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SOME CLASSES OF TOPOLOGICAL SPACES WITH UNIQUE QUASI-UNIFORMITY

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ABSTRACT. We follow P. Fletcher and W. F. Lindgren's work in the study of topological spaces with a unique quasi-uniformity by generalizing some of their results and constructing larger classes of *uqu* spaces which contain some of their examples as a particular case.

1. **Introduction**. If the symmetry requirement is dropped from the definition of uniform space, then we obtain the concept of quasi-uniform space. Similar to the uniform case, every quasi-uniform space $(X, \mathcal{U})$ has a subjacent topological space $(X, \tau(\mathcal{U}))$, where a fundamental system of neighbourhoods of the point x is formed by the sets $U(x) = \{y \in X : (x, y) \in U\}$, $U \in \mathcal{U}$.

One of the most attractive results in the theory of quasi-uniform spaces is that every topological space is quasi-uniformizable. If (X, τ) is a topological space, then W. J. Pervin showed in [5] that the sets $S_G = (G \times G) \cup (X \sim G \times X)$, where G is an open set, form a subbase of a certain transitive quasi-uniformity, usually referred to as Pervin's quasi-uniformity.

Since every topological space admits a compatible quasi-uniformity, it seems rather natural to try to characterize those topological spaces which admit a unique quasi-uniformity (*uqu* spaces). Totally bounded quasi-uniformities play a fundamental role in the treatment of this problem; it might be interesting to recall that, different from the uniform case, the notions of precompact and totally bounded quasi-uniformity are not equivalent, it is easily noticed by looking at the definitions that total boundedness strictly implies precompactness:

— a quasi-uniform space $(X, \mathcal{U})$ is precompact if and only if for each $U \in \mathcal{U}$ there is a finite subset $A \subset X$ such that $X = \cup\{U(a) : a \in A\}$.

— $(X, \mathcal{U})$ is totally bounded if and only if for each $U \in \mathcal{U}$ there is a finite cover of X, $A_1, A_2, \ldots, A_n$, such that $A_i \times A_i \subset U$, $1 \leqslant i \leqslant n$.

P. Fletcher and W. F. Lindgren have studied the problem of giving a suitable purely topological characterization of *uqu* spaces (see [1], [2], [3] and [4]). The following two definitions are due to Lindgren:

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— In a topological space X, an open cover $\mathcal{L}$ is said to be a *Q-Cover* if for each point $x \in X$, its trace respect to $\mathcal{L}$, that is $T(x, \mathcal{L}) = \cap\{L \in \mathcal{L} : x \in L\}$, is an open set.

— A topological space is *Q-Finite* whenever all *Q-covers* are finite.

An idea of the much restricted area occupied by *uqu* spaces inside the class of topological spaces is given to us by Lindgren, when he shows in |3| that the classes of *uqu*, *Q-Finite* and Hereditarily Compact spaces are ordered by set-inclusion. Also, he gave (in [3] and [4]) two examples of *uqu* spaces which have infinite topology, thus destroying P. Fletcher's conjecture that every *uqu* space should have finite topology.

Our aim here is to generalize those two examples finding larger classes of *uqu* spaces of infinite topology. The paper is divided in two sections, the first one is devoted to a characterization of the cofinite *uqu* spaces, the second gives us a rather simple way to find *uqu* spaces with topology of any desired cardinality. For this purpose, we make use of Corollary 3.6 of [3], where Lindgren states that a sufficient condition for a topological space to be *uqu* is that every compatible quasi-uniformity should be totally bounded (notice that the condition is necessary as well, since Pervin's quasi-uniformity is always totally bounded).

2. **Cofinite *uqu* spaces**. Lindgren's first example of a *uqu* infinite space is the set of real numbers R equipped with the cofinite topology (see [4], example 2.8). We prove that this is a particular one of a more general case.

2.1 LEMMA. *If a topological space X is Q-Finite then it is a Baire space.*

PROOF. Let $(G_n)_{n=1}^{\infty}$ be a strictly decreasing sequence of dense open sets. We shall show that the set $\cap_{n=1}^{\infty} G_n$ is dense. Let H be a proper open set. For each n, let $H_n = H \cap G_n$. We may suppose that $(H_n)_{n=1}^{\infty}$ is also strictly decreasing (otherwise, it is trivial). If $H \cap (\cap_{n=1}^{\infty} G_n) = \cap_{n=1}^{\infty} H_n = \emptyset$, then the family $\mathcal{L} = \{H_n\}_{n=1}^{\infty} \cup \{X\}$ would be an infinite *Q-cover*.

2.2. PROPOSITION. *If X is a cofinite space, then X is uqu if and only if it is Baire.*

PROOF. Since *uqu* implies *Q-Finite*, necessity follows from the previous lemma. For the sufficiency, we make use of Lindgren's Corollary 3.6 of [3]. Let $\mathcal{U}$ be a compatible quasi-uniformity. If $\mathcal{U}$ were not totally bounded, there would be a $U \in \mathcal{U}$ such that $X \sim \cap\{U(x) : x \in X\}$ is an infinite set (otherwise, if $A = \cap\{U(x) : x \in X\}$, then $\{A\} \cup \{\{x\} : x \in X \sim A\}$ would be a finite cover of X satisfying $A \times A \subset U$, $\{x\} \times \{x\} \subset U$, for each $x \in X \sim A$). Thus, we may select a countably infinite set $B = \{b_n : n \geqslant 1\}$ contained in $X \sim A$. Now, since B is dense, for each $V \in \mathcal{U}$, we have $V^{-1}(B) = X$. Besides, for each $n \geqslant 1$, there is $a_n \in X$ such that $(a_n, b_n) \notin U$, and then, if we choose $V \in \mathcal{U}$ such that $V \circ V \subset U$, $V(a_n) \cap V^{-1}(b_n) = \emptyset$; thus, for each n, $V^{-1}(b_n)$ is a finite set. This would imply that X is countably infinite since $X = V^{-1}(B)$. However, since X has the cofinite topology this would contradict the fact that X is Baire.

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2.3 COROLLARY. *A cofinite space is uqu if and only if it is not countably infinite.*

We must honestly remark that the sufficiency part of the proof of Proposition 2.2 is an exact adaptation of Lindgren's proof of Example 2.8 of [4].

3. **Linearly ordered *uqu* spaces**. Lindgren's second example of an infinite *uqu* space is the set $X = [0, 1)$ with the topology $\mathcal{L} = \{\emptyset, X\} \cup \{[0, 1/n) : n \geqslant 1\}$, see [3], Example 3.2. We find a suitable generalization of this example.

3.1. LEMMA. *In a well-ordered set $(X, \leqslant)$, every decreasing sequence must be finite.*

3.2. PROPOSITION. *Let $(X, \leqslant)$ be a well-ordered set with a greatest element p. For each $x \in X$, let $T(x) = \{y \in X : x < y\}$ (open tail). Let $\mathcal{L}$ be the family consisting of $\emptyset$, X and all open tails of X. $(X, \mathcal{L})$ is a uqu topological space.*

PROOF. By making use of the well-ordering, it is straightforward to show $\mathcal{L}$ to be a topology in X. Let $\mathcal{U}$ be any compatible quasi-uniformity, we shall show that $\mathcal{U}$ is totally bounded. Let $U, V \in \mathcal{U}$ such that $V \circ V \subset U$. It is evident that, for each $x \in X$, the closed tail $T^*(x) = \{y \in X : x \leqslant y\}$ is contained in $\operatorname{int}(V(x))$. Also, by means of Lemma 3.1, every increasing open sequence is finite (i.e., $(X, \mathcal{L})$ is hereditarily compact). We construct by induction the following open sequence:

$$G_1 = \operatorname{int}(V(p)),$$

$$\text{for all } n \geqslant 1, \qquad x_n = \begin{cases} 1, & \text{if } G_n = X, \\ x, & \text{if } G_n = T(x), \end{cases}$$

$$\text{for all } n \geqslant 2, \qquad G_n = \operatorname{int}(V(x_{n-1})).$$

The sequence $(G_n)_{n=1}^{\infty}$ is easily seen to be increasing, therefore it only has a finite number of distinct elements: that is, $G_1 \subset G_2 \subset \ldots \subset G_n = G_m$, for all $m \geqslant n$. Moreover, $G_n = X$, otherwise, $G_n = T(x_n) = G_{n+1} = \operatorname{int}(V(x_n))$ would imply $x_n \in T(x_n)$.

The sets $A_1 = G_1$, $A_i = G_i \sim G_{i-1}$, for $i = 2, 3, \ldots, n$, obviously cover X and we now show that $A_i \times A_i \subset U$, $1 \leqslant i \leqslant n$: If $(x, y) \in A_1 \times A_1$, then $(p, y) \in V$ and since $x \leqslant p$, $(x, p) \in V$, so $(x, y) \in V \circ V \subset U$. If $(x, y) \in A_i \times A_i$, $2 \leqslant i \leqslant n - 1$, then $x, y \in G_i \sim G_{i-1} = \operatorname{int}(V(x_{i-1})) \sim T(x_{i-1})$, thus, $(x_{i-1}, y) \in V$ and, since $x_{i-1} \in T^*(x) \subset V(x)$, $(x, x_{i-1}) \in V$, so $(x, y) \in V \circ V \subset U$.

Finally, if $(x, y) \in A_n \times A_n$, then $x, y \in G_n \sim G_{n-1}$, thus, since $\operatorname{int}(V(x_{n-1})) = G_n = X$ and $x \notin G_{n-1} = T(x_{n-1})$ implies that $x_{n-1} \in T^*(x) \subset V(x)$, we have $(x_{n-1}, y) \in V$ and $(x, x_{n-1}) \in V$, so $(x, y) \in V \circ V \subset U$.

3.3. COROLLARY. For every cardinal $m \geqslant 2$, there is a *uqu* topological space whose topology has cardinal m.

Next we obtain another result which has Proposition 3.2 as a particular case. Its proof is quite similar to that of the mentioned proposition. As a consequence we have that Lindgren's second example is also contained in the following result.

3.4. Proposition. *Let $(X, \leqslant)$ be a linearly-ordered set with a greatest element p. Let Y be a non-empty subset of X which is well-ordered by the restriction of $\leqslant$. For each $x \in X$, let $T(x) = \{z \in X : x < z\}$. If $\mathscr{L} = \{\emptyset, X\} \cup \{T(y) : y \in Y\}$, then $(X, \mathscr{L})$ is a uqu topological space.*

Proof. The well-ordering in Y proves $\mathscr{L}$ to be a topology in X. If $\mathscr{U}$ is a compatible quasi-uniformity and $U, V \in \mathscr{U}$ are such that $V \circ V \subset U$, we form the following increasing open sequence:

$$G_1 = \operatorname{int}(V(p)),$$

$$\text{for } n \geqslant 2, \qquad G_n = \begin{cases} X, & \text{if } G_{n-1} = X, \\ \operatorname{int}(V(y)), & \text{if } G_{n-1} = T(y). \end{cases}$$

By Lemma 3.1, applied to $(Y, \leqslant)$, the sequence $(G_n)_{n=1}^{\infty}$ has a finite number of distinct elements: $G_1 \subset G_2 \subset \ldots \subset G_n = X$.

The sets $A_1 = G_1$, $A_i = G_i \sim G_{i-1}$, for $i = 2, 3, \ldots, n$, cover X and $A_i \times A_i \subset U$, $1 \leqslant i \leqslant n$.

3.5. Corollary. *Lindgren's example: $X = [0, 1)$, with the topology $\tau = \{\emptyset, X\} \cup \{[0, 1/n) : n \geqslant 2\}$, is a particular case of the uqu spaces considered in Proposition* 3.4.

Proof. We consider the real interval $X = [0, 1)$ with the anti-usual ordering $\geqslant$. Obviously, $(X, \geqslant)$ is a linearly-ordered set with 0 as greatest element. Now, let $Y = \{1/n : n \geqslant 2\}$, which is a well-ordered subset of $(X, \geqslant)$. It is quite evident that, for each $n \geqslant 2$, $[0, 1/n) = T(1/n)$, and so $\tau = \{\emptyset, X\} \cup \{T(1/n) : n \geqslant 2\}$.

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References

1. P. Fletcher and W. F. Lindgren, *Quasi-uniformities with a transitive base*. Pacific J. Math. **43**(3), (1972), pp. 619–631.

2. P. Fletcher and W. F. Lindgren, *Quasi-uniform spaces*. Marcel Dekker, Inc. New York (1982).

3. W. F. Lindgren, *Topological spaces with a unique compatible quasi-uniformity*. Canad. Math. Bull. **14**(3) (1971), pp. 369–372.

4. W. F. Lindgren, *Topological spaces with a unique quasi-uniform structure*. Arch. der Math. XXII (1971), pp. 417–419.

5. W. J. Pervin, *Quasi-uniformization of topological spaces*. Math. Ann. **147** (1962), pp. 316–317.

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