A COSINE FUNCTIONAL EQUATION IN HILBERT SPACE

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Throughout this paper R denotes the set of all real numbers, $m(K)$ the Lebesgue measure of $K \subseteq R$, H a Hilbert space, $L(H)$ the set of all linear continuous mappings of H into H, endowed with the usual structure of a Banach space.

We consider the mapping F of the set R into $L(H)$ such that

$$F(x + y) + F(x - y) = 2F(x)F(y) \tag{1}$$

holds for all $x, y \in R$. In **(2)** we have solved this equation under the assumption that H is of finite dimension. In this paper we prove that a weak measurability of F implies its weak continuity in the case of separable Hilbert space. In Theorem 2 we prove that every weakly continuous solution of (1) in the set of normal transformations has the form $F(x) = \cos(xN)$, where the normal transformation N does not depend on x.

We start with a preliminary lemma.

LEMMA 1. *Let K be a linear Lebesgue measurable set such that $0 < m(K) < +\infty$. There exists a number $a > 0$ with the property that for every $x \in (-a, a)$ there are $s_1(x)$, $s_2(x)$, $s_3(x) \in K$ such that $s_1(x) = s_2(x) - x/2 = s_3(x) - x$.*

Proof. Let $u(x)$ be the function defined on the set of all real numbers R by the equation $u(x) = m(K \cap (K - x/2) \cap (K - x))$. If $\chi(t)$ denotes the characteristic function of the set K then

$$\begin{aligned}|u(x) - u(0)| &= |\textstyle\int \chi(t)[\chi(t + x/2)\chi(t + x) - \chi(t)\chi(t + x) \\ &\qquad\qquad + \chi(t)\chi(t + x) - \chi(t)]dt\,| \\ &\leqslant \textstyle\int |\chi(t + x/2) - \chi(t)|dt + \int|\chi(t + x) - \chi(t)|dt.\end{aligned}$$

Since the right side tends to zero as $x \to 0$ we find the function $u(x)$ continuous in $x = 0$. Since $u(0) = m(K) \neq 0$, there exists a constant $a > 0$ such that $u(x) \neq 0$ for all $x \in (-a, a)$. But $u(x) \neq 0$ implies $K \cap (K - x/2) \cap (K - x) \neq \phi$. Hence for each $x \in (-a, a)$ there are $s_1(x)$, $s_2(x)$, $s_3(x) \in K$ such that $s_1(x) = s_2(x) - x/2 = s_3(x) - x$ and hence Lemma 1 is proved.

THEOREM 1. *Let F be a mapping of R into $L(H)$ which satisfies* (1) *for every $x, y \in R$.*

Suppose that: (1) *there is an interval $I = [a, b] \subseteq R$ such that the restriction of F to I is weakly measurable;*

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(2) *if* $F(x)f = 0$, *almost everywhere, then* $f = 0$; (3) H *is a separable Hilbert space.*

Then F is weakly continuous on R.

Proof. We divide the proof into three parts.

1. The function F is measurable on R. (1) implies:

$$F\left(x - \frac{b-a}{2}\right) = 2F(x)F\left(\frac{b-a}{2}\right) - F\left(x + \frac{b-a}{2}\right).$$

When x runs through the interval $[a, \frac{1}{2}(a+b)]$ then $x + \frac{1}{2}(b-a)$ runs over the interval $[\frac{1}{2}(a+b), b]$. Since $F(y)$ is measurable on each of these intervals we find that $F(y)$ is measurable on the interval $[a - \frac{1}{2}(b-a), a]$. Thus, the measurability of the function F on the interval I implies the measurability of this function on the interval $I' = [a - \frac{1}{2}(b-a), b]$. The way by which I' is obtained from I enables us to deduce that the function F is measurable on the set $(-\infty, b)$. For $x = 0$ (1) implies that F is an even function. Thus the function F is measurable on the set of all real numbers.

2. The function F is locally bounded. The separability of H implies immediately that $x \to ||F(x)||$ is a measurable function, hence there is a measurable set $K \subset R$ of strictly positive measure such that $L = \sup ||F(x)|| < +\infty$, $(x \in K)$. We assert that $||F(x)||$ is bounded on every finite interval. Since the function F is an even function we can, without loss of generality, assume that $K \subseteq [0, +\infty]$. If we put $x + y$ instead of y in (1) we get: $F(x) = 2F(x+y)F(y) - F(x+2y)$. This implies:

$$||F(x)|| \leqslant 2||F(x+y)|| \,.\, ||F(y)|| + ||F(x+2y)||. \tag{2}$$

For $x = y$ (1) implies: $F(2x) = 2F^2(x) - E$ and this gives:

$$||F(2x)|| \leqslant 2||F(x)||^2 + 1. \tag{3}$$

From (2) and (3) we get:

$$||F(x)|| \leqslant 2||F(x+y)|| \,.\, ||F(y)|| + 2||F(y + \tfrac{1}{2}x)||^2 + 1. \tag{4}$$

According to Lemma 1 there exists a number $a > 0$ with the property that for every $x \in (0, a)$ a number y can be found such that $y, y + \frac{1}{2}x, y + x \in K$. If $x \in (0, a)$ and if y is the corresponding element of K then (4) implies: $||F(x)|| \leqslant 4L^2 + 1$ for every $x \in (0, a)$. Thus the function $||F(x)||$ is bounded on the interval $(0, a)$. This and (3) imply that $||F(x)||$ is bounded on the interval $(0, 2a)$. From this we infer that the function $||F(x)||$ is bounded on every finite interval of the type $(0, b)$, $(b > 0)$. Since F is an even function we have that it is bounded on every finite interval.

3. The function F is weakly continuous. Since the function $F(x)$ is measurable and locally bounded, the functional

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$$\int_a^b (F(x)f, g)\, dx \tag{5}$$

is a bounded linear function on H for any $a, b \in R$ and $g \in H$. There is, therefore, a unique element $g_{ab} \in H$ such that:

$$\int_a^b (F(x)f, g)\, dx = (f, g_{ab})$$

for every $f \in H$. Let H' denote the set of all g_{ab}. We assert that H' is dense everywhere on H. In fact, let $h \in H$, $h \perp H'$, that is, let

$$\int_a^b (F(x)h, g)\, dx = 0 \tag{6}$$

for all $g \in H$ and for all numbers a and b. For given, but arbitrary g, (6) implies:

$$(F(x)\, h, g) = 0 \tag{7}$$

for $x \notin S_g$ where $mS_g = 0$. Let $A = \{g_1, g_2, g_3, \ldots,\}$ be a countable set dense in H and let

$$S = \bigcup_{n=1}^{\infty} S_{g_n}.$$

According to (7) we have

$$(F(x)h, g_n) = 0 \tag{8}$$

for all $x \notin S$. Since A is dense in H (8) implies $F(x)h = 0$ for every $x \notin S$, that is, almost everywhere. The requirement of Theorem 1 implies $h = 0$, that is, the set H' is dense in H.

If we put $2F(y)f$ instead of f in (5) and if we use (1) we find:

$$2(F(y)f, g_{ab}) = \int_{a+y}^{b+y} (F(x)f, g)\, dx + \int_{a-y}^{b-y} (F(x)f, g)\, dx. \tag{9}$$

If y_k tends to y_0, then (9) implies: $(F(y_k)f, h) \to (F(y_0)f, h)$ for every $h \in H'$. Since the sequence $F(y_k)f$ is bounded and since H' is dense in H we find

$$(F(y_k)f, g) \to (F(y_0)f, g)$$

for each pair $f, g \in H$, that is, $F(y_k)$ tends weakly to $F(y_0)$ whenever y_k tends to y_0. This proves that F is weakly continuous. Q.e.d.

THEOREM 2. *Let $N(x)$ be a mapping of R into $L(H)$ which satisfies* (1) *for every $x, y \in R$.*

Suppose that: (1) *$N(x)$ is a normal transformation for every $x \in R$;* (2) *if $N(x)f = 0$, almost everywhere, then $f = 0$;* (3) *$N(x)$ is weakly continuous.*

Then a bounded self-adjoint transformation B and self-adjoint transformation A which commutes with B can be found in such a way that

$$N(x) = \tfrac{1}{2}[\exp(ixN) + \exp(-ixN)] = \cos(xN)$$

holds for all x where $N = A + iB$.

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Proof. I. As in Theorem 1 we have

$$\int_a^b (N(x)f, g)\, dx = (f, g_{ab}).$$

We assert that the set H' of all g_{ab} is dense in H. In fact if h is an element of H which is orthogonal on H', then (6) holds for all $a, b \in R$ and $g \in H$. The continuity of function $(N(x)f, g)$ together with (6) imply (7) for every $x \in R$ and for every $g \in H$. From here we get $N(x)h = 0$ for all x which implies $h = 0$. Thus the set H' is dense in H. Using (1) we obtain:

$$\left(\frac{N(x) - E}{x} f, g_{ab}\right) = \frac{1}{2x}\left[\int_b^{b+x} (N(u)f, g)\, du + \int_b^{b-x} (N(u)f, g)\, du - \int_a^{a+x} (N(u)f, g)\, du - \int_a^{a-x} (N(u)f, g)\, du\right]$$

which implies:

$$\lim_{x\to 0} \left(\frac{N(x) - E}{x} f, g_{ab}\right) = 0$$

for every $g_{ab} \in H'$ and for every $f \in H$. From here it follows that the sequence

$$\frac{N^*(x) - E}{x} h$$

converges weakly to zero for every $h \in H'$, when $x \to 0$. There exists, therefore, a number $M(h)$ such that:

$$||[N(2^{-n}) - E]\, h|| \leqslant 2^{-n} M(h).$$

This implies that the series

$$\sum_{n=1}^{\infty} ||[N(2^{-n}) - E]h||^2 \tag{10}$$

is convergent for every $h \in H'$.

II. The fact that $N(x)$ is an even function implies that $N(x)$ and $N(y)$ commute one with another for every couple of real numbers x and y. Now we consider the functional equation (1) only for x and y from the set

$$G = \{r | r = 2^{-l}k,\, l, k = 0, \pm 1, \pm 2, \ldots ,\}.$$

Since G is countable and since $N(r)$ and $N(r')$ $(r, r' \in G)$ commute we find (**4**, p. 67),

$$N(r) = \int_R f(\xi, r) E(\Delta_\xi) \tag{11}$$

where $E(\Delta)$ is a real spectral measure and the function $f(\xi, r)$ is $E(\Delta)$-measurable and finite everywhere for every $r \in G$. If we put (11) in (1) we get:

$$f(\xi, r + r') + f(\xi, r' - r) = 2f(\xi, r)f(\xi, r') \tag{12}$$

for all $r, r' \in G$ and for almost all ξ (G is countable!). Using (11) we can write (12) in the form:

$$\lim_{n\to\infty} \int_R \sum_{k=1}^{n} |f(\xi, 2^{-n}) - 1|^2 ||E(\Delta_\xi)h||^2.$$

From the above it follows that the series

$$\sum_{n=1}^{\infty} |f(\xi, 2^{-n}) - 1|^2 \tag{13}$$

is convergent almost everywhere with respect to the measure $||E(\Delta)h||^2$. Since the set H' is dense in H the series (13) is convergent almost everywhere with respect to $E(\Delta)$. Thus

$$f(\xi, 2^{-n}) \to 1 \tag{14}$$

almost everywhere with respect to $E(\Delta)$. It follows from (14) and (12) that

$$f(\xi, r) = \tfrac{1}{2}[\exp ir\phi(\xi) + \exp(-ir\phi(\xi))] \tag{15}$$

hold true almost everywhere in ξ and for all $r \in G$ (see (**2**, Lemma 4)). Here $\phi(\xi)$ is $E(\Delta)$-measurable and everywhere finite complex-valued function. Thus the transformations

$$N = \int_R \phi(\xi)E(\Delta_\xi),\ A = \int_R [Re\phi(\xi)]E(\Delta_\xi) \text{ and } B = \int_R [Im\phi(\xi)]E(\Delta_\xi) \tag{16}$$

are defined. Since

$$||N(r)|| = \text{ess sup } |f(\xi, r)| < +\infty$$

for every $r \in G$, we find:

$$\text{ess sup } |\text{Im } \phi(\xi)| < +\infty,$$

that is, the transformation B is bounded. Then (16), (15), and (11) imply:

$$N(r) = \tfrac{1}{2}[\exp(irN) + \exp(-irN)] = \cos(rN)$$

for every $r \in G$. By the weak continuity and the fact that the set G is dense on R we find: $N(x) = \cos(xN)$ for every $x \in R$.

Remark 1. If we consider a mapping $r \to N(r)$ of the set G in the set $L(H)$ such that:

(1) $N(r)$ is a normal transformation;
(2) $N(r + r') + N(r' - r) = 2N(r)N(r')$ for all $r, r' \in G$, and
(3) $\lim ||N(1/2^n) - E|| = 0$

then $N(r) = \cos(rN)$, where normal transformation N does not depend on r. Indeed the representation (11) holds in this case too. Since $||N(r)|| = $ ess sup $|f(\xi, r)|$ (14) also holds. This together with (11) leads to (12) and consequently to (15), from which $N(r) = \cos(rN)$ follows.

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