INEQUALITIES FOR A CLASS OF TERMINATING GENERALISED HYPERGEOMETRIC FUNCTIONS

by T. M. MACROBERT

(Received 3rd February, 1949)

§ 1. *Introductory.* By applying Gauss's Theorem it can be seen that, if n is a positive integer and α is not integral,

$$F\left(\begin{matrix} -n,\ \delta-n\ ; & 1 \\ \alpha-2n & \end{matrix}\right)=\frac{\Gamma(\alpha-2n)\,\Gamma(\alpha-\delta)}{\Gamma(\alpha-n)\,\Gamma(\alpha-\delta-n)}=\frac{\Gamma(1-\alpha+n)\,\Gamma(1-\alpha+\delta+n)}{\Gamma(1-\alpha+2n)\,\Gamma(1-\alpha+\delta)}$$

$$=\frac{\Gamma(\tfrac{1}{2})\,\Gamma(1-\alpha+n)\,\Gamma(1-\alpha+\delta+n)\,2^{\alpha-2n}}{\Gamma(\tfrac{1}{2}-\tfrac{1}{2}\alpha+n)\,\Gamma(1-\tfrac{1}{2}\alpha+n)\,\Gamma(1-\alpha+\delta)},$$

so that

$$F\left(\begin{matrix} -n,\ \delta-n\ ; & 1 \\ \alpha-2n & \end{matrix}\right)\sim\frac{2^{\alpha}\Gamma(\tfrac{1}{2})}{\Gamma(1-\alpha+\delta)}\cdot\frac{n^{\frac{1}{2}-\alpha+\delta}}{2^{2n}}. \qquad \text{(1)}$$

In section 2 it will be proved that, if

$$F(n)\equiv F\left(\begin{matrix} -n,\ \delta-n,\ \gamma-2n\ ; & 1 \\ \alpha-2n,\ \beta-2n & \end{matrix}\right),$$

where α and β are not integers,

$$|\,F(n)\,|\leqq M\,\frac{n^{\mu}}{2^{2n}}, \qquad \text{(2)}$$

where μ and M are constants independent of n.

Now, by the Ratio Test, the series

$$\sum_{n=1}^{\infty}\frac{n^{\mu}}{2^{2n}}\,x^{n}$$

converges absolutely if $|\,x\,|<4$. Hence, by the Comparison Test, the series

$$\sum_{n=1}^{\infty}F(n)\,x^{n},$$

also converges absolutely if $|\,x\,|<4$.

The formulae

$$\frac{1}{\Gamma(z+1)}\sim\frac{e^{z}}{\sqrt{(2\pi)}\,z^{z+\frac{1}{2}}}, \qquad \text{(3)}$$

$$\frac{\Gamma(z+\alpha)}{\Gamma(z+\beta)}\sim z^{\alpha-\beta}, \qquad \text{(4)}$$

where $-\pi<\text{amp}\,z<\pi$ will be required in the proof; in these formulae the convergence is uniform if

$$-\pi+\epsilon\leqq\text{amp}\,z\leqq\pi-\epsilon.$$

The proof can easily be extended to more general hypergeometric functions of the type $F(n)$.

A similar discussion of the function

$$F\left(\begin{matrix} -n,\ \alpha,\ \beta\ ;\ 1 \\ \gamma-\tfrac{1}{2}n,\ \delta-\tfrac{1}{2}n \end{matrix}\right)$$

will be found in section 3.

https://doi.org/10.1017/S2040618500032913 Published online by Cambridge University Press

§ 2. *Proof by Contour Integration.* The contour $DOABCD$ (Fig. 1) consists of DA, the part of the x-axis from $-n^2$, where n is a large positive integer, to $n+\frac{1}{2}$, indented above the x-axis at the points $0, 1, 2, \ldots, n$, the segment AB of the line $x=n+\frac{1}{2}$, B being the point where the line meets the circle $|z|=n^2$, and the arc BCD of that circle. Consider the integral

$$\int e^{2\pi i z}\pi \operatorname{cosec}(\pi z) f(z)\, dz,$$

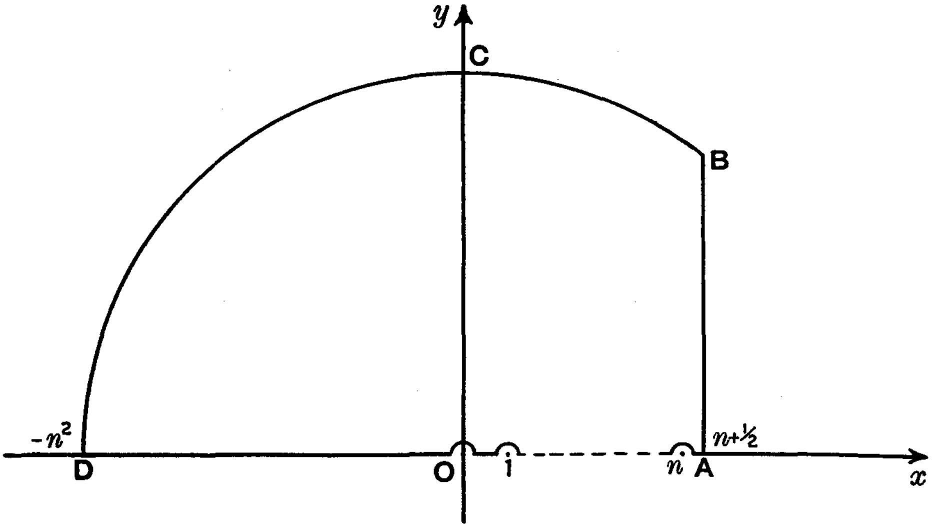

FIG. 1

taken round the contour, where

$$f(z)=\frac{\Gamma(1-\alpha+2n-z)\,\Gamma(1-\beta+2n-z)}{\Gamma(1+z)\,\Gamma(n+1-z)\,\Gamma(n+1-\delta-z)\,\Gamma(1-\gamma+2n-z)} \quad \ldots\ldots\ldots\ldots(5)$$

$$=-\frac{\sin \pi z}{\pi}\,\frac{\Gamma(-z)\,\Gamma(1-\alpha+2n-z)\,\Gamma(1-\beta+2n-z)}{\Gamma(n+1-z)\,\Gamma(n+1-\delta-z)\,\Gamma(1-\gamma+2n-z)}. \quad \ldots\ldots\ldots\ldots(6)$$

If n is large enough all the singularities of the integrand will lie outside the contour and the value of the integral will be zero. Thus

$$0=-\pi i\kappa(n)F(n)+P\int_{-n^2}^{n+\frac{1}{2}} e^{2\pi i x}\pi \operatorname{cosec}(\pi x) f(x)\,dx+J_1+J_2,$$

where J_1 and J_2 are the integrals along AB and BCD respectively and

$$\kappa(n)=\frac{\Gamma(1-\alpha+2n)\,\Gamma(1-\beta+2n)}{\Gamma(1+n)\,\Gamma(1-\delta+n)\,\Gamma(1-\gamma+2n)} \quad \ldots\ldots\ldots\ldots(7)$$

$$=\frac{\Gamma(\frac{1}{2}-\frac{1}{2}\alpha+n)\,\Gamma(1-\frac{1}{2}\alpha+n)\,\Gamma(\frac{1}{2}-\frac{1}{2}\beta+n)\,\Gamma(1-\frac{1}{2}\beta+n)\,2^{-\alpha-\beta+2n}}{\Gamma(1+n)\,\Gamma(1-\delta+n)\,\Gamma(\frac{1}{2}-\frac{1}{2}\gamma+n)\,\Gamma(1-\frac{1}{2}\gamma+n)\,\Gamma(\frac{1}{2})\,2^{-\gamma}}$$

$$\sim\frac{2^{\gamma-\alpha-\beta}}{\Gamma(\frac{1}{2})}\cdot\frac{2^{2n}}{n^{\alpha+\beta-\gamma-\delta+\frac{1}{2}}}, \quad \ldots\ldots\ldots\ldots(8)$$

by (4).

Hence, on taking imaginary parts, we have

$$F(n)=\frac{1}{\kappa(n)}\int_{-n^2}^{n+\frac{1}{2}} 2\cos(\pi x) f(x)\,dx+\frac{1}{\pi\kappa(n)}I(J_1+J_2).$$

https://doi.org/10.1017/S2040618500032913 Published online by Cambridge University Press

Now, in the integral replace $2\cos(\pi x)$ by $e^{i\pi x}+e^{-i\pi x}$, separate the two parts and replace them by integrals round $DCBA$ and the reflection of $DCBA$ in the x-axis respectively. Then

$$F(n)=\frac{1}{\pi\kappa(n)}I(J_1+J_2)-\frac{1}{\kappa(n)}(I_1+I_2+I_3+I_4),$$

where I_1 and I_2 are the integrals of $e^{i\pi z}f(z)$ along AB and the arc BCD respectively and I_3 and I_4 are the integrals of $e^{-i\pi z}f(z)$ along the reflections of AB and the arc BCD in the x-axis respectively.

Now

$$I_1=(-1)^{n+1}\int_0^{n^2\sin\phi}\frac{e^{-\pi y}\Gamma(\frac{1}{2}-\alpha+n-iy)\,\Gamma(\frac{1}{2}-\beta+n-iy)}{\Gamma(\frac{3}{2}+n+iy)\,\Gamma(\frac{1}{2}-iy)\,\Gamma(\frac{1}{2}-\delta-iy)\,\Gamma(\frac{1}{2}-\gamma+n-iy)}\,dy,$$

where $\cos\phi=(n+\frac{1}{2})/n^2$, so that, when $n\to\infty$, $\phi\to\frac{1}{2}\pi$.

The factors $1/\Gamma(\frac{1}{2}-iy)$ and $1/\Gamma(\frac{1}{2}-\delta-iy)$ are finite and independent of n for finite values of y; while, when y is large, by (3),

$$\frac{1}{\Gamma(\frac{1}{2}-iy)\,\Gamma(\frac{1}{2}-\delta-iy)}\sim\frac{e^{-1-\delta-2iy}}{2\pi(-\frac{1}{2}-iy)^{-iy}(-\frac{1}{2}-\delta-iy)^{-\delta-iy}}$$
$$=\frac{e^{-1-\delta-2iy}e^{-\chi y}e^{i\psi\delta-\psi y}}{2\pi\,|\,\frac{1}{2}+iy\,|^{-iy}\,|\,\frac{1}{2}+\delta+iy\,|^{-\delta-iy}},$$

where $\chi=\tan^{-1}(2y)$, in the third quadrant, and $\psi=\tan^{-1}\{y/(\frac{1}{2}+\delta)\}$, in the third or fourth quadrant. Thus, when $y\to\infty$, χ and ψ both $\to-\frac{1}{2}\pi$, and therefore

$$\frac{e^{-\pi y}\,|\,\frac{1}{2}+\delta+iy\,|^{-\delta}}{|\,\Gamma(\frac{1}{2}-iy)\,\Gamma(\frac{1}{2}-\delta-iy)\,|}$$

tends to a definite limit when $y\to\infty$. This function is therefore bounded for large values of y, and consequently for $0\leqq y\leqq\infty$.

Thus

$$\frac{e^{-\pi y}}{|\,\Gamma(\frac{1}{2}-iy)\,\Gamma(\frac{1}{2}-\delta-iy)\,|\,n^{\rho}},$$

where ρ is the larger of 0 and 2δ, is bounded for $0\leqq y\leqq n^2\leqq\infty$.

Again, from (4),

$$\frac{\Gamma(\frac{1}{2}-\beta+n-iy)}{\Gamma(\frac{1}{2}-\gamma+n-iy)}\sim(n-iy)^{\gamma-\beta}=(n^2+y^2)^{\frac{1}{2}\gamma-\frac{1}{2}\beta}e^{-i\omega(\gamma-\beta)},$$

where $\omega=\tan^{-1}(y/n)$, in the first quadrant. Therefore

$$\left|\frac{\Gamma(\frac{1}{2}-\beta+n-iy)}{\Gamma(\frac{1}{2}-\gamma+n-iy)}\right|\frac{1}{n^{\sigma}},$$

where σ is the larger of 0 and $2\gamma-2\beta$, is bounded for $0\leqq y\leqq n^2\leqq\infty$.

Next

$$\frac{\Gamma(\frac{1}{2}-\alpha+n-iy)}{\Gamma(\frac{3}{2}+n+iy)}\sim\frac{e^{1+\alpha+2iy}(-\frac{1}{2}-\alpha+n-iy)^{-\alpha+n-iy}}{(\frac{1}{2}+n+iy)^{1+n+iy}}$$
$$\sim e^{2iy}(n^2+y^2)^{-\frac{1}{2}-\frac{1}{2}\alpha-iy}e^{-i\omega(1-\alpha+2n)},$$

where $\omega=\tan^{-1}(y/n)$ in the first quadrant.

Therefore

$$\left|\frac{\Gamma(\frac{1}{2}-\alpha+n-iy)}{\Gamma(\frac{3}{2}+n+iy)}\right|\frac{1}{n^{\tau}},$$

where τ is the larger of 0 and $-2-2\alpha$, is bounded for $0\leqq y\leqq n^2\leqq\infty$.

Thus the modulus of the integrand, divided by $n^{\rho+\sigma+\tau}$, is bounded for $0 \leqq y \leqq n^2 \leqq \infty$. Therefore, since the range of integration is of length $n^2 \sin \phi$,

$$\left| \frac{I_1}{\kappa(n)} \right| < M_1 \frac{n^{\mu_1}}{2^{2n}},$$

where M_1 is a definite positive number independent of n.

Similar results hold for I_3 and J_1.

Again,

$$I_2 = -\frac{1}{\pi} \int_{\phi}^{\pi} e^{i\pi z} \sin(\pi z) \frac{\Gamma(-z)\Gamma(1-\alpha+2n-z)\Gamma(1-\beta+2n-z)}{\Gamma(n+1-z)\Gamma(n+1-\delta-z)\Gamma(1-\gamma+2n-z)} n^2 e^{i\theta} i \, d\theta,$$

where $z = n^2 e^{i\theta}$ and $-z = n^2 e^{i(\theta-\pi)}$. Here, when n is large,

$$\left| \frac{\Gamma(-z)\Gamma(1-\alpha+2n-z)\Gamma(1-\beta+2n-z)}{\Gamma(n+1-z)\Gamma(n+1-\delta-z)\Gamma(1-\gamma+2n-z)} \right| \sim | -z |^{\gamma-\alpha-\beta+\delta-1} = n^{2\gamma-2\alpha-2\beta+2\delta-2}.$$

Hence

$$\left| \frac{I_2}{\kappa(n)} \right| < M_2 \frac{n^{\gamma-\alpha-\beta+\delta+\frac{1}{2}}}{2^{2n}},$$

where M_2 is a definite positive number independent of n.

Similar results hold for J_2 and I_4.

Thus, finally, we arrive at the inequality (2).

§ 3. *Discussion of a Second Type of Generalised Hypergeometric Function.* It is proposed to prove that, if

$$F(n) = F\begin{pmatrix} -n, \alpha, \beta ; 1 \\ \gamma - \frac{1}{2}n, \delta - \frac{1}{2}n \end{pmatrix}, \quad \text{.......(9)}$$

where n is a positive integer, then

$$| F(n) | \leqq M n^{\mu} 2^n, \quad \text{.......(10)}$$

M and μ being constants independent of n.

It then follows that the series

$$\sum_{n=1}^{\infty} F(n) x^n$$

is absolutely convergent for $|x| < \frac{1}{2}$.

Let m be a positive integer greater than the larger of $-\alpha$ and $-\beta$. Let the contour $ABCA$ be formed of the segment AB of the x-axis from $m+\frac{1}{2}$ to n^2, where n is large, indented above the axis at $m+1, m+2, \ldots, n$, the part of the circle $|z| = n^2$ above the x-axis from B to the point C, where it crosses the ordinate at A, and the line CA.

Now, consider the integral

$$\int e^{2i\pi z} f(z) \, dz,$$

taken round the contour of Fig. 2, where

$$f(z) = \frac{\Gamma(-z)\Gamma(\alpha+z)\Gamma(\beta+z)}{\Gamma(n+1-z)\Gamma(\gamma-\frac{1}{2}n+z)\Gamma(\delta-\frac{1}{2}n+z)} \quad \text{.......(11)}$$

$$= (-1)^{n+1} \frac{\Gamma(-n+z)\Gamma(\alpha+z)\Gamma(\beta+z)}{\Gamma(1+z)\Gamma(\gamma-\frac{1}{2}n+z)\Gamma(\delta-\frac{1}{2}n+z)} \quad \text{.......(12)}$$

$$= -\frac{\pi}{\sin \pi z} \frac{\Gamma(\alpha+z)\Gamma(\beta+z)}{\Gamma(1+z)\Gamma(n+1-z)\Gamma(\gamma-\frac{1}{2}n+z)\Gamma(\delta-\frac{1}{2}n+z)}. \quad \text{.....(13)}$$

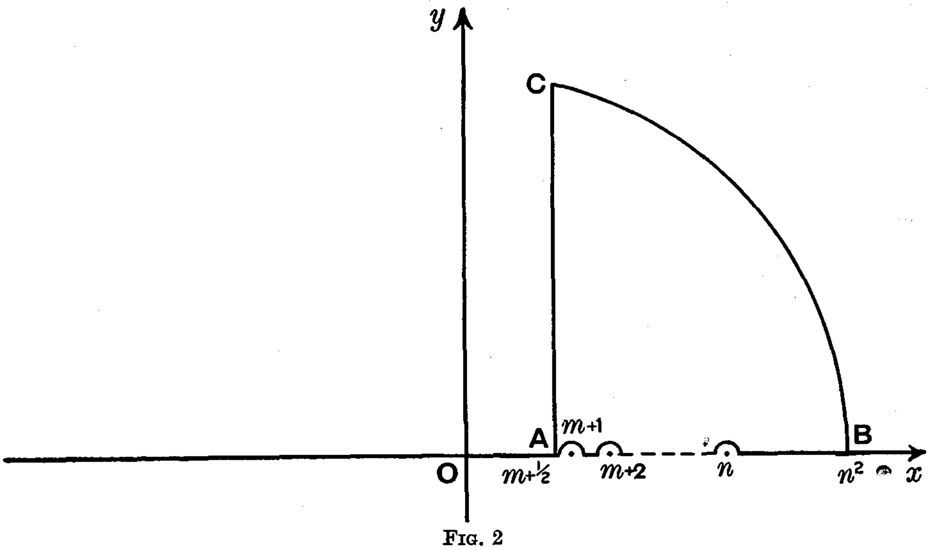

FIG. 2

Since all the singularities of the integrand lie outside the contour, the value of the integral is zero, and therefore

$$0 = i\pi\kappa(n)\{F(n) - \textit{the first } m+1 \textit{ terms of the series}\} + P\int_{m+\frac{1}{2}}^{n^2} e^{2i\pi x} f(x)\,dx + J_1 + J_2,$$

where

$$\kappa(n) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(n+1)\Gamma(\gamma-\frac{1}{2}n)\Gamma(\delta-\frac{1}{2}n)}, \quad \text{.............................(14)}$$

so that

$$|\kappa(n)| < D2^{-n} n^{\frac{1}{2}-\gamma-\delta}, \quad \text{...(15)}$$

D being a constant independent of n, and J_1 and J_2 are the integrals of

$$e^{2i\pi z} f(z)$$

along BC and CA respectively.

Hence, on equating imaginary parts, we have

$$E(n) \equiv F(n) - \textit{the first } m+1 \textit{ terms of the series}$$

$$= -\frac{1}{\pi\kappa(n)}\int_{m+\frac{1}{2}}^{n^2} 2\cos\pi x \sin\pi x\, f(x)\,dx - \frac{1}{\pi\kappa(n)} I(J_1+J_2)$$

$$= \frac{1}{\kappa(n)}\int_{m+\frac{1}{2}}^{n^2} (e^{i\pi x} + e^{-i\pi x})\phi(x)\,dx - \frac{1}{\pi\kappa(n)} I(J_1+J_2),$$

where

$$\phi(z) = \frac{\Gamma(\alpha+z)\Gamma(\beta+z)}{\Gamma(1+z)\Gamma(n+1-z)\Gamma(\gamma-\frac{1}{2}n+z)\Gamma(\delta-\frac{1}{2}n+z)}. \quad \text{.....................(16)}$$

Thus

$$E(n) = -\frac{1}{\kappa(n)}(I_1+I_2+I_3+I_4) - \frac{1}{\pi\kappa(n)} I(J_1+J_2),$$

where I_1 and I_2 are the integrals of

$$e^{i\pi z}\phi(z)$$

along BC and CA respectively, and I_3 and I_4 are the integrals of

$$e^{-i\pi z}\phi(z)$$

along the reflections in the x-axis of BC and CA respectively.

On referring to (13) and (12) and applying (4) it can be seen that, on BC,

$$| e^{i\pi z}\phi(z) | < G | z |^{\alpha+\beta-\gamma-\delta-1},$$

where G is a constant independent of n. Thus the moduli of the integrals J_1 and I_1 are each less than

$$Hn^{2\alpha+2\beta-2\gamma-2\delta},$$

where H is a constant independent of n. A similar result holds for I_3.

Again, on AC,

$$\left|\frac{\Gamma(\alpha+z)}{\Gamma(1+z)}\right| < Ly^{\alpha-1} \leqq Nn^{|2\alpha-2|},$$

where $0 \leqq y \leqq n^2$ and L and N are constants independent of n. Also, from (3),

$$\left|\frac{\Gamma(-n+z)\,\Gamma(\beta+z)}{\Gamma(\gamma-\frac{1}{2}n+z)\,\Gamma(\delta-\frac{1}{2}n+z)}\right| < R\left|\frac{(iy-n)^{iy-n}(iy)^{iy}}{(iy-\frac{1}{2}n)^{2iy-n}}\right| n^{\tau},$$

where R and τ are constants independent of n,

$$= R\,\frac{(y^2+\frac{1}{4}n^2)^{\frac{1}{2}n}e^{-\psi y}e^{-\frac{1}{2}\pi y}}{(y^2+n^2)^{\frac{1}{2}n}e^{-2\chi y}}\,n^{\tau},$$

where $\psi = \tan^{-1}(-y/n)$ in the second quadrant and $\chi = \tan^{-1}(-2y/n)$, also in the second quadrant. When $y \to n^2 \to \infty$, ψ and χ both $\to \frac{1}{2}\pi$. Thus the expression is less than

$$Qn^{\tau},$$

where Q is a constant independent of n.

It follows that the moduli of the integrals J_2 and I_2 are each less than

$$Sn^{\sigma},$$

where σ and S are constants independent of n.

A similar result holds for I_4.

Hence, finally, (10) is obtained.

UNIVERSITY OF GLASGOW