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THE HOMOLOGY OF UNIFORM SPACES

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1. Introduction. In the past thirty years, algebraic topologists have developed a great body of knowledge concerning the category of topological spaces. By contrast, corresponding problems in the category of uniform spaces have been barely touched. Lubkin [**8**] studied the notion of a covering space in the category of generalized uniform spaces, and suggested that much of algebraic topology could be profitably studied in this category. Deming [**2**] discussed the fundamental group of a generalized uniform space, and related it to the first Čech homology group. A slightly different version of Čech cohomology was defined by Kuzminov and Svedov in [**7**] and related to the dimension theory of uniform spaces.

In this paper we will use the definition of a uniform space due to Tukey [**11**], in which a uniform structure Ω is a filter in the set of covers of X, partly ordered by refinement, such that any cover in Ω has a star refinement in Ω. The collection of uniform spaces and uniformly continuous maps form a category. To every uniformity Ω, we may associate a completely regular topology $\mathbf{T}\Omega$ by defining the closure of a set A to be the intersection of the sets $\mathrm{St}(A, \alpha)$ as α ranges over Ω. Since a uniformly continuous function is *a priori* continuous, we may set $\mathbf{T}f = f$ and regard $\mathbf{T}$ as a functor from the category of uniform spaces to the category of topological spaces.

$\mathbf{T}$ can be factored through the intermediate category of proximity spaces. We regard a proximity structure on X as being specified by a relation $\ll$ between subsets of X satisfying the usual axioms (see [**10**]). Given a uniformity Ω, the associated proximity $\mathbf{P}\Omega$ is defined by asserting that $A \ll B$ means that $\mathrm{St}(A, \alpha) \subset B$ for some α in Ω. Setting $\mathbf{P}f = f$, we may regard $\mathbf{P}$ as a functor from the category of uniform spaces to the category of promimity spaces. The usual process of topologizing a proximity space is also functorial – call this functor $\mathbf{T}'$ – and it is trivial to check that $\mathbf{T} = \mathbf{T}' \circ \mathbf{P}$.

Given a completely regular topological space (X, τ), let $\mathscr{U}(\tau)$ denote the set of uniformities on X whose associated topology is τ. This is a lattice with largest element (the fine uniformity), which is partitioned into equivalence classes by the proximity relation. Each equivalence class has a smallest element which is totally bounded, and therefore has a completion whose associated topological space is compact. This well-known correspondence between Hausdorff compactifications of (X, τ) and proximity structures or

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totally bounded uniform structures which induce τ, is a standard tool in the study of compactifications.

In section 2 we define the Čech theory on the category of uniform spaces and remark on the analogs in this context, of the standard basic properties of this theory. In section 3 we show that the Čech groups of a uniform space are naturally isomorphic to the groups of its completion, whence in particular, the homology of a compactification of X is the homology of the associated totally bounded uniformity. In section 4 we show that there is a huge collection of homologically different metric uniformities on the real line, which induce the usual topology.

2. Čech homology theory. We assume the reader is familiar with the Čech theory constructed in [**5**]. In brief, one constructs the groups of a topological space X with topology τ by first associating to each open cover of X a schema (or abstract simplicial complex) N_α, called the nerve of the cover. The system of open covers is directed by refinement and whenever β refines α there are a number of simplical maps from N_β to N_α, all of which are contiguous and therefore induce the same homomorphisms on the homology and cohomology groups. This gives rise to a direct system of cohomology groups and an inverse system of homology groups. The limits of these systems are called the Čech homology and cohomology groups of X, and denoted respectively by $H_*(X, \tau)$ and $H^*(X, \tau)$. If X is a uniform space with uniformity Ω, we do exactly the same thing, except that we take the limits over the system of open uniform covers, instead of all open covers. The resulting groups will be denoted by $H_*(X, \Omega)$ and $H^*(X, \Omega)$. If (X, Ω) and (Y, Ψ) are uniform spaces and $f: X \to Y$ is uniformly continuous, then for each open uniform cover β of T, $\alpha = f^{-1}(\beta)$ is an open uniform cover of X, whence homomorphisms $f_* : \check{H}_*(X, \Omega) \to \check{H}_*(X, \Psi)$ and $f^* : \check{H}^*(Y, \Psi) \to \check{H}^*(X, \Omega)$ are induced in precisely the same fashion as in the topological situation. The relative groups of a pair (X, A) are defined in the analogous fashion, with the limits being taken over pairs of open uniform covers.

Except for the homotopy and excision axioms, the discussion of the Eilenberg and Steenrod axioms can be carried over word for word to the uniform setting. Two uniformly continuous maps $f, g = (X, \Omega) \to (Y, \Psi)$ are uniformly homotopic if there is a uniformly continuous $F : X \times I \to \mathrm{R}$ such that $F(x, 0) = f(x)$ and $F(x, 1) = g(x)$, where the unit interval I carries the unique uniformity compatible with the usual topology and $X \times I$ carries the product uniformity. Note that f and g can be homotopic as maps of the associated topological spaces, without being uniformly homotopic. It is true that uniformly homotopic maps induce the same homomorphism, and the proof is essentially the same as in the topological case but technically easier since the product uniformity on $X \times I$ has as a base $\mathscr{B}_1 \times \mathscr{B}_2$ where $\mathscr{B}_1$ and $\mathscr{B}_2$ are bases for the uniformities on X and I respectively.

As in the topological case, we have an excision property $H(X, B) \approx$

$H(X - A, B - A)$ for both homology and cohomology, provided $A \ll B$. (This seems to be the proper analog of the less restrictive topological condition $\bar{A} \subset B^\circ$.)

We remark that Dowker [**3**] has shown that the Čech and Alexander cohomology theories and the Čech and Vietoris homology theories are naturally isomorphic when based on the same family of coverings. The Alexander cochain complex $C_n(X, \Omega)$ is the quotient group of all functions from X^{n+1} to the coefficient group G, modulo the subgroup of all locally zero functions (f is locally zero if for some uniform open cover α, $f(x_0, \ldots, x_n) = 0$ whenever some set of α contains $\{x_0, \ldots, x_n\}$), with the standard coboundary. Specializing to the case $n = 0$, we have

2.1. THEOREM. *Let G be a nontrivial abelian group, with the discrete uniformity. $H^\circ(X, \Omega)$ consists of all uniformly continuous functions from X into the coefficient group G.*

The notion of connectedness for uniform spaces was introduced by Lubkin in [**8**] and shown by Deming [**2**] to be equivalent to this condition: (X, Ω) is connected if every uniformly continuous function from X to a discrete uniform space with more than one point, is constant. (For example, the space of rational numbers is connected in the usual metric uniformity, though rather badly disconnected in the induced topology.) Thus, we have

COROLLARY. *A uniform space is connected if and only if $H^\circ(X, \Omega)$ is isomorphic to the (nontrivial) coefficient group G.*

3. Homology and completion. In 1960 Alfsen and Fenstad [**1**] gave a construction of the completion of a uniform space based on the notion of a regular filter. In a proximity space, a filter $\mathscr{F}$ is said to be regular if for any A in $\mathscr{F}$ there is a B in $\mathscr{F}$ such that $B \ll A$, and in a uniform space a filter is regular if it is regular in the induced proximity. Every filter $\mathscr{F}$ has a largest regular subfilter $\mathscr{F}^\#$ called the envelope of $\mathscr{F}$. If $\mathscr{F}$ is Cauchy, then $\mathscr{F}^\#$ is the smallest Cauchy subfilter of $\mathscr{F}$, so that the minimal Cauchy filters are exactly the regular Cauchy filters.

Given a separated uniform space (X, Ω), let $\tilde{X}$ denote the set of regular Cauchy filters in X, and for each subset A of X denote by $S(A)$ the set of regular Cauchy filters containing A. Associate to each open cover α in Ω a cover $\tilde{\alpha}$ of $\tilde{X}$ which consists of the sets $S(V)$ as V ranges over α. The collection of these covers is a base for a complete separated uniformity Ω on X, and the "inclusion" map f which assigns to each point its filter of neighbourhoods, is a uniform embedding of X as a dense subset of $\tilde{X}$. The verification of these assertions is a straightforward restatement of the standard arguments in the language of Tukey uniformities, and is left to the reader.

The point of this construction lies in the observations that $\alpha = f^{-1}(\tilde{\alpha})$ and that α and $\tilde{\alpha}$ have isomorphic nerves. Since the covers $\{\tilde{\alpha}\}$ are cofinal in $\tilde{\Omega}$, we have immediately

3.1. THEOREM. *The inclusion map of a uniform space into its completion induces homology and cohomology isomorphisms.*

We remark that if these inclusion maps are regarded as defining a natural transformation Φ from the identity to the completion functor, then $\mathbf{H}\Phi$ is a natural equivalence. Thus any sort of algebraic functor which is defined for uniform spaces by defining it on nerves of covers and taking limits, will be unable to distinguish between a uniform space and its completion.

3.2. *Example.* If X is locally compact and Ω_L is the uniformity with a base of open covers α such that each set in α has compact closure or compact complement, then $H(X, \Omega_L) = \check{H}(\mathring{X})$, where $\mathring{X}$ is the one-point compactification of X. (This is Theorem 6.9 on page 272 of [**5**].) This is because Ω_L is the relative uniformity on X induced by the unique uniformity on $\mathring{X}$, and $\mathring{X}$ is the completion of (X, Ω_L).

3.3. *Example.* If X is a metric space, then the finest uniformity compatible with the metric topology has as base, all open covers, and the Čech groups of the fine uniformity thus are the same as the Čech groups of X. We do not know whether it is generally true that the Čech groups of a completely regular topological space X are the same as the Čech groups of the corresponding fine uniformity Ω_F.

3.4. *Example.* Let X be completely regular, and let Ω_{SC} be the uniformity generated by $\{f^{-1}(\alpha)\}$ where α is a uniform cover in the metric uniformity on the reals, and $f \in C^*(X)$. Then Ω_{SC} is totally bounded and its completion is the Stone-Čech compactification βX. Thus $H(X, \Omega_{SC}) = \check{H}(\beta X)$. If X is normal, then Ω_{SC} is generated by all finite open covers. In particular, if X is the reals, then Dowker [**4**] has shown that $\check{H}^1(\beta X)$ is the quotient group $C(X)/C^*(X)$, while, of course, $\check{H}(X)$ is acyclic. It follows that the homology of the uniformities Ω_F and Ω_{SC} are in this case quite different, although the uniformities belong to the same proximity class.

4. Uniformities for the real line. In this section we prove two theorems about the homology of the real line as a uniform space. The first says that the reals are acyclic in the usual metric uniformity, just as they are in the topological setting based on all open covers. Note that the reals are contractible as a topological space (and hence uniformly contractible in the fine uniformity); however, Isbell [**6**] has shown that the reals are *not* uniformly contractible in the usual metric uniformity.

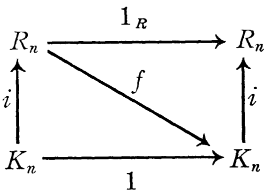
The second theorem shows that we can find a uniformity for the real line which induces the usual topology, which has any finite sequence $G_1, \ldots, G_n$ of finitely generated abelian groups as its homology groups $H_1(X), \ldots, H_n(X)$. The uniformities constructed to prove this theorem are both metric and totally bounded, hence each is the only element in its proximity class.

4.1. THEOREM. *The set of real numbers and the set of rational numbers with the metric uniformities are acyclic.*

Proof. For each positive integer n, let U_n be the relation $(x, y) \in U_n$ if and only if $|x - y| \leqq 1/n$. U_n is a closed symmetric entourage, and the sequence $U_1, U_2, \ldots, U_n, \ldots$, is a basis for the metric uniformity on the set R of reals (and the trace of these entourages on $Q \times Q$ is a base for the metric uniformity on the set Q of rationals). Let Q_n and R_n be the schemas in which $\{x_0, \ldots, x_m\}$ is a simplex if and only if $(x_i, x_j) \in U_n$ for all $0 \leqq i, j \leqq m$. Note that as a schema R_n is isomorphic to the nerve of the cover $\{U_{2n}[x] : x \in R\}$.

We claim that Q_n and R_n are both acyclic for each n, whence, since they are cofinal in the inverse system of schemas, it will follow that both Q and R are acyclic. To show this let K_n denote the set of all rationals of the form k/n. With the same relation U_n, this is a subschema of both Q_n and R_n. As a schema, it is isomorphic to the set of integers, with the simplexes being singletons and the adjacent integers. K_n is acyclic for all n, since its realization is *topologically* isomorphic to the reals.

Define a map $f : R_n \rightarrow K_n$ by $f(x) = k/n$ if $x \in [k/n - 1/2n, k/n + 1/2n)$. f is a map in the category of schemas, since if two points are within $1/n$ they can be mapped no further apart than two adjacent points of K_n.



$f \circ i = 1$, where i is the inclusion map. We claim that $i \circ f$ and 1_R have a common acyclic carrier. Let $\{x_0, \ldots, x_m\} = s$ be a simplex (with $x_0 < x_1 < \ldots < x_m$) and let $\phi(s)$ be the full subcomplex of R_n with vertices $x_0, \ldots, x_m$ and $y_1 = k/n = f(x_0)$ and also $y_2 = (k + 1)/n$ if $f(x_m) = y_2$. Now $\phi(s)$ is a simplex except in two cases

(i) $x_0 < x_1 < \ldots < x_k \leqq y_1 < x_{k+1} < \ldots < x_m < y_2$, $|y_2 - x_0| > 1/n$

(ii) $y_1 < x_0 < \ldots < x_k \leqq y_2 < x_{k+1} < \ldots < x_m$, $|x_m - y_1| > 1/n$.

Let us assume the first alternative (the argument is the same for the other alternative) and note that if we delete y_2 we have a simplex L of R_n. Let $g : \phi(s) \rightarrow L$ be defined by $g(y_2) = x_m$ and $g(t) = t$ otherwise. g is a "retraction" and $j \circ g$, where $j : L \rightarrow \phi(s)$ is the inclusion map, is contiguous to $1_{\phi(s)}$. Hence L is a "deformation retract" of $\phi(s)$, and since L is acyclic, so is $\phi(s)$.

It is well-known that if $G_1, G_2, \ldots, G_n$ is a sequence of finitely generated abelian groups, then there exists a compact polyhydron X such that $H_0(X) = Z$, $H_n(X) = G_n$ for $n = 1, 2, \ldots, N$. We will prove the following:

4.2. THEOREM. *Given any sequence $G_1, G_2, \ldots, G_N$ of finitely generated abelian groups, there is a (totally bounded, metric) uniformity Ω inducing the* *usual topology on the set of real numbers* R, *such that* $H_0(R, \Omega) = Z$ *and* $H_n(R, \Omega) = G_n$ *for* $n = 1, 2, \ldots, N$.

By Theorem 3.1 it suffices to show that there is a metrizable compactification of the reals with these holology groups. In [**9**] Simon describes a large class of compactifications of the reals. His main result is that if X is a compact Hausdorff space, and $g : [0, \infty) \to X$ is a continuous function such that $g([a, \infty))$ is dense in X for every $a > 0$, then there is a compactification (Y, f) of $[0, \infty)$ such that $Y - f([0, \infty))$ is homeomorphic to X. In fact, setting $h(x) = x/1 + x$ (so $h : [0, \infty) \to$ onto $[0, 1)$), we define

$$f : [0, \infty) \to X \times I \quad \text{by} \quad f(x) = (g(x), h(x)) \quad \text{and} \quad Y = \overline{\text{imf}}.$$

We can regard these compactifications as compactifications of R, since $(0, \infty)$ is homeomorphic to $(-\infty, \infty)$, so we are in a sense compactifying R by adding a point to the left end and X to the right end.

The above theorem follows immediately from these two observations:

LEMMA 1. *If* (Y, f) *is a "Simon compactification" of* $[0, \infty)$ *with* $Y - f([0, \infty)) = X$, *then the inclusion* $i : X \to Y$ *induces homology and cohomology isomorphisms.*

LEMMA 2. *If* X *is any connected, finite simplicial complex, then there exists a Simon compactification* (Y, f) *of* $[0, \infty)$ *such that* $Y - f([0, \infty)) \simeq X$.

The first lemma is a consequence of the continuity property of the Čech groups. We define $Y_n = Y - f([0, n))$ and let $i_n : Y_n \to Y_{n-1}$ be the inclusion map. Then since Y_{n+1} is a strong deformation retract of Y_n for each n (the retracting homotopy pulls the image of $[n, n + 1]$ in Y_n onto the end point $f(n + 1)$), the homomorphisms $i_n^* : \check{H}_*(Y_n) \to \check{H}_*(Y_{n-1})$ are isomorphisms, and therefore

$$\check{H}_*(X) = \lim_{\leftarrow} \check{H}_*(Y_n) = \check{H}_*(Y).$$

To establish Lemma 2, we need only produce a map $g : [0, \infty) \to X$, such that $g([a, \infty))$ is dense in X for each $a > 0$. We will assume that X has been realized as a geometric polyhedron in some finite dimensional Euclidean space. We construct g by defining a sequence of continuous functions $g_n : [n, n + 1] \to X$ such that

(1) $g_n(n) = g_n(n + 1) =$ a fixed "base vertex" x_0 of X for all n,

(2) each point of X is within $1/n$ of $\text{Im } g_n$,

and "pasting" them together – i.e., $g(x) = g_n(x)$ if $x \in [n, n + 1]$. It is clear that g will have the desired property.

To describe g_n we build an "edge loop" $\langle x_0 x_1 \rangle \langle x_1 x_2 \rangle \ldots \langle x_{N-1}, x_N \rangle$ in X and associate to each "edge" $\langle x_m, x_{m+1} \rangle$ a maximal simplex σ_m which has $\langle x_m, x_{m+1} \rangle$ as an edge, so that the list of associated maximal simplexes $\sigma_1, \sigma_2, \ldots, \sigma_N$ includes (perhaps with repetition) all the maximal simplexes of X. Now $X = \bigcup_{i=1}^N \sigma_i$ and each σ_i is totally bounded, so we can find a finite subset

$S_n{}^i$ of σ_i such that each point of σ_i is within $1/n$ of some points of $S_n{}^i$. Clearly the piecewise linear path σ_i which starts at x_{i-1} and goes through every point of $S_n{}^i$ in some order will lie in σ_i by convexity and pass within $1/n$ of each point of σ_i. Since σ_i begins at x_{i-1} and ends at x_i, the path $l_1 * l_2 * \ldots * l_N$, suitably reparametrized, can be taken to be g_n.

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