0^+} I_1 = \int_D |u_1(t) - u_2(t)| \, dx - \int_D |u_{01} - u_{02}| \, dx \tag{3.3}$$

for any $t \in [0, T]$ a.s. in Ω . For $L \geq 0$ being the Lipschitz constant of F we have

$$\begin{aligned}
 |I_3| &\leq \frac{1}{\delta} \int_{\{|u_1 - u_2| < \delta\}} |F(u_1) - F(u_2)| |\nabla(u_1 - u_2)| \, dx \, ds \\
 &\leq \frac{L}{\delta} \int_{\{|u_1 - u_2| < \delta\}} |u_1 - u_2| |\nabla(u_1 - u_2)| \, dx \, ds \\
 &\leq L \int_{\{|u_1 - u_2| < \delta\}} |\nabla(u_1 - u_2)| \, dx \, ds,
 \end{aligned}$$

and it follows, by arguments similar to the ones on p. 6, that $\lim_{\delta \rightarrow 0^+} I_3 = \int_{\{u_1 = u_2\}} |\nabla(u_1 - u_2)| \, dx \, ds = 0$ a.s. in Ω and the result holds true. \square

In particular, proposition 3.1 implies that whenever a solution (in the sense of definition 1.1) to (1.1) exists, it is unique. The following lemma (see, e.g. [8, lemma 1.1, pp. 144–145]) contains a suitable necessary and sufficient condition for strong convergence:

LEMMA 3.2. *Let V be a Polish space equipped with the Borel σ -algebra. A sequence of V -valued random variables (X_n) converges in probability if and only if for every pair of subsequences (X_l) and (X_m) there exists a joint subsequence (X_{l_k}, X_{m_k}) which converges for $k \rightarrow \infty$ in law to a probability measure μ such that $\mu(\{(w, z) \in V \times V \mid w = z\}) = 1$.*

LEMMA 3.3. *The sequence of piecewise affine approximations (\hat{u}_N) converges in probability on (Ω, \mathcal{F}, P) and there exists a strong solution to (1.1).*

Proof. We apply lemma 3.2 to prove the assertion. Let (\hat{u}_K) , and (\hat{u}_L) be a pair of subsequences of (\hat{u}_N) . Then, repeating the arguments of §2.3, it follows that the

random vector $(Y_{K,L})$ defined by

$$(u_K^r - \widehat{u}_K, \widehat{u}_K, \widehat{u}_K - \widehat{M}_K, M_K, \widehat{M}_K, \Phi_K, u_L^r - \widehat{u}_L, \widehat{u}_L, \widehat{u}_L - \widehat{M}_L, M_L, \widehat{M}_L, \Phi_L, W)$$

is tight on the appropriate product space, and therefore relatively compact; thus we can extract a joint subsequence (Y_{K_j,L_j}) which converges in law. Thus, in particular, there exist probability measures μ_1, μ_2 such that $(\mu_1, \mu_2) = \lim_{j \rightarrow \infty} \mathcal{L}(\widehat{u}_{K_j}, \widehat{u}_{L_j})$. Now we continue as in §§ 2.3, 2.4 and 2.5: passing to a new probability space $(\Omega', \mathcal{F}', P')$ and not changing notation for random variables with the same law, we find random variables u_∞^1, u_∞^2 such that $\mathcal{L}(u_\infty^1) = \mu_1, \mathcal{L}(u_\infty^2) = \mu_2$ and, since we have the strong convergences of \widehat{M}_N, M_N and Φ_N , it implies that $\widehat{M}_{K_j}, M_{K_j}, \widehat{M}_{L_j}, M_{L_j}$ and Φ_{K_j}, Φ_{L_j} are converging to the same limits and

$$u_\infty^i(t) - u_0 - \int_0^t \operatorname{div}(a(x, u_\infty^i, \nabla u_\infty^i) + F(u_\infty^i)) \, ds = \int_0^t \Phi_\infty \, dW_\infty(s)$$

for $i = 1, 2$, where W_∞ is a cylindrical (\mathcal{F}_t^∞) -Wiener process and (\mathcal{F}_t^∞) is the augmentation of the filtration $\sigma(\Phi(s), W_\infty(s), u_\infty^1(s), u_\infty^2(s))_{0 \leq s \leq t}, t \in [0, T]$. Now, from proposition 3.1 it follows that $u_\infty^1 = u_\infty^2$ in $L^2(\Omega'; \mathcal{C}([0, T]L^2(D)))$ and for the joint law (μ_1, μ_2) on $\mathcal{C}([0, T]; W^{-1,p'}(D))^2$ it follows that

$$(\mu_1, \mu_2) \left(\{(w, z) \in \mathcal{C}([0, T]; W^{-1,p'}(D)) \times \mathcal{C}([0, T]; W^{-1,p'}(D)) \mid w = z\} \right) = 1$$

and the convergence in probability of the sequence (\widehat{u}_N) follows from lemma 3.2. Again, since we have the strong convergence of \widehat{M}_N, M_N and Φ_N in Ω , it is possible to recover the steps of the proof of existence of a solution in Ω and therefore a strong solution exists. □

Appendix A. On the regularization of the initial condition

LEMMA A.1. *For any positive τ , there exists $u_0^\tau \in W_0^{1,p}(D)$ such that (u_0^τ) converges to u_0 in $L^2(D)$ for $\tau \rightarrow 0^+$ and satisfies $\frac{1}{2}\|u_0^\tau\|_{L^2(D)}^2 + \tau\|\nabla u_0^\tau\|_p^p \leq \frac{1}{2}\|u_0\|_{L^2(D)}^2$.*

Proof. Denote by u^τ the unique solution in $W_0^{1,p}(D)$ to the problem $u^\tau - \tau\Delta_p u^\tau = u_0$. Using the solution u^τ as a test function in the equation, one gets

$$\|u^\tau\|_{L^2(D)}^2 + \tau\|\nabla u^\tau\|_p^p = \int_D u_0 u^\tau \, dx \leq \frac{1}{2}\|u_0\|_{L^2(D)}^2 + \frac{1}{2}\|u^\tau\|_{L^2(D)}^2,$$

and consequently, up to a not relabelled subsequence, (u^τ) converges weakly in $L^2(D)$ to a given v with the additional information that $\|u^\tau\|_{L^2(D)} \leq \|u_0\|_{L^2(D)}$. Therefore, for any $\varphi \in \mathcal{D}(D)$, one has

$$\int_D u_0 \varphi \, dx = \int_D u^\tau \varphi \, dx + \tau \int_D |\nabla u^\tau|^{p-2} \nabla u^\tau \nabla \varphi \, dx \rightarrow \int_D v \varphi \, dx$$

and therefore (u^τ) converges to $v = u_0$ in the sense of distributions and weakly in $L^2(D)$. As a consequence, the whole sequence (u^τ) converges to u_0 weakly in $L^2(D)$ and strongly in $L^2(D)$ thanks to the uniform convexity property and the above inequality. Therefore, $u_0^\tau := u^\tau$ for all $\tau > 0$. □

A.1. The lemma of Brezis–Lieb

LEMMA A.2. *Let (Ω, \mathcal{F}, P) be a probability space $(u_n) \subset L^2(\Omega; X)$ where X is a separable Banach space such that $u_n \rightarrow u$ in X a.s. in Ω for some $u \in L^2(\Omega; X)$ and $\sup_n \mathbb{E}\|u_n\|_X^2 =: M < +\infty$. Then,*

$$\lim_{n \rightarrow \infty} \mathbb{E}(\|u_n\|_X^2 - \|u_n - u\|_X^2) = \mathbb{E}\|u\|_X^2. \tag{A.1}$$

In particular, without using any argument of uniform convexity, a.s. convergence of (u_n) to u in X and $\lim_{n \rightarrow \infty} \mathbb{E}\|u_n\|_X^2 = \mathbb{E}\|u\|_X^2$ yields the convergence of (u_n) to u in $L^2(\Omega; X)$.

Proof. For $n \in \mathbb{N}^*$ and $\varepsilon > 0$ we define $X_n := \left| \|u_n\|_X^2 - \|u_n - u\|_X^2 - \|u\|_X^2 \right|$, $Y_n^\varepsilon := (X_n - \varepsilon\|u_n\|_X^2)^+$. For all $n \in \mathbb{N}^*$ we have

$$\begin{aligned} X_n &\leq \|u\|_X^2 + \left| \|u_n\|_X - \|u_n - u\|_X \right| (\|u_n\|_X + \|u_n - u\|_X) \\ &\leq \|u\|_X^2 + \|u\|_X (\|u_n\|_X + \|u_n - u\|_X) \\ &\leq \|u\|_X^2 + 2\|u\|_X \|u_n\|_X + \|u\|_X^2 \leq \|u\|_X^2 + \frac{1}{\varepsilon} \|u\|_X^2 + \varepsilon \|u_n\|_X^2 + \|u\|_X^2 \\ &= \left(2 + \frac{1}{\varepsilon} \right) \|u\|_X^2 + \varepsilon \|u_n\|_X^2 \end{aligned} \tag{A.2}$$

and from (A.2) it follows that there exists a constant $C_\varepsilon \geq 0$ not depending on $n \in \mathbb{N}^*$ such that $0 \leq Y_n^\varepsilon \leq C_\varepsilon \|u\|_X^2$ for all $n \in \mathbb{N}^*$, a.s. in Ω . Since $\lim_{n \rightarrow \infty} \|u_n - u\|_X^2 = 0$ a.s. in Ω , it follows that $\lim_{n \rightarrow \infty} Y_n^\varepsilon = 0$. Combining these two results, we get that the convergence also holds in $L^1(\Omega)$. Now, $\mathbb{E}[X_n] = \mathbb{E}[X_n - \varepsilon\|u_n\|_X^2] + \varepsilon\mathbb{E}\|u_n\|_X^2 \leq \mathbb{E}[Y_n] + \varepsilon M$, and therefore it follows that $\limsup_{n \rightarrow \infty} \mathbb{E}[X_n] \leq \varepsilon M$, for any $\varepsilon > 0$. Therefore we get $\limsup_{n \rightarrow \infty} \mathbb{E}[X_n] \leq 0$ and this yields (A.1). \square

A.2. Some technical tools

A.2.1. *Laws and subsets*

LEMMA A.3. *Let (Ω, A, P) and $(\tilde{\Omega}, \tilde{A}, \tilde{P})$ be complete probability spaces, E, F be separable Banach spaces with a continuous embedding of F in E and consider $X : \Omega \rightarrow E$ and $\tilde{X} : \tilde{\Omega} \rightarrow E$, two random variables with the same law on E .*

If X takes values in F , then it is a random variable with values in F . Moreover, \tilde{X} is also a random variable with values in F , with the same law on F .

Proof. Assume that $X(\Omega) \subset F$. By [16, theorem 1.1, p. 5], $\mathcal{B}(F) \subset \mathcal{B}(E)$ and $X : \Omega \rightarrow F$ is a random variable. Moreover, $F \in \mathcal{B}(E)$ and $0 = P[X \notin F] = \tilde{P}[\tilde{X} \notin F]$ thus it follows that $\tilde{X}(\tilde{\Omega}) \subset F$. The same argument yields that $\tilde{X} : \tilde{\Omega} \rightarrow F$ is a random variable and that X and \tilde{X} have the same law on F , namely the restriction of their law to F . \square

A.2.2. *On predictability* Let (Ω, A, P) be a complete probability space, E, F be separable Banach spaces, and $X : \Omega \times [0, T] \rightarrow E$ a process with continuous trajectories and denote by (\mathcal{F}_t) its natural filtration.

LEMMA A.4. For any $t \in [0, T]$, denote by $X_{|[0,t]} : \Omega \rightarrow \mathcal{C}([0, t]; E)$, $\omega \mapsto X(\omega, \cdot)$. Then, $\sigma\{X_{|[0,t]}\}$, the sigma algebra generated by the random variable $X_{|[0,t]}$, is \mathcal{F}_t .

Proof. Since X is continuous with respect to t and adapted to (\mathcal{F}_t) , it is progressively measurable, thus $\forall t \in [0, T]$, $X : \Omega \times [0, t] \rightarrow E$ is $\mathcal{F}_t \times B(0, t)$ measurable. Therefore, $X : \Omega \times [0, t] \rightarrow E$ is a Carathéodory function on $(\Omega, \mathcal{F}_t) \times [0, t]$ to E , and thanks to [1, theorem 1], the random variable $X_{|[0,t]} : \Omega \rightarrow \mathcal{C}([0, t]; E)$, $\omega \mapsto X(\omega, \cdot)$ is \mathcal{F}_t -measurable. Hence, $\sigma\{X_{|[0,t]}\} \subset \mathcal{F}_t$.

On the other hand, for any $s \in [0, t]$, the mapping $\delta_s : \mathcal{C}([0, t]; E) \rightarrow E, g \mapsto g(s)$ is continuous. Thus, $\omega \mapsto X(\omega, s) = \delta_s(X_{|[0,t]}(\omega))$ is $\sigma\{X_{|[0,t]}\}$ measurable for any such s and $\sigma\{X_{|[0,t]}\} = \mathcal{F}_t$. \square

Denote by $Y : \Omega \rightarrow F$ a \mathcal{F}_t -measurable random variable. Then, Y is $\sigma\{X_{|[0,t]}\}$ measurable and, by [9, lemma 1.13 p. 7],³ there exists a Borel function h from $\mathcal{C}([0, t], E)$ to F such that $Y = h(X_{|[0,t]})$.

Denote by ψ the nonnegative mapping defined on $\mathcal{C}([0, T]; E) \times F$ by $\psi(u, v) = \|h[u_{|[0,t]}] - v\|_F$ where $u_{|[0,t]}$ denotes the restriction to $[0, t]$ of u . The mapping $u \mapsto u_{|[0,t]}$ is continuous from $\mathcal{C}([0, T]; E)$ into $\mathcal{C}([0, t]; E)$, and h is a Borel function from $\mathcal{C}([0, t]; E)$ into F . Therefore, $(u, v) \mapsto h[u_{|[0,t]}] - v$ is a Borel function from $\mathcal{C}([0, T]; E) \times F$ into F , hence ψ is a Borel function.

Consider another complete probability space $(\tilde{\Omega}, \tilde{\mathcal{A}}, \tilde{P})$, a random variable $\tilde{Y} : \tilde{\Omega} \rightarrow F$, and a pathwise continuous process $\tilde{X} : \tilde{\Omega} \times [0, T] \rightarrow E$ with its natural filtration $(\tilde{\mathcal{F}}_t)$.

LEMMA A.5. If (X, Y) and (\tilde{X}, \tilde{Y}) have the same law, \tilde{Y} is $\tilde{\mathcal{F}}_t$ -measurable.

Proof. Indeed, $\mathbb{E}[\psi(X, Y)] = \tilde{\mathbb{E}}[\psi(\tilde{X}, \tilde{Y})]$ and since $0 = \mathbb{E}[\psi(X, Y)]$, one has also $\tilde{\mathbb{E}}[\psi(\tilde{X}, \tilde{Y})] = 0$. Therefore, $\tilde{Y} = h(\tilde{X}_{|[0,t]})$ and \tilde{Y} is $\sigma\{\tilde{X}_{|[0,t]}\}$, thus $\tilde{\mathcal{F}}_t$ -measurable since what has been developed above for X holds also for \tilde{X} . \square

A similar result holds concerning continuous processes:

LEMMA A.6. If $Y : \Omega \times [0, T] \rightarrow F$ and $\tilde{Y} : \tilde{\Omega} \times [0, T] \rightarrow F$ are continuous processes such that Y is (\mathcal{F}_t) -adapted and $(X, Y), (\tilde{X}, \tilde{Y})$ have the same law μ on $\mathcal{C}([0, T]; E) \times \mathcal{C}([0, T]; F)$, then \tilde{Y} is $(\tilde{\mathcal{F}}_t)$ -predictable.

Proof. Indeed, Y is also progressively measurable and, for any $t, Y_{|[0,t]} : (\Omega, \mathcal{F}_t) \rightarrow \mathcal{C}([0, t], F)$, $\omega \mapsto Y(\omega, \cdot)$ is measurable, thus $\sigma\{X_{|[0,t]}\}$ measurable. Then, there exists a Borel function h_t such that $Y_{|[0,t]} = h_t(X_{|[0,t]})$. By using the above idea, for given $(u, v) \in \mathcal{C}([0, T]; E) \times \mathcal{C}([0, T]; F)$, $\psi(u, v) = \|h[u_{|[0,t]}] - v_{|[0,t]}\|_{\mathcal{C}([0,t],F)}$ is a non-negative Borel function and $0 = \mathbb{E}[\psi(X_{|[0,t]}, Y_{|[0,t]})] = \tilde{\mathbb{E}}[\psi(\tilde{X}_{|[0,t]}, \tilde{Y}_{|[0,t]})]$. Thus, $\tilde{Y}_{|[0,t]}$ is $\tilde{X}_{|[0,t]}$ measurable, thus $\tilde{\mathcal{F}}_t$ measurable, and, as a Carathéodory function, it is progressive, then predictable. \square

³[9, lemma 1.13, p. 7]: fix two measurable functions f and g from a measurable space (Ω, \mathcal{A}) into some measurable Borel spaces (S, \mathcal{S}) and (T, \mathcal{T}) . f is g -measurable (i.e. $\sigma(f) \subset \sigma(g)$) if and only if there exists some measurable mapping $h : T \rightarrow S$ with $f = h \circ g$.

A.3. From Ω to Ω'

In this section, we are interested in proving that the time discretization scheme is preserved by changing the probability space. Following the notations of §§ 2.3.1 and 2.3.2, by Skorokhod representation, we consider a vector of random elements $Y'_N = (u'_N, \widehat{u}'_N, \widehat{u}'_N - \widehat{M}'_N, M'_N, \widehat{M}'_N, \Phi'_N, W'_N)$ with the same law ν_N as Y_N on \mathcal{X} . Let us notice that thanks to Appendix A.2.1, Y_N and Y'_N have both the same law on $[L^2(Q_T) \cap L^p(0, T; W_0^{1,p}(D))]^2 \times W^{1,q,p'}(0, T; W_0^{1,q}(D), W^{-1,p'}(D)) \times [\mathcal{C}([0, T]; L^2(D))]^2 \times L^2(0, T; HS(L^2(D))) \times \mathcal{C}([0, T]; U)$ denoted \mathcal{X}' where $q = \min(2, p)$. Note that the semi-implicit Euler–Maruyama scheme (2.1) means, for any k ,

$$[\widehat{u}_N - \widehat{M}_N](t_{k+1}) - [\widehat{u}_N - \widehat{M}_N](t_k) - \int_{t_k}^{t_{k+1}} \operatorname{div}(a(x, u_N^r, \nabla u_N^r) + F(u_N^r)) \, dt = 0,$$

and since for any $\varphi \in \mathcal{C}_c^\infty(D)$, the following mapping Ψ defined by

$$\Psi(Y_N) = \left| \int_D \varphi [\widehat{u}_N - \widehat{M}_N](t_{k+1}) - \varphi [\widehat{u}_N - \widehat{M}_N](t_k) + \int_{t_k}^{t_{k+1}} (a(x, u_N^r, \nabla u_N^r) + F(u_N^r)) \nabla \varphi \, dt \, dx \right|,$$

is a positive Borel function on \mathcal{X}' , one gets that

$$[\widehat{u}'_N - \widehat{M}'_N](t_{k+1}) - [\widehat{u}'_N - \widehat{M}'_N](t_k) - \int_{t_k}^{t_{k+1}} \operatorname{div}(a(x, u'_N, \nabla u'_N) + F(u'_N)) \, dt = 0.$$

For a fixed $N \in \mathbb{N}^*$, u_N^r is a $W_0^{1,p}(D)$ -valued, right continuous step function on the fixed time discretization $(t_k)_{k=0}^N$: it can be identified with an element of $W_0^{1,p}(D)^N$ by the Borel measurable mapping $\sum_{k=1}^N u_k 1_{[t_{k-1}, t_k)} \mapsto (u_1, \dots, u_N)$. By Appendix A.2.1, one concludes that u'_N follows the same structure: $\sum_{k=1}^N u'_k 1_{[t_{k-1}, t_k)}$ and

$$\int_{t_k}^{t_{k+1}} \operatorname{div}(a(x, u'_N, \nabla u'_N) + F(u'_N)) \, dt = \tau \operatorname{div}(a(x, u'_{k+1}, \nabla u'_{k+1}) + F(u'_{k+1})).$$

By a similar reasoning, Φ'_N , \widehat{u}'_N and \widehat{M}'_N are respectively right continuous step function and piecewise affine continuous functions for the two last ones. Taking into account those information, one gets that the discretization is conserved by changing the probability space, i.e.

$$u'^{k+1} - u'^k - \tau \operatorname{div}(a(x, u'^{k+1}, \nabla u'^{k+1}) + F(u'^{k+1})) = \Phi'_k \Delta_{k+1} W_N.$$

Then, by Appendix A.2.2, since Φ_k is \mathcal{F}_{t_k} measurable where (\mathcal{F}_t) is the filtration generated by W , then, Φ'_k will be \mathcal{F}'_{t_k} measurable where (\mathcal{F}'_t) is the filtration generated by W_N . Therefore, the time discretization (2.1) is totally recovered and by the uniqueness of lemma 2.1, u'_k is \mathcal{F}'_{t_k} measurable for any k .

REMARK A.1. Note that, as a by-product, one gets that

$$\begin{aligned}\mathbb{E}'\|\Phi'_N\|_{L^2(0,T;HS(L^2(D)))}^2 &= \mathbb{E}\|\Phi_N\|_{L^2(0,T,HS(L^2(D)))}^2 \rightarrow \mathbb{E}\|\Phi\|_{L^2(0,T;HS(L^2(D)))}^2, \\ \mathbb{E}'\|\widehat{M}'_N\|_{\mathcal{C}([0,T];L^2(D))}^2 &= \mathbb{E}\|\widehat{M}_N\|_{\mathcal{C}([0,T];L^2(D))}^2 \rightarrow \mathbb{E}\left\|\int_0^\cdot \Phi dW(t)\right\|_{\mathcal{C}([0,T];L^2(D))}^2\end{aligned}$$

for $N \rightarrow \infty$,

$$\begin{aligned}\mathbb{E}' \sup_{t \in [0,T]} \|\Phi'_N(t)\|_{HS(L^2(D))}^2 &= \mathbb{E} \sup_{t \in [0,T]} \|\Phi_N(t)\|_{HS(L^2(D))} \leq \mathbb{E}\|\Phi\|_{\mathcal{C}([0,T];HS(L^2(D)))}^2, \\ \mathbb{E}'\|W_N\|_{\mathcal{C}([0,T];U)}^2 &= \mathbb{E}\|W\|_{\mathcal{C}([0,T];U)}^2\end{aligned}$$

for all $N \in \mathbb{N}^*$ and all the *a priori* estimates developed in lemmas 2.2 to 2.6 remain true on $(\Omega', \mathcal{F}', P')$.

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