J. Austral. Math. Soc. (Series A) 62 (1997), 371–397

ON THE ACTION OF THE UNITARY GROUP ON THE PROJECTIVE PLANE OVER A LOCAL FIELD

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(Received 29 January 1996; revised 30 September 1996)

Communicated by R. Howlett

Abstract

Let G be a unitary group of rank one over a non-archimedean local field K (whose residue field has a characteristic $\neq 2$). We consider the action of G on the projective plane. A $G(K)$ equivariant map from the set of points in the projective plane that are semistable for every maximal K split torus in G to the set of convex subsets of the building of $G(K)$ is constructed. This map gives rise to an equivariant map from the set of points that are stable for every maximal K split torus to the building. Using these maps one describes a $G(K)$ invariant pure affinoid covering of the set of stable points. The reduction of the affinoid covering is given.

1991 *Mathematics subject classification* (*Amer. Math. Soc.*): primary 32P05; secondary 14M17, 20G25.

Introduction

Let K be a non-archimedean local field. We assume that the characteristic of the residue field of K is $\neq 2$. For a separable algebraic extension $L \supset K$ of degree two, we consider the action of the unitary group $SU_3(L)$ on $\mathbb{P}^2_L$. The rank of the unitary group is assumed to be one.

Let Y^{ss} and Y^s be the subspaces of $\mathbb{P}^2_L$ consisting of the points that are semistable and stable, respectively, for every maximal K-split torus $S \subset SU_3(L)$. Here one takes the S-linearization coming from the (unique) $SU_3(L)$-linearization of some ample line bundle on $\mathbb{P}^2_L$. All ample line bundles give the same set of (semi-) stable points, since they are all powers of the ample line bundle $\mathcal{O}(1)$.

Let B denote the Bruhat-Tits building of $SU_3(L)$. Since the rank of $SU_3(L)$ is one, the building is a tree.

This work is supported by the Australian Research Council.

We construct a map $I : Y^{ss} \longrightarrow \{\text{convex subsets of } B\}$ that is $SU_3(L)$-equivariant. A complete description of the convex subsets that are in the image of I is given (See Theorems 5.10 and 6.2). In particular we prove that $I(x)$ is bounded if and only if $x \in Y^s$.

The map I is then used to construct a pure affinoid covering of the rigid analytic space Y^s. The components of the reduction of Y^s with respect to this affinoid covering are proper. There is a 1-1 correspondance between these components and certain bounded convex subsets of the building B (See Theorem 7.5). In the last paragraph we describe the components of the reduction of Y^s. Giving a reduction of Y^s is equivalent to giving a formal scheme over L^0, the ring of integers of L, that has as its generic fibre the analytic space Y^s and as its closed fibre the reduction of Y^s.

Let $\Gamma \subset SU_3(L)$ be a discrete and co-compact subgroup. Then Y^s/Γ is a separated rigid analytic space. Since the group Γ has infinitely many orbits on the components of the reduction of Y^s, the quotient is not proper. Moreover we do not expect that the quotient itself can be compactified (See 8.13). This is in contrast with similar spaces considered in [4]. There one always assumes that the sets of stable and semistable points coincide. Then the quotient is proper. On the quotients of the spaces considered in [4] there exist no non-constant meromorphic functions (except for some cases related to Drinfeld's symmetric space). On the quotients Y^s/Γ of the space Y^s studied here there do exist non-constant meromorphic functions. This will be treated in a forthcoming paper.

Finally I would like to thank Marius van der Put and King Lai for their help and encouragement and Michel Gros for asking too many questions about $SU_3(L)$.

1. The group $SU_3(L)$ and its building

1.1. Notation.

(1) K: a non-archimedean local field with $\text{Char}(\bar{K}) \neq 2$.
(2) $\bar{K}$: the residue field of K.
(3) $L \supset K$: a separable algebraic extension of K of degree 2.
(4) π: a generator of the maximal ideal of L^0.
(5) τ: the generator of $\text{Gal}(L/K)$; we write $\bar{x} := \tau(x)$ for $x \in L$.
(6) K^0, L^0: the ring of integers of K, L.
(7) $V_0 \cong (L^0)^3$: an L^0-module with on it the unitary form $f(x, y) = x_1\bar{y}_2 + x_2\bar{y}_1 + 2x_0\bar{y}_0$.
(8) $V := V_0 \otimes L$: on it we have the unitary form $f \otimes L$ which we also denote by f. Note that if $\text{Char}(K) \neq 2$, every unitary form on V that gives rise to a unitary group of rank one can be brought into the form f after a suitable choice of the basis.

(9) G: the linear algebraic group defined over K^0 that acts on V_0 preserving f. Since we have not defined the module V_0 over K^0, the action of G on V_0 is not defined over K^0. However one can define a linear action of G on $V_0 \times V_0$ over K^0 by letting $g \in G$ act as $g \times \tau(g)$ (See also 2.1).

(10) $G(K^0) = SU_3(L^0)$ and $G(K) = SU_3(L)$.

(11) $S \subset G$: the torus in G that is diagonal with respect to the coordinates x_0, x_1, x_2 of V_0.

(12) $S(K) \cong K^*$: the maximal K-split torus in $G(K)$ coming from S.

(13) Z: the centraliser in G of S. One has $Z(K) \cong L^*$. The subgroup $Z(K) \subset G(K)$ consists of all the elements that act diagonally with respect to the coordinates x_0, x_1, x_2.

1.2. The building of $SU_3(L)$. The Bruhat-Tits building B of $SU_3(L)$ is a tree. We give a combinatorial description of B. The vertices of B correspond 1-1 with equivalence classes of certain L^0-submodules of $V \cong L^3$. The equivalence relation is given by:

$$M \sim N \quad \text{if and only if} \quad \exists(\lambda \in L^*) \quad \text{such that} \quad M = \lambda \cdot N$$

for $M, N \subset V$ L^0-modules. One denotes the equivalence class of M by $[M]$.

Let e_0, e_1, e_2 be an L-basis of V such that the unitary form f has the standard form $f(x, y) = x_1\bar{y}_2 + x_2\bar{y}_1 + 2x_0\bar{y}_0$ with respect to this basis.

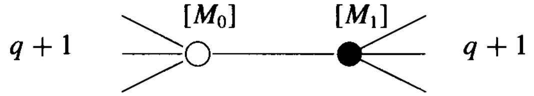
One takes the following two L^0 submodules in V:

$$M_0 := \langle e_0, e_1, e_2\rangle, \qquad M_1 := \langle e_0, \pi e_1, e_2\rangle.$$

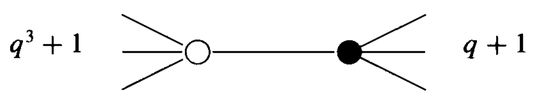
The building B is given by :

vertices:	$SU_3(L)$ images of $[M_0]$ and $[M_1]$
edges (or chambers) :	$SU_3(L)$ images of $\{[M_0], [M_1]\}$.

The tree B depends on whether L/K is ramified or not. Let $q := \sharp\bar{K}$; then B has the following form:



if L/K is ramified;



if L/K is unramified.

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1.3. The root system. The root system of $SU_3(L)$ is of type BC_1. One has for the maximal K split torus $S(K) \cong K^*$ four additive groups $U_{\pm 2\alpha} \subset U_{\pm\alpha}$ in $SU_3(L)$. The group U_α consists of the elements $u_\alpha(a,b)$ and $U_{-\alpha} = \{u_{-\alpha}(a,b) \in SU_3(L)\}$ which are as follows:

$$u_\alpha(a,b)\begin{cases} e_1 \to e_1 \\ e_2 \to e_2 + ae_0 + be_1 \\ e_0 \to e_0 - 2\bar{a}e_1 \end{cases} \qquad u_{-\alpha}(a,b)\begin{cases} e_1 \to e_1 + ae_0 + be_2 \\ e_2 \to e_2 \\ e_0 \to e_0 - 2\bar{a}e_2. \end{cases}$$

In both cases a and b satisfy: $2a\bar{a} + b + \bar{b} = 0$.

Now $U_{2\alpha} \subset U_\alpha$ is $U_{2\alpha} = \{u_\alpha(0,b) \mid b+\bar{b} = 0\}$ and $U_{-2\alpha} = \{u_{-\alpha}(0,b) \mid b+\bar{b} = 0\}$.

1.4. The affine root system. Let v denote the additive valuation of L with $v(\pi) = 1$. One defines the following subgroups of $U_{\pm\alpha}$ and $U_{\pm 2\alpha}$ for $n \in \mathbb{Z}$:

(1) $$U_{n+\alpha} := \{u_\alpha(a,b) \in U_\alpha \mid v(b) \geq 2n\};$$

(2) $$U_{n+2\alpha} := \{u_\alpha(0,b) \in U_{2\alpha} \mid v(b) \geq n\};$$

(3) $$U_{n-\alpha} := \{u_{-\alpha}(a,b) \in U_{-\alpha} \mid v(b) \geq 2n\};$$

(4) $$U_{n-2\alpha} := \{u_{-2\alpha}(0,b) \in U_{-2\alpha} \mid v(b) \geq n\}.$$

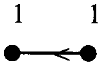
Note that $v(b) \geq 2n$ implies $v(a) \geq n$, since $2a\bar{a} + b + \bar{b} = 0$. If $b + \bar{b} = 0$, then $b = c \cdot \gamma$, for $c \in K$ and $\gamma \in L$ fixed such that $\gamma + \bar{\gamma} = 0$. If L/K is unramified, one can choose γ s.t $v(\gamma) = 0$. Hence $\{v(b) \mid b + \bar{b} = 0\} = \mathbb{Z}$.

If L/K is ramified one can choose γ such that $v(\gamma) = 1$. Hence $\{v(b) \mid b + \bar{b} = 0\} = \{2n + 1 \mid n \in \mathbb{Z}\}$ in this case.

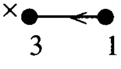
This gives us the following affine roots for $S(K) \cong K^*$ in $SU_3(L)$:

$2n+1 \pm 2\alpha$,	$n \pm \alpha$,	$n \in \mathbb{Z}$	if	L/K	is ramified;
$n \pm 2\alpha$,	$n \pm \alpha$,	$n \in \mathbb{Z}$	if	L/K	is unramified.

The affine Dynkin diagrams for these root systems are:



if L/K is ramified ($C - BC_1$);



if L/K is unramified ($C - BC_1^{IV}$).

(For more details see [3, §10.1] or [7, §1.16].)

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1.5. Parahoric subgroups. In the building B there is an apartment A associated to the maximal K-split torus $S(K) \cong K^*$. Its vertices correspond to $[M_n]$, $n \in \mathbb{Z}$ where $M_{2n} := \langle e_0, \pi^n e_1, \pi^{-n} e_2 \rangle$ and $M_{2n+1} := \langle e_0, \pi^{n+1} e_1, \pi^{-n} e_2 \rangle$. We will denote the vertex in A belonging to $[M_n]$ by n.

One easily sees that $U_{n+2\alpha}$ stabilizes the vertices $m \leq n$ and $U_{n+\alpha}$ the vertices $m \leq n/2$. Moreover, $U_{n-2\alpha}$ stabilizes the vertices $m \geq -n$ and $U_{n-\alpha}$ the vertices $m \geq -n/2$.

The parahoric subgroups are the stabilizers of the vertices and edges in B. They are generated by $Z(K^0)$ and the additive groups stabilizing it.

For the edge $\{0, 1\}$ one finds as stabilizer the group generated by $Z(K^0)$, $U_{1+\alpha}$, $U_{0-\alpha}$, $U_{1+2\alpha}$ and $U_{0-2\alpha}$, if L/K is unramified. If L/K is ramified this parahoric subgroup is generated by the groups $Z(K^0)$, $U_{1+\alpha}$, $U_{0-\alpha}$, $U_{1+2\alpha}$ and $U_{1-2\alpha}$.

1.6. The subgroups $SU_2(L) \subset SU_3(L)$. The elements $u_{\pm\alpha}(0, b)$, $b+\bar{b} = 0$ generate a subgroup $SU_2(L) \subset SU_3(L)$. Since $SU_2(L) \cong SL_2(K)$, this gives an embedding of the $SL_2(K)$ building in the building B of $SU_3(L)$. Note that both groups have the same rank and that both buildings are trees.

The torus $S(K)$ is contained in $SU_2(L)$. It acts on the apartment A belonging to $S(K) \cong K^*$. The additive groups for K^* in $SU_2(L)$ are $U_{\pm 2\alpha}$. The associated affine roots are $n \pm 2\alpha$ if L/K is unramified, and $2n + 1 \pm 2\alpha$ if L/K is ramified. Hence the vertices of A which are also vertices in the $SL_2(K)$ building are n if L/K is unramified and $2n + 1$ if L/K is ramified.

We define $B_2 := \bigcup_{g \in SU_2(L)} g(A) \subset B$. Then $B_2 \subset B$ is the $SL_2(K)$ building. This embedding $B_2 \hookrightarrow B$ is *simplicial* if L/K is unramified. If L/K ramifies, then every $SL_2(K)$ chamber consists of two chambers (for $SU_3(L)$ in B).

The embedding $B_2 \hookrightarrow B$ is unique since a maximal K-split torus $K^* \subset SU_2(L) \subset SU_3(L)$ determines a unique apartment $A \subset B$.

The subgroup $SU_2(L)$ preserves the decomposition $< e_0 > \oplus < e_1, e_2 >$ of V and the unitary form $x_1 \bar{y}_2 + x_2 \bar{y}_1$ on $< e_1, e_2 >$. From this it also follows that the apartment $A \subset B$ is contained in exactly one $SU_2(L)$ sub-building.

2. The action of the torus on $\mathbb{P}^2_L$

2.1. Preliminaries. The group G acts on $\mathbb{P}^2_{L^0} \cong \mathbb{P}(V_0)$. This action is not defined over K^0, but over L^0. However, there exists a scheme $\Xi \subset \mathbb{P}(V_0 \times V_0)$ defined over K^0 such that $\Xi \otimes L^0 = \mathbb{P}(V_0 \times < 0 >) \dot{\cup} \mathbb{P}(< 0 > \times V^0) \cong \mathbb{P}^2_{L^0} \dot{\cup} \mathbb{P}^2_{L^0}$. The action of G on Ξ is defined over K^0. The group $\mathrm{Gal}(L/K)$ permutes the two components $\mathbb{P}^2_{L^0}$.

The torus $S \subset G$ acts on $X := \mathbb{P}(V_0) \cong \mathbb{P}^2_{L^0}$. This action is defined over L^0. We

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make the following definitions:

$$X_S^{ss} := \text{set of semistable points for } S = \{x \in \mathbb{P}^2_{L^0} \mid x_0^2 \text{ or } x_1x_2 \text{ is invertible}\},$$
$$X_S^{s} := \text{set of stable points for } S = \{x \in \mathbb{P}^2_{L^0} \mid x_1x_2 \text{ is invertible}\}.$$

Note that the centraliser $Z(K^0) \cong (L^0)^*$ of $S(K^0)$ acts on both spaces.

Let f be a homogeneous polynomial of degree n. Then f is called *invertible* at $x \in X_S^{ss}$ if $|f(x)| = \max\{|x_i|^n \bigm| i = 0, 1, 2\}$. In particular this means that $f \neq 0$ on the closed fibre of X_S^{ss}. Furthermore one has:

$$X_S^{ss} \otimes L = \{x \in \mathbb{P}^2_L \mid x_0^2 \neq 0 \quad \vee \quad x_1x_2 \neq 0\} = \mathbb{P}^2_L - \{(0,1,0), (0,0,1)\},$$
$$X_S^{s} \otimes L = \{x \in \mathbb{P}^2_L \mid x_1x_2 \neq 0\} = \mathbb{P}^2_L - \{(x_0, x_1, 0), (x_0, 0, x_2)\}.$$

2.2. Analytifications. To each algebraic variety corresponds a rigid analytic variety which has the same set of closed points (See [2] or [1]). We denote the analytic varieties corresponding to $X_S^s \otimes L$ and $X_S^{ss} \otimes L$ by Y_A^s and Y_A^{ss} respectively. Here A is the apartment in B belonging to $S(K)$.

We also need some analytic spaces corresponding to X_S^{ss} and X_S^s. The set of points of these spaces consists of their closed fibres. They are:

$$\begin{aligned} Y_{0,A}^{ss} &:= \text{the completion of } X_S^{ss} \text{ along the closed fibre} \\ &= \{x \in Y_A^{ss} \mid |x_1x_2/x_0^2| \le 1,\ |x_1/x_0| \le 1,\ |x_2/x_0| \le 1\} \\ &\quad \cup \{x \in Y_A^{ss} \mid |x_0^2/x_1x_2| \le 1,\ |x_1/x_2| = 1\}, \\ Y_{0,A}^{s} &:= \text{the completion of } X_S^{s} \text{ along the closed fibre} \\ &= \{x \in Y_A^{ss} \mid |x_0^2/x_1x_2| \le 1,\ |x_1/x_2| = 1\}. \end{aligned}$$

Here the suffix 0 corresponds to the vertex $0 \in A$. In fact we need similar analytic subspaces for every simplex, that is, vertex or edge, $\sigma \in A$. This is done as in [4, §3.3, §3.4].

First we analytify the torus $S \otimes K$. From now on S will denote the analytification of $S \otimes K$. For each simplex $\sigma \in A$ one defines the affinoid subspace $S_\sigma \subset S$ by:

$$S_\sigma := \operatorname{Sp}\left(K\langle \pi^n \chi \mid n \in \mathbb{Z}, \quad \chi \in \chi(S), \quad n + \chi \ge 0 \quad \text{on} \quad \sigma\rangle\right).$$

For this definition one has to identify the apartment A with the dual of $\chi(S) \otimes \mathbb{R}$. Equivalently one can also define S_σ by $S_\sigma := \{s \in S \mid s \cdot 0 \in \sigma\}$. Here one identifies $A \cong \mathbb{R}$, and $s \in S$ acts on A by translation by $2 \cdot v(s_1)$.

Our torus S has, with respect to the coordinates x_0, x_1, x_2, a diagonal form $s = \operatorname{diag}(s_0, s_1, s_2)$ with $s_0 = 1$, $s_1s_2 = 1$. Hence one may put $s_1 = t$, $s_2 = t^{-1}$ and $s_0 = 1$. For the standard chamber $\sigma_0 = \{0, 1\} = \{[M_0], [M_1]\}$ one has

$$S_{\sigma_0} = \operatorname{Sp}\left(K\langle t, \pi t^{-2}\rangle\right).$$

As in [4] one defines:

$$Y^s_{\sigma,A} := S_\sigma \cdot Y^s_{0,A} = \{s \cdot x \mid s \in S_\sigma, x \in Y^s_{0,A}\}; \qquad Y^{ss}_{\sigma,A} := S_\sigma \cdot Y^{ss}_{0,A}.$$

Note that in [4] one restricts to the case where the variety X satisfies $X^s = X^{ss}$. Therefore the spaces $Y^s_{\sigma,A}$ and $Y^{ss}_{\sigma,A}$ coincide. Since in our case $Y^s_{\sigma,A} \neq Y^{ss}_{\sigma,A}$, we cannot use these spaces to construct an affinoid covering of Y^s_A. For that we need to do a little more.

First one needs a definition.

DEFINITION 2.3. For $x \in Y^{ss}_A$ we define the *interval of S semistability* by:

$$I_A(x) := \overline{\{s^{-1} \cdot 0 \mid s \cdot x \in Y^{ss}_{0,A},\ s \in S\}} \subset A$$

Here $s \in S$ means $S \in S(K^{\mathrm{alg}})$, where K^{alg} is the algebraic closure of K. Hence if one puts $A \cong \mathbb{R}$, the points $s^{-1} \cdot 0$ are in $\mathbb{Q}$. This is the reason one takes the closure $\overline{\{s^{-1} \cdot 0 \mid \dots\}}$ instead of just $\{\dots\}$.

This map I_A which associates to a point $x \in Y^{ss}_A$ a subset of A replaces the function $v_{X,T,L} : Y^{ss}_A \to A$ used in [4]. One has:

PROPOSITION 2.4. *Let $x \in Y^{ss}_A$. Then:*

(1) $I_A(s \cdot x) = s \cdot I_A(x)$ *for all $s \in S$.*
(2) $I_A(x) \subset A$ *is convex.*
(3) $I_A(x) = A$ *if and only if $x = (1, 0, 0)$.*
(4) $I_A(x) \subset A$ *is a half-apartment if and only if $x \in Y^{ss}_A - Y^s_A$, $x \neq (1, 0, 0)$.*
(5) $I_A(x) \subset A$ *is bounded if and only if $x \in Y^s_A$.*

PROOF. Part 1 follows directly from the definition of $I_A(x)$. To prove the other statements, we will describe $I_A(x)$ for all $x \in Y^{ss}_A$.

If $x = (1, 0, 0)$ then $x \in Y^{ss}_{0,A}$. Furthermore x is a fixed point for S. This proves $I_A(x) = A$.

If $x \in Y^{ss}_A$ and $|x_0^2| \le |x_1 x_2|$, then $x \in Y^s_A$. We may assume $|x_1| = |x_2|$ after replacing x by $s \cdot x$ for suitable $s \in S$. Now $x \in Y^s_{0,A}$ and one easily sees that $I_A(x)$ consist only of the point 0.

If $x = (x_0, x_1, 0)$ with $x_0, x_1 \neq 0$, then by using 1 we may assume $|x_0| = |x_1|$. Then $s \cdot x \in Y^{ss}_{0,A}$ if and only if $|s_1| \le 1$. So $I_A(x)$ is a half-apartment in this case. The case $x = (x_0, 0, x_2)$ is similar.

The only points in Y^{ss}_A not yet treated are those with $|x_1 x_2| < |x_0^2|$ and $x_1 x_2 \neq 0$. It is enough to treat those x with $|x_1| = |x_2| < |x_0|$. Now $s \cdot x \in Y^{ss}_{0,A}$ if and only if $|s_1 x_1| \le |x_0|$ and $|s_2 x_2| \le |x_0|$. Since $|s_1| = |s_2^{-1}|$ one finds $|x_2/x_0| \le |s_1| \le |x_0/x_1|$ in this case. Hence $I_A(x)$ is a bounded interval in A.

We now have treated all points $x \in Y_A^{ss}$. One easily verifies that statements 2 to 5 hold.

PROPOSITION 2.5. *For $x \in Y_A^{ss}$ one has:*

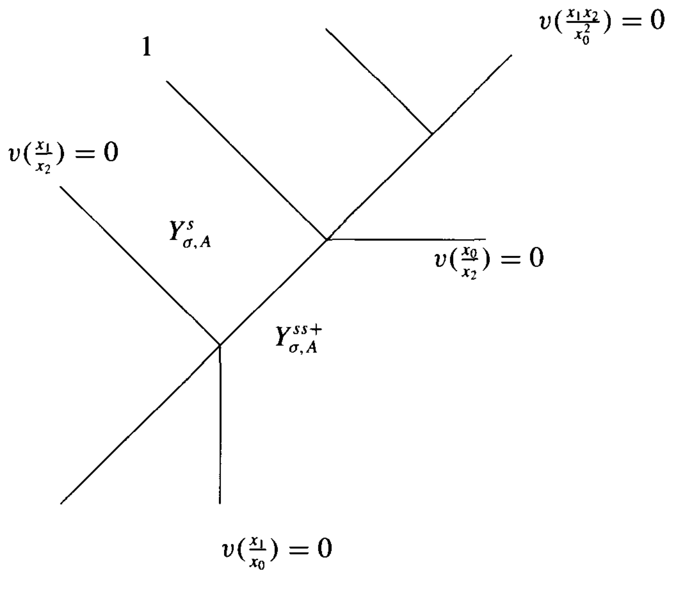
(1) $x \in Y_{\sigma,A}^{ss}$ *if and only if* $I_A(x) \cap \sigma \neq \emptyset$.

(2) $x \in Y_{\sigma,A}^{s}$ *if and only if* $I_A(x)$ *is a point contained in* σ.

PROOF. Part 1 follows directly from the definitions. Part 2 follows from $x \in Y_{0,A}^{s}$ if and only if $I_A(x) = \{0\}$.

REMARK 2.6. The space $Y_{\sigma,A}^{s}$ is affinoid, but $Y_{\sigma,A}^{ss}$ is not. One can cover $Y_{\sigma,A}^{ss}$ by the affinoid subspaces $Y_{\sigma,A}^{s}$ and $Y_{\sigma,A}^{ss+}$. Here $Y_{\sigma,A}^{ss+} := \{x \in Y_{\sigma,A}^{ss} \mid |x_1x_2/x_0^2| \leq 1\}$. The covering $\{Y_{\sigma,A}^{s}, Y_{\sigma,A}^{ss+}\}$ of $Y_{\sigma,A}^{ss}$ is pure.

Let $\mathscr{C}_A^s := \{Y_{\sigma,A}^{s} \mid \sigma \in A\}$ and $\mathscr{C}_A^{ss} := \{Y_{\sigma,A}^{s}, Y_{\sigma,A}^{ss+} \mid \sigma \in A\}$. In the figure below we draw the covering $\mathscr{C}_A^{ss}$ using the values of $v(x_i/x_j)$, where v denotes the valuation of L.



PROPOSITION 2.7.

(1) (a) $\bigcup_{\sigma\in A} Y_{\sigma,A}^{s} = \{x \in Y_A^{ss} \mid I_A(x) \text{ is a point}\} = \{x \in Y_A^{ss} \mid |x_0^2/x_1x_2| \leq 1\} \subsetneq Y_A^{s}$.

(b) *The covering $\mathscr{C}_A^s$ is pure.*

(c) *The components of the reduction of* $\bigcup_{\sigma\in A} Y^s_{\sigma,A}$ *with respect to the covering* $\mathscr{C}^s$ *are not proper.*

(2) (a) $\bigcup_{\sigma\in A} Y^{ss}_{\sigma,A} = Y^{ss}_A$.

(b) *The covering* $\mathscr{C}^{ss}_A$ *of* Y^{ss}_A *is not pure.*

PROOF. Parts 1(a) and 2(a) are a direct consequence from the previous proposition. The other parts of the proposition follow more or less immediately from the picture of the covering $\mathscr{C}^{ss}_A$ given above, using [8, §2]. Note that S translates along the line $v(x_0^2/x_1x_2) = 0$ in the picture above.

REMARK 2.8. The proposition above shows that one cannot use the affinoids $Y^s_{\sigma,A}$ and $Y^{ss+}_{\sigma,A}$ to get a good affinoid covering of Y^s_A or Y^{ss}_A. Before we give a pure affinoid covering of Y^s_A we need to know $I_A(x)$ in more detail. First we need a definition.

DEFINITION 2.9. For $x \in Y^s_A$ the interval $I_A(x)$ has two extremal points P_1 and P_2. There exists a unique point $P_3 \in A$ which has equal distance to both P_1 and P_2. Hence one can define:

$$v_A(x) := P_3 = (P_1 + P_2)/2.$$

Note that one cannot extend v_A to Y^{ss}_A.

As before v will denote the additive valuation of L, such that $v(\pi) = 1$, extended to the algebraic closure of L (or K). Also we identify A with $\mathbb{R}$ such that the vertices correspond to the integers as before. The interval with extremal points P_1 and P_2 will be denoted by $[P_1, P_2]_A :=$ convex hull of $\{P_1, P_2\}$. Using this definition we obtain:

PROPOSITION 2.10. *For* $x \in Y^s_A$ *one has:* $v_A(x) = v(x_1/x_2)$ *and*

$$I_A(x) = \begin{cases} [2v(x_0/x_2), 2v(x_1/x_0)]_A & \text{if} \quad \left|x_1x_2/x_0^2\right| \le 1, \\ \{v_A(x)\} & \text{if} \quad \left|x_0^2/x_1x_2\right| \le 1. \end{cases}$$

PROOF. First we remark that $s \in S$ acts on A by translation with $v(s_1/s_2) = 2v(s_1)$. So the descriptions in the proposition satisfy $v_A(s\cdot x) = s\cdot v_A(x)$ and $I_A(s\cdot x) = s\cdot I_A(x)$ as they should.

Hence it is sufficient to prove the theorem for $x \in Y^s_A$ with $|x_1| = |x_2|$. If $|x_1x_2| \ge \left|x_0^2\right|$ then $I_A(x) = \{0\}$. Hence $v_A(x) = 0$ and the proposition holds in this case.

If $|x_1x_2| \le \left|x_0^2\right|$ and $|x_1| = |x_2|$, then $s \in S$ with $s\cdot x \in Y^{ss}_{0,A}$ satisfy $|s_1x_1| \le |x_0|, |s_2x_2| \le |x_0|$. Hence one has $|x_2/x_0| \le |s_1| \le |x_0/x_1|$. Now $s^{-1}\cdot 0 \in I_A(x)$. Hence one finds $v(x_1/x_0) \ge v(s_1^{-1}) \ge v(x_0/x_2)$. Hence one has $I_A(x) = [2v(x_0/x_2), 2v(x_1/x_0)]_A$ in this case.

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COROLLARY 2.11. *For $x \in Y_A^s$ one has:*

$$I_A(x) = \{z \in A \mid \operatorname{dist}(v_A(x), z) \leq \max(0, v(x_1x_2/x_0^2))\}.$$

PROOF. If $v(x_1x_2/x_0^2) \leq 0$ then $I_A(x)$ is the point $v_A(x)$. Hence the proposition holds in this case.

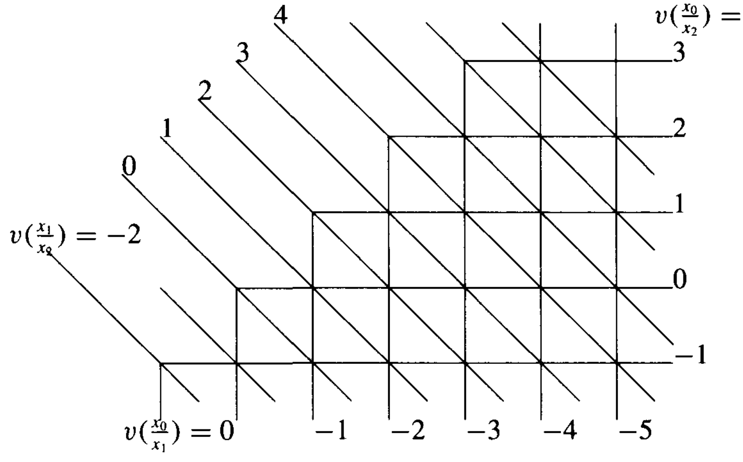
If $v(x_1x_2/x_0^2) \geq 0$, then $I_A(x) = [2v(x_0/x_2), 2v(x_1/x_0)]_A$ and $v_A(x) = v(x_1/x_2)$. Now $2v(x_1/x_0) - v(x_1/x_2) = v(x_1^2/x_0^2 \cdot x_2/x_1) = v(x_1x_2/x_0^2)$. So the proposition also holds in this case.

3. A pure affinoid covering of Y_A^s

3.1. Since the 'natural' affinoids $Y_{\sigma,A}^s$ do not cover all of Y_A^s we have to construct some additional affinoids. One wants the covering to be $Z(K)$ invariant.

Such affinoids can be constructed in the following way. Let $\phi : \{x \in Y_A^s \mid x_0 \neq 0\} \to \mathbb{R}^2$ be the map $x \to (v(x_0/x_1), v(x_0/x_2))$. The inverse image of a bounded convex polyhedron in $\mathbb{R}^2$ whose faces are contained in rational lines is an affinoid subspace of Y_A^s. In particular if one covers $\mathbb{R}^2$ by polyhedra as above such that the intersection of two such polyhedra is a face of both and such that each 0-dimensional face (that is, vertex) is contained in only finitely many polyhedra, then the corresponding affinoid covering will be pure (See [8, Lemma 2.4]).

In the figure below we give the polyhedra in $\mathbb{R}^2$ corresponding to the affinoid covering $\mathscr{C}_A$ of Y_A^s we will use.



If one knows the point $\phi(x)$ for $x \in Y_A^s$, then one can easily determine $v_A(x)$ and $I_A(x)$. In fact, for $x, y \in Y_A^s$ with $|x_1x_2/x_0^2| \leq 1$ one has $\phi(x) = \phi(y)$ if and only if $I_A(x) = I_A(y)$.

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The affinoid covering $\mathscr{C}_A$ consists of the following affinoids:

$$Y\{\rho_1, \rho_2, \sigma\}_A := \{x \in Y_A^s \mid I_A(x) \subset \text{convex hull of}\{\rho_1, \rho_2\},$$
$$I_A(x) \cap \rho_i \neq \emptyset, i = 1, 2, v_A(x) \in \sigma\}.$$

Here $\sigma \in A$ is a chamber and $\rho_i \subset A$ is the union of two neighbouring chambers bounded by two vertices of type 0, that is, $\rho_i := [2n_i, 2n_i+2]_A, n_i \in \mathbb{Z}$. For notational reasons we will write $\rho_i \in A$ for $\rho_i \subset A$. We only consider triples ρ_1, ρ_2, σ such that $Y\{\rho_1, \rho_2, \sigma\}_A$ is non-empty. Once one fixes ρ_1 and ρ_2 there are exactly four choices for σ such that $Y\{\rho_1, \rho_2, \sigma\}_A$ is non-empty. Of these four choices, two correspond to polyhedra in the picture above. The other two correspond to one-dimensional faces if $\rho_1 = \rho_2$ and to vertices if $\rho_1 \neq \rho_2$. By allowing σ to be a vertex one can obtain the faces of the polyhedra in the skew lines. If one allows one of the ρ_i to be a vertex of type 0, one obtains the faces of the polyhedra that are contained in the horizontal and vertical lines in the picture above.

Note that if $\sigma \subset \rho$ one has:

$$Y\{\rho, \rho, \sigma\}_A = \{x \in Y_A^s \mid I_A(x) \subset \rho, \ v_A(x) \in \sigma\} \supset Y_{\sigma,A}^s.$$

The reason that one uses the subsets $\rho_i \subset A$, instead of chambers in the definition of the affinoids, is that this choice associates L^0-submodules of L^3 to the vertices of the polyhedra in the picture. Had we used chambers, those modules would only have been defined over a finite extension of L^0. Our choice will therefore ensure that the components of the reduction of Y^s will correspond to L^0-modules (See also Remark 8.11).

In the proposition below we state all the relevant properties of our affinoid covering $\mathscr{C}_A$.

PROPOSITION 3.2. (1) $\bigcup_{\rho_1,\rho_2 \in A} Y\{\rho_1, \rho_2, \sigma\} = \{x \in Y_A^s \mid v_A(x) \in \sigma\}$.
(2) $\bigcup_{\rho_1,\rho_2,\sigma \in A} Y\{\rho_1, \rho_2, \sigma\} = Y_A^s$.
(3) *The affinoid covering* $\mathscr{C}_A = \{Y\{\rho_1, \rho_2, \sigma\}_A \mid \rho_1, \rho_2, \sigma \in A\}$ *is pure.*
(4) *The components of the reduction of* Y_A^s *with respect to* $\mathscr{C}_A$ *are proper.*
(5) *There is a* 1-1 *correspondence between components of the reduction and integer intervals* $[2n, 2m]_A \subset A, n, m \in \mathbb{Z}$.

PROOF. Statements (1) and (2) are obvious. The statements (3) and (4) are proved as in [8, §2]. The last statement of the proposition follows from the fact that the extremal points of the polyhedra in the figure above correspond with the integer intervals $[2n, 2m]_A$ in A.

4. The action of $SU_3(L)$ on $\mathbb{P}^2_L$

4.1. In this section we study the affinoids Y^s_σ and Y^{ss}_σ. We then define an interval $I(x)$ of semistability for x with respect to $SU_3(L)$, and study the connection between Y^{ss}_σ and $I(x)$ for $x \in Y^{ss}$. Analogous to [4, §3.5, 3.6], one defines:

DEFINITION 4.2. $Y^s := \bigcap_{g\in SU_3(L)} g(Y^s_A)$ and $Y^{ss} := \bigcap_{g\in SU_3(L)} g(Y^{ss}_A)$ Let $A' \subset B$ be an apartment and $\sigma' \in A'$ a simplex. One can find $\sigma \in A$ and $g \in SU_3(L)$ such that $g(\sigma) = \sigma'$ and $g(A) = A'$. Then one takes:

$$Y^s_{\sigma',A'} := g(Y^s_{\sigma,A}) \quad \text{and} \quad Y^{ss}_{\sigma',A'} := g(Y^{ss}_{\sigma,A}).$$

Moreover one needs:

$$Y^s_\sigma := \bigcap_{A'\ni\sigma} Y^s_{\sigma,A'} \quad \text{and} \quad Y^{ss}_\sigma := \bigcap_{A'\ni\sigma} Y^{ss}_{\sigma,A'}.$$

As in [4] the subspaces $Y^s_\sigma \subset Y^s_{\sigma,A}$ and $Y^{ss}_\sigma \subset Y^{ss}_{\sigma,A}$ are nice open subdomains.

DEFINITION 4.3. As in [4, §3.6] we define a function r_{A_1,A_2}, which is useful for studying the affinoids defined above. Let $A_1 = g_1(A)$ and $A_2 = g_2(A)$ with $g_i \in SU_3(L)$. For $z \in \mathbb{P}^2_L$ define:

$$r_{A_1,A_2}(z) := \max\{\left|g_1^* x_1 g_1^* x_2(z)\right|, \left|g_1^* x_0^2(z)\right|\}/\max\{\left|g_2^* x_1 g_2^* x_2(z)\right|, \left|g_2^* x_0^2(z)\right|\}.$$

This function now has, mutatis mutandis, the same properties in our situation as in [4].

In the lemma below we state the properties of r_{A_1,A_2} which are either obvious or for which the proof is exactly as in [4, §3.6].

LEMMA 4.4. (a) *$r_{gA,A}(x)$ is well defined for $x \in Y^{ss}_A$.*
(b) *$r_{gA,A}(x)$ only depends on A and gA and not on the choice of $g \in SU_3(L)$.*
(c) *$r_{ghA,hA}(x) \cdot r_{hA,A}(x) = r_{ghA,A}(x)$ and $r_{gA,A}(x) = (r_{A,gA}(x))^{-1}$.*
(d) *$x \in Y^{ss}_{\sigma,A}$ implies $r_{gA,A}(x) \leq 1\ \forall (g \in P_\sigma)$.*
Here P_σ denotes the stabiliser of σ in $SU_3(L)$.

DEFINITION 4.5. Define:

$$r(x) := \begin{cases} 0 & \text{if } x \notin Y^{ss}_A, \\ \inf\{r_{gA,A}(x) \mid g \in G(K)\} & \text{if } x \in Y^{ss}_A. \end{cases}$$

One has:

PROPOSITION 4.6. (a) *$x \in Y^{ss}_{\sigma,gA}$ and $r_{gA,A}(x) = r(x) > 0$ if and only if $x \in Y^{ss}_\sigma$.*

(b) *$x \in Y^{ss}$ if and only if $r(x) > 0$.*

PROOF. The proofs of similar statements in [4, 3.6(d), (f)] remain valid in our case, mutatis mutandis.

DEFINITION 4.7. To understand the analytic space Y^{ss} better, define, for $x \in Y^{ss}$, the *interval of $SU_3(L)$-semistability* by

$$I(x) := \{z \in B \mid \forall(A \ni z)\ z \in I_A(x)\} \subset B.$$

Here $I_{gA}(x)$, $g \in SU_3(L)$, is defined by $I_{gA}(x) := g(I_A(g^{-1}(x)))$. This is well defined, since $t(I_A(t^{-1}(x))) = I_A(x)$ for $t \in Z(K)$.

From the definition one gets:

$$0 \in I(x) \text{ if and only if } x \in Y_0^{ss}.$$

A close look at the proof of in [4, 3.6(f)] gives us:

PROPOSITION 4.8. *$x \in Y_\sigma^{ss}$ if and only if*

$$\forall(A_1, A_2 \ni \sigma) \qquad I_{A_1}(x) \cap \sigma = I_{A_2}(x) \cap \sigma \neq \emptyset.$$

PROOF. It is sufficient to proof it for $\sigma \in A$ a chamber. Let $z \in I_A(x) \cap \sigma$; since $x \in Y_{\sigma,A}^{ss}$, such a z exist. It is now sufficient to prove, for all $z \in I_A(x) \cap \sigma$, that $z \in I_{gA}(x) \cap \sigma$ for all $g \in P_\sigma$. It is sufficient to prove it only in case $z \in \mathbb{Q} \subset A \cong \mathbb{R}$.

We take a finite extension $K' \supset K$ such that $z \in 2v((K')^*)$. Now there exists an element $s \in S(K') \cong (K')^*$ such that $z = s^{-1} \cdot 0$. We put $Y_{z,A}^{ss} := s^{-1} \cdot (Y_{0,A}^{ss} \otimes K')$.

Clearly we have $z \in I_A(x)$ if and only if $x \in Y_{z,A}^{ss}$.

We put $Y_z^{ss} := \bigcap_{g \in P_\sigma} g(Y_{z,A}^{ss})$. Since $r_{gA,A}(x) \leq 1, \forall g \in P_\sigma$, and $r_{A,A}(x) = r(x) = 1$ by Proposition 4.6(a), we must have $r_{gA,A}(x) = 1$ for all $g \in P_\sigma$.

Hence $x \in Y_z^{ss}$. In particular $x \in Y_{z,gA}^{ss}$ and therefore $z \in I_{gA}(x)$ for all $g \in P_\sigma$. This proves the proposition.

PROPOSITION 4.9. (1) *$x \in Y_\sigma^{ss}$ if and only if $I(x) \cap \sigma \neq \emptyset$;*

(2) *Let $R(x) := \{A' \subset B \mid r_{A',A}(x) = r(x)\}$. For $x \in Y^{ss}$ one has:*

$$I(x) = \bigcup\nolimits_{A' \subset R(x)} I_{A'}(x).$$

(3) *$x \in Y_\sigma^s$ if and only if $I(x)$ is a point contained in σ.*

PROOF. The first statement follows directly from Proposition 4.8. The second statement is a direct consequence of Propositions 4.8 and 4.6. The third part is clear from Definition 4.7.

COROLLARY 4.10. (1) *$I(x)$ is convex.*
(2) *$x \in Y^s$ if and only if $I(x)$ is bounded.*

We will omit the proof of this corollary, since it is also a trivial consequence of Theorem 6.2 below. Now we can state what remains true of [4, Theorem 3.6] in our case.

PROPOSITION 4.11. (1) $Y^{ss} = \bigcup_{\sigma \in B} Y^{ss}_{\sigma}$.
(2) (a) $Y^s_{\sigma_1} \cap Y^s_{\sigma_2} = \emptyset$ *if* $\sigma_1 \cap \sigma_2 = \emptyset$ *and equals* $Y^s_{\sigma_3}$ *if* $\sigma_1 \cap \sigma_3 = \sigma_3$.
(b) $Y^{ss}_{\sigma_1} \cap Y^{ss}_{\sigma_2} \neq \emptyset \quad \forall \sigma_1, \sigma_2 \in B$.
(3) *The covering $\mathscr{C}^s := \{Y^s_\sigma \mid \sigma \in B\}$ is pure. It covers the space*

$$\{x \in Y^s \mid I(x) \text{ is a point}\} \subsetneq Y^s.$$

PROOF. Except for 2(b), everything follows easily from the previous propositions. As for 2(b), we note that for $\sigma_1, \sigma_2 \in B$ one can find x such that $I(x) \cap \sigma_1 \neq \emptyset$ and $I(x) \cap \sigma_2 \neq \emptyset$ as follows.

Clearly if $\sigma_1, \sigma_2 \in A$ one can find x such that $\sigma_1 \cap I_A(x) \neq \emptyset$ and $\sigma_2 \cap I_A(x) \neq \emptyset$. By choosing x carefully one may assume that $A \subset R(x)$. Hence $I_A(x) \subset I(x)$. So we have constructed a point $x \in Y^{ss}_{\sigma_1} \cap Y^{ss}_{\sigma_2}$.

5. The action of $SU_2(L)$ on $\mathbb{P}^2_L$

5.1. Before determining the interval of $SU_3(L)$ semistability for $x \in Y^{ss}$ it is useful to study the interval of semistability with respect to the subgroup $SU_2(L) \subset SU_3(L)$. One has an $SU_2(L^0)$-equivariant map $\varphi : \mathbb{P}^2_{L^0} - \{(1,0,0)\} \to \mathbb{P}^1_{L^0}$, given by $\varphi(x_0, x_1, x_2) = (x_1, x_2)$. One can use the action of $SU_2(L)$ on $\mathbb{P}^1_L$ to study the action of $SU_2(L)$ on $\mathbb{P}^2_L$. Since $SU_2(L) \cong SL_2(K)$ the space of points in $\mathbb{P}^1_L$ which are stable for all maximal K-split tori in $SU_2(L)$ is essentially Mumford's upper halfplane Ω_1.

5.2. The action of $SU_2(L)$ on $\mathbb{P}^1_L$. Let $SU_2(L^0)$ act on $\mathbb{P}^1_{L^0}$ respecting the unitary form $\tilde{f}(x,y) = x_1\bar{y}_2 + x_2\bar{y}_1$. For the torus $T \subset SU_2(L^0)$ acting diagonally for the coordinates x_1, x_2, the sets of stable and semistable points $(\mathbb{P}^1)^s_T$ and $(\mathbb{P}^1)^{ss}_T$ coincide, that is, $(\mathbb{P}^1)^s_T = (\mathbb{P}^1)^{ss}_T$. In particular, all the results of [4] apply to our situation.

Let B_2 be the building of $SU_2(L) \cong SL_2(K)$ and $A \subset B_2$ the apartment belonging to $T(K)$. Let $\mathscr{Z}_A$ be the analytic space corresponding to $(\mathbb{P}^1_{L^0})^{ss}_T \otimes L$ and $\mathscr{Z}_{0,A}$ the completion of $(\mathbb{P}^1)^{ss}_T$ along the closed fibre.

$$\text{One has:} \qquad \mathscr{Z}_A = \{x \in \mathbb{P}^1_L \mid x_1 \neq 0 \wedge x_2 \neq 0\},$$
$$\mathscr{Z}_{0,A} = \{x \in \mathscr{Z}_A \mid |x_1/x_2| = 1\}.$$

The interval of T-stability is given by:

$$I_A(x)_{\mathbb{P}^1} := \overline{\{t^{-1} \cdot 0 \mid t \cdot x \in \mathscr{X}_{0,A}, \quad t \in T\}}.$$

Here $x \in \mathscr{X}_A$ and T denotes the analytification of $T \otimes K$.

Define: $\mathscr{X} := \bigcap_{g \in SU_2(L)} g(\mathscr{X}_A)$ and for $x \in \mathscr{X}$ take as interval of $SU_2(L)$ semistability

$$I(x)_{\mathbb{P}^1} := \{z \in B_2 \mid \forall (A \ni z) \quad z \in I_A(x)_{\mathbb{P}^1}\}.$$

Note that $\mathscr{X} = \tilde{\Omega}_1 := \mathbb{P}^1_L - \{x \mid x_1/x_2 = c \cdot \gamma, c \in K \text{ or } x_2 = 0\}$. Here $\gamma \in L$ is an element such that $\gamma + \bar{\gamma} = 0$, $\gamma \neq 0$.

From $(\mathbb{P}^1)^s_T = (\mathbb{P}^1)^{ss}_T$ one directly concludes the following:

PROPOSITION 5.3. (1) *For $x \in \mathscr{X}_A$ the interval $I_A(x)_{\mathbb{P}^1}$ is a point.*
(2) *For $x \in \mathscr{X}$ the interval $I(x)_{\mathbb{P}^1}$ is a point.*

DEFINITION 5.4. Let $H \subset B$ be the $SU_2(L)$ building regarded as a subcomplex of the $SU_3(L)$ building B. Let $A \subset H$ be an apartment. We define the following spaces:

$$Y_H^{ss} := \bigcap_{g \in SU_2(L)} g(Y_A^{ss});$$
$$Y_H^{s} := \bigcap_{g \in SU_2(L)} g(Y_A^{s}).$$

For $x \in Y_H^{ss}$ one can now define the interval of $SU_2(L)$ semistability by:

$$I_H(x) := \{z \in H \mid \forall (A \ni z \wedge A \subset H) \quad z \in I_A(x)\}.$$

We take $\varphi : \mathbb{P}^2_{L^0} - \{(1,0,0)\} \to \mathbb{P}^1_{L^0}$, $\varphi(x_0, x_1, x_2) = (x_1, x_2)$ as before. We will also denote the map $\varphi \otimes L : \mathbb{P}^2_L - \{(1,0,0)\} \to \mathbb{P}^1_L$ by φ.

PROPOSITION 5.5. $Y_H^s = \varphi^{-1}(\tilde{\Omega}_1) = \varphi^{-1}(\mathscr{X})$.

PROOF. One easily sees that if $A \subset H$, then $Y_A^s = \varphi^{-1}(\mathscr{X}_A)$. Now the proposition follows from the definitions of Y_H^s and $\mathscr{X}$ and the fact that φ is $SU_2(L)$-equivariant.

PROPOSITION 5.6. *For $x \in Y_A^s$ one has $v_A(x) = I_A(\varphi(x))_{\mathbb{P}^1}$ and*

$$I_A(x) = \left\{z \in A \mid \operatorname{dist}(z, I_A(\varphi(x))_{\mathbb{P}^1}) \leq \max\left(0, v\left(x_1x_2/x_0^2\right)\right)\right\}.$$

PROOF. Since $s \cdot v_A(x) = v_A(s \cdot x)$ and φ is S-equivariant it is sufficient to proof it for x with $|x_1| = |x_2|$. Then $v_A(x) = 0$ and clearly $\varphi(x) \in \mathscr{X}_{0,A}$. Hence $v_A(x) = I_A(\varphi(x))_{\mathbb{P}^1} = 0$ in this case.

Now the second statement follows immediately from Corollary 2.11.

DEFINITION 5.7. For $y \in B$ and $r \geq 0$ define

$$C(y, r) := \{z \in B \mid \operatorname{dist}(y, z) \leq r\}.$$

PROPOSITION 5.8. *Let $x \in Y_H^{ss}$ and let $0 \in A$ be the usual vertex. Then*

$$C(0, r) \cap A \subset I_H(x) \quad \textit{implies} \quad C(0, r) \cap H \subset I_H(x).$$

PROOF. If $r = 0$ the statement is trivial, so we assume $r > 0$. Since $C(0, r) \cap A \subset I_A(x)$ we have $v(x_1 x_2/x_0^2) \geq r > 0$. The fact that 0 is in the centre of $C(0, r)$ actually gives us $v(x_i/x_0) \geq r/2 > 0$, $i = 1, 2$.

If $h \in P_0 \cap SU_2(L)$, then $v(h^* x_i(x)/x_0) \geq \min(v(x_1/x_0), v(x_2/x_0)) \geq r/2$, since $x \in Y_{0,A}^{ss}$. Hence $C(0, r) \cap hA \subset I_{hA}(x)$. Since $0 \in I_H(x)$ and $0 \in hA$, we use proposition 4.9 (2) to obtain $I_{hA}(x) \subset I_H(x)$ for all $h \in P_0 \cap SU_2(L)$. Therefore $C(0, r) \cap H \subset I_H(x)$.

DEFINITION 5.9. For $x \in Y_H^s$, we put $v_H(x) := I(\varphi(x))_{\mathbb{P}^1}$. We say A *determines* $I_H(x)$ if:

(1) $\forall (A' \subset H) \quad I_A(x) \subset A'$ implies $I_{A'}(x) = I_A(x)$.
(2) $x \in Y_H^s$ implies $v_H(x) \in A$.
$x \in Y_H^{ss} - Y_H^s$ implies $I_A(x)$ is not bounded.

THEOREM 5.10. *Let $x \in Y_H^{ss}$ and suppose A determines $I_H(x)$. Then:*

(1) *$x \in Y_H^s$ implies $I_H(x) = C(v_H(x), \max(0, v(x_1 x_2/x_0^2))) \cap H$.*
(2) *$x = (1, 0, 0)$ implies $I_H(x) = H$.*
(3) *$x \in Y_H^{ss} - Y_H^s$, $x \neq (1, 0, 0)$ implies $I_H(x) = \bigcup_{g \in U_{2\alpha}} g(I_A(x))$. Here $U_{2\alpha} \subset SU_2(L)$ stabilises the limit point of the half-apartment containing $I_A(x)$.*

PROOF. (1) Since $I_A(\varphi(x))_{\mathbb{P}^1} = I(\varphi(x))_{\mathbb{P}^1}$ for all A containing $I(\varphi(x))_{\mathbb{P}^1}$, we have $v_A(x) = v_H(x)$ for all $A \subset H$ such that $v_H(x) \in A$. Furthermore, $|h^* x_i/x_i(x)| = 1$, $i = 1, 2$ for all $h \in P_\sigma \cap SU_2(L)$. Here $v_H(x) \in \sigma \in A$.

Hence $I_{hA}(x) = C(v_H(x), \max(0, v(x_1 x_2/x_0^2))) \cap hA$ for all $h \in P_\sigma \cap SU_2(L)$. Furthermore $v_H(x) \in hA$ implies $I_{hA}(x) \subset I_H(x)$. Now it is clear that $I_H(x) = H \cap C(v_H(x), \max(0, v(x_1 x_2/x_0^2)))$.
(2) If $x = (1, 0, 0)$ then $I_A(x) = A$. Since x is a fixed point for the action of $SU_2(L)$, we easily conclude $I_H(x) = H$.
(3) Take $x \in Y^{ss} - Y^s$ such that $I_A(x)$ is a half-apartment and A determines $I_H(x)$. Let $U_{2\alpha}$ be the additive group in $SU_2(L) \subset SU_3(L)$ stabilising the limit point of $I_A(x)$. Let $g \in U_{2\alpha}$. Then $A \cap gA$ is again a half-apartment. Clearly $I_A(x) \cap gA \subset I_{gA}(x)$. Let y be the extremal point of $I_A(x) \subset A$. We may assume that the vertex $0 \in A \cap gA$

and $0 \in I_A(x)$. If $y \in gA$ then $I_{gA}(x) = I_A(x)$, since A determines $I_H(x)$. So let us assume that $y \notin gA$.

If $y \notin gA$ then $C(0, \text{dist}(0, y)) \cap A \subset I_H(x)$. Hence $C(0, \text{dist}(0, y)) \cap H \subset I_H(x)$ by Proposition 5.8. In particular $gI_A(x) \subset I_{gA}(x)$. From the assumption that A determines $I_H(x)$ one easily concludes $I_{gA}(x) = g(I_A(x))$. Clearly $I_{gA}(x) \subset I_H(x)$. Now statement (3) follows.

6. The intervals of $SU_3(L)$-semistability

6.1. Let $x \in Y^{ss}$. The interval of $SU_3(L)$ semistability $I(x)$ is convex. In particular one can find an apartment $A \subset B$ satisfying the following conditions:

(1) $\forall A' \subset B \quad I_A(x) \subset A'$ implies $I_{A'}(x) = I_A(x)$

(2) $x \in Y^s$ implies $|I_A(x)| = \max\{|I_{A'}(x)| \mid I_{A'}(x) \subset I(x)\}$

$x \in Y^{ss} - Y^s$ implies $I_A(x)$ is not bounded.

If A satisfies these conditions we say that A *determines* $I(x)$. One easily sees that $I_A(x) \subset I(x)$ if A determines $I(x)$.

The apartment A is contained in a unique $SU_2(L)$ sub-building $H \subset B$. From the definitions it follows immediately that A determines $I_H(x)$. In fact one has:

THEOREM 6.2. *Let $x \in Y^{ss}$ and assume that $A \subset B$ determines $I(x)$. If H is the $SU_2(L)$ sub-building that contains A then $I(x) = I_H(x) \subset H \subset B$.*

PROOF. From $I_A(x) \subset I(x)$ we conclude $1 = r_{A,A}(x) = r(x)$. If $\sigma \in A$ with $\sigma \cap I_A(x) \neq \emptyset$, then $r_{hA,A}(x) = r(x) = 1$ for all $h \in P_\sigma$. Hence $I_{hA}(x) \subset I(x)$. Taking the union of the $I_{hA}(x)$ for all $h \in P_\sigma \cap SU_2(L)$, for all $\sigma \in A$ such that $\sigma \cap I_A(x) \neq \emptyset$, we obtain $I_H(x)$. So $I_H(x) \subset I(x)$. Doing the same for all $h \in P_\sigma$, σ as above we find $I_H(x) = I(x) \cap H$.

Suppose $I(x) \cap H \neq I(x)$. Then there exists a vertex v and an apartment A' such that $A' \cap H = \{v\}$ $(\text{char}(\bar{K}) \neq 2)$, $v \in I(x) \cap H$ with $I_{A'}(x) \neq \{v\}$. The embedding of $H \subset B$ is such that there exists $A' \subset B$ such that $A' \cap H = \{v\}$ only if v is of type 0. So it is sufficient to treat the case $v = 0 \in A \subset H$.

Firstly we assume that 0 is an extremal point of $I_H(x)$ considered as a subset of H. If $I_{A'}(x) \neq \{0\}$ there exists an apartment A'' such that $A'' \supset I_A(x)$ and $I_{A''}(x) \cap I_{A'}(x) \neq \{0\}$. Hence $I_A(x) \neq I_{A''}(x)$. This cannot be, since we assumed that A determined $I(x)$.

Now we assume that 0 is not an extremal point of $I_H(x)$ considered as a subset of H. Now $A' = hA$ with $h \in P_0$. We may assume $h = u_\alpha(a, b)u_{-\alpha}(c, d)$ with $|a| = |c| = 1$. Since $hA \cap A = \{0\}$ we also need to assume $|cb + a| = 1$ (this

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implies $|a - \bar{c}^{-1}| = |b + (c\bar{c})^{-1}| = 1$.) Explicit calculations for h as above show that $I_{hA}(x) = \{0\}$. Hence $I(x) \subset H$ and the theorem follows.

REMARK 6.3. If A and A' are two apartments that determine $I(x)$, then the $SU_2(L)$-buildings H and H' containing A and A', respectively, might be different. However we still have: $I_H(x) = I_{H'}(x)$. In particular, $I_H(x) \subset H \cap H'$.

DEFINITION 6.4. Let $x \in Y^s$ and suppose A determines $I(x)$. Then we define $v_B(x) := v_A(x)$. Note that this does not depend on the choice of the apartment A that determines $I(x)$.

In the next proposition we show how $I_A(x)$ is related to $I(x)$ for any apartment A in the building B.

PROPOSITION 6.5. *Let $A \subset B$ be an apartment and $x \in Y^{ss}$. Then*

(1) $I_A(x) = I(x) \cap A$, *if $A \cap I(x) \neq \emptyset$.*
(2) *$I_A(x)$ is the vertex in A closest to $I(x)$, if $A \cap I(x) = \emptyset$.*

PROOF. The first statement is a direct consequence of Proposition 4.9(2). So we only have to prove the second statement.

Let $A \subset B$ be an apartment such that $A \cap I(x) = \emptyset$. Let $\sigma \in A$ be a chamber such that $I_A(x) \cap \sigma \neq \emptyset$. There exists $g \in P_\sigma$ such that $gA \cap I(x) \neq \emptyset$. Hence $I_{gA}(x) = I(x) \cap gA$, according to statement (1). Furthermore it follows from Proposition 4.6(a) that $r_{gA,A}(x) < 1$.

Clearly we can choose g in either $P_\sigma \cap U_\alpha$ or $P_\sigma \cap U_{-\alpha}$. We will only treat the case where $g \in P_\sigma \cap U_\alpha$, since the other case is similar. Without loss of generality we may assume that A is our standard apartment and that σ is the chamber corresponding to the modules $[M_0]$ and $[M_{-1}]$. Hence $g = u_\alpha(a, b)$ with $|a|, |b| \leq 1$.

We will firstly prove that $I_A(x)$ consists of a single point. Let us assume that $I_A(x)$ is not a point. Then $I_A(x) \cap \sigma = [2v(x_0/x_2), 2v(x_1/x_0)]_A \cap [-1, 0]_A \neq \emptyset$. In particular $v(x_0/x_2) \leq 0$. Hence $|x_2/x_0| \leq 1$. Furthermore $|x_1x_2/x_0^2| < 1$. Since $r_{gA,A}(x) < 1$, we must have $|g^*x_0/x_0(x)| = |(x_0 - ax_2)/x_0| < 1$. Since $|a| \leq 1$ and $|x_2/x_0| \leq 1$, we must have $|a| = 1$ and $|x_2/x_0| = 1$. Since $|x_1x_2/x_0^2| < 1$, we also have $|x_1/x_0| < 1$.

We have $g^*x_1 = x_1 + 2\bar{a}x_0 - bx_2$. From $|(x_0 - ax_2)/x_0| < 1$ and $|x_0/x_2| = 1$ it follows that $|2\bar{a}x_0 - bx_2/x_0| = 1$. Furthermore $|g^*x_1/x_0(x)| = |g^*x_2/x_0(x)| = 1$ and $|g^*x_0/x_0(x)| < 1$. So $I_{gA}(x)$ is the vertex 0. This contradicts our assumption that $A \cap I(x) = \emptyset$. Therefore $I_A(x)$ has to be a point.

Now let us assume that $I_A(x)$ consists of a single point. Our assumptions are such that we must show that $I_A(x) = \{0\}$, since this is clearly the vertex in σ closest to

$I(x)$. We must furthermore show that 0 is the vertex in A closest to $I(x)$. Since $r_{gA,A}(x) < 1$ and $g^*x_2 = x_2$, we must have $|g^*x_1/x_1(x)| < 1$.

If $|x_0^2/x_1x_2(x)| < 1$, then $|b| = 1$ and $|x_2/x_1(x)| = 1$. Hence $I_A(x) = \{0\}$. Since $|b| = 1$, it is clear that 0 is the vertex in A closest to $I(x)$.

If $|x_0^2/x_1x_2(x)| = 1$ then $|g^*x_0/x_0(x)| < 1$. Hence $|x_1/x_0(x)| = |x_2/x_0(x)| = 1$. Again we conclude that $I_A(x) = \{0\}$, and furthermore $|a| = 1$. Hence again 0 is the vertex in A closest to $I(x)$. This concludes the proof of the proposition.

7. A pure affinoid covering of Y^s

7.1. The description of $I(x)$ given above, enables us to give a pure affinoid covering of Y^s. The affinoids used will be nice open affinoid subspaces of the affinoids $Y\{\rho_1, \rho_2, \sigma\}_A$. The components of the reduction of Y^s with respect to this pure affinoid covering $\mathscr{C}$ will be in 1-1 correspondance with certain convex subsets of the building.

DEFINITION 7.2. Let $A \subset B$ be an apartment. Let $H \subset B$ be the $SU_2(L)$ subbuilding determined by A.

An *A-stable polyhedron* $\Delta_A \subset A$ is the convex hull of two vertices of type 0 $\tau_1, \tau_2 \in A$. We write $\Delta_A = [\tau_1, \tau_2]_A$.

Let $\Delta_A = [\tau_1, \tau_2]_A$ be an A-stable polyhedron and let $v_A(\Delta_A)$ denote the center of Δ_A. Then $v_A(\Delta_A)$ is the unique point $z \in A$ such that $\mathrm{dist}(\tau_1, z) = \mathrm{dist}(\tau_2, z)$. Suppose $v_A(\Delta_A) \in \sigma \in A$. We call $\Delta = \bigcup_{g\in P_\sigma \cap SU_2(L)} g(\Delta_A)$ a *stable polyhedron*. Here $SU_2(L)$ is the group acting on $H \subset B$. We write $\Delta = [\tau_1, \tau_2]$. Note that $\Delta \subset H$.

The stable polyhedron Δ is uniquely determined by τ_1 and τ_2. If we take another apartment $A' \ni \tau_1, \tau_2$, the corresponding $SU_2(L)$ sub-building $H' \subset B$ contains Δ, that is, $\Delta \subset H \cap H'$.

The center of Δ is denoted by $v_B(\Delta)$.

We say A *determines* Δ if $\Delta = [\tau_1, \tau_2]$ with $\tau_i \in A$. Note that $v_B(\Delta) \in A$ if A determines Δ.

DEFINITION 7.3. For $g \in SU_3(L)$ we put $Y\{g(\rho_1), g(\rho_2), g(\sigma)\}_{gA} := g(Y\{\rho_1, \rho_2, \sigma\}_A)$. Now define:

$$Y\{\rho_1, \rho_2, \sigma\} := \{x \in Y\{\rho_1, \rho_2, \sigma\}_A \mid A \quad \text{determines} \quad I(x)\}.$$

If $\rho_1 = \rho_2 =: \rho$ and $\sigma \subset \rho$, then:

$$Y\{\rho, \rho, \sigma\} = \{x \in Y^s \mid I(x) \subset \rho,\ v_B(x) \in \sigma\} \supset Y^s_\sigma.$$

If $\rho_1 \neq \rho_2$ then we write $\rho_i = [\tau_1^i, \tau_2^i]$ with $\mathrm{dist}(\tau_1^1, \tau_1^2) = \mathrm{dist}(\tau_2^1, \tau_2^2) - 4$. Then

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$$Y\{\rho_1, \rho_2, \sigma\} = \{x \in Y^s \mid [\tau_1^1, \tau_1^2] \subset I(x) \subset [\tau_2^1, \tau_2^2], \quad v_B(x) \in \sigma\}.$$

We denote the covering $\{Y\{\rho_1, \rho_2, \sigma\} \mid \rho_i, \sigma \in B\}$ by $\mathscr{C}$.

Note that the affinoid covering given here differs from the one given in [9]. There are some mistakes in [9]: the covering given there is wrong. (Luckily [9] is not easily obtained outside Japan.)

PROPOSITION 7.4. *Let $R : Y\{\rho_1, \rho_2, \sigma\}_A \to R(Y\{\rho_1, \rho_2, \sigma\}_A)$ denote the canonical reduction map. Then there exists an open affine set $V \subset R(Y\{\rho_1, \rho_2, \sigma\}_A)$ such that $R^{-1}(V) = Y\{\rho_1, \rho_2, \sigma\}$.*

In particular $Y\{\rho_1, \rho_2, \sigma\}$ is affinoid and its canonical reduction is V.

PROOF. We prove the proposition by giving an explicit description of $Y\{\rho_1, \rho_2, \sigma\} \subset Y\{\rho_1, \rho_2, \sigma\}_A$ as an open subset. For a convex subset $\Delta \subset B$ we denote by $P(\Delta)^-$ the (not pointwise) stabiliser of Δ in $SU_3(L)$.

Firstly we treat the case $\rho_1 = \rho_2$. We take:

$$W := \{x \in Y\{\rho_1, \rho_1, \sigma\}_A \mid \forall (g \in P_\sigma) \ |g^* x_i / x_i(x)| = 1, \ i = 1, 2\}$$

Firstly we will show that $W = Y\{\rho_1, \rho_1, \sigma\}$. If $x \in Y\{\rho_1, \rho_1, \sigma\}_A$ then $I_A(x) \subset I(x)$ or $r_{gA,A}(x) < 1$. Hence $|g^*x_1 g^*x_2 / x_1 x_2(x)| \le 1$, since $\left|I_{gA}(x)\right| \ge |I_A(x)|$ or $r_{gA,A}(x) < 1$. If $x \in Y\{\rho_1, \rho_1, \sigma\}$ then $|g^*x_1 g^*x_2/x_1x_2(x)| = 1$ for all $g \in P_\sigma$, since $I_{gA}(x) = g(I_A(x))$. Furthermore for each $g \in P_\sigma$ there are apartments A' and A'' that correspond to the coordinates g^*x_1, x_2 and g^*x_2, x_1, respectively. Therefore if x is in $Y\{\rho_1, \rho_1, \sigma\}_A$ then for all $g \in P_\sigma$ $|g^*x_i/x_i(x)| \le 1$, $i = 1, 2$, and furthermore $W = Y\{\rho_1, \rho_1, \sigma\}$. Now it is clear that $R(W)$ is an open affine subset of $R(Y\{\rho_1, \rho_1, \sigma\}_A)$. Clearly $W = R^{-1}(R(W))$, so the statement is true if $\rho_1 = \rho_2$.

Suppose $\rho_1 \neq \rho_2$. As before we write $\rho_i = [\tau_1^i, \tau_2^i]$ with $\mathrm{dist}(\tau_1^1, \tau_1^2) + 4 = \mathrm{dist}(\tau_2^1, \tau_2^2)$. The set of extremal vertices of the stable polyhedron $[\tau_1^1, \tau_1^2]$ will be denoted by $\mathscr{V} := \{V_1 \cdots V_S\}$. Let $A_i \subset H$ denote an apartment containing V_i. Now the proof is similar to in the case $\rho_1 = \rho_2$ using the following subsets of $Y\{\rho_1, \rho_2, \sigma\}_A$:

$$\begin{aligned}
Z_0 &:= Y\{\rho_1, \rho_2, \sigma\}_A \\
Z_1 &:= \{x \in Z_0 \mid \quad v_H(x) = v_A(x)\} \\
&= \{x \in Z_0 \mid \forall (g \in P_\sigma \cap SU_2(L)) \ |g^* x_i / x_i(x)| = 1, \quad i = 1, 2\} \\
Z_2 &:= \{x \in Z_1 \mid \forall (H' \supset [\tau_2^1, \tau_2^2]) \ v_{H'}(x) = v_H(x)\} \\
&= \{x \in Z_1 \mid \forall (g \in P([\tau_2^1, \tau_2^2])^- \cap P_\sigma) \, |g^* x_i / x_i(x)| = 1, \quad i = 0, 1, 2\} \\
Z_3 &:= \{x \in Z_2 \mid I(x) = I_H(x)\} \\
&= \{x \in Z_2 \mid \forall (V \in \mathscr{V}) \forall (g \in P_V) g A_j \cap H = \{V\} \text{ implies } \left|g^* x_1 g^* x_2 / x_0^2(x)\right| = 1\} \\
&= Y\{\rho_1, \rho_2, \sigma\}.
\end{aligned}$$

Now $R(Z_3) \subset R(Z_2) \subset R(Z_1) \subset R(Z_0)$. Furthermore $R(Z_i) \subset R(Z_{i-1}), i = 1, 2, 3$ is an open and affine subset and $R^{-1}(R(Z_i)) = Z_i$. This proves the proposition.

THEOREM 7.5. (1) $\bigcup_{\rho_1, \rho_2, \sigma \in B} Y\{\rho_1, \rho_2, \sigma\} = Y^s$.
(2) *The covering* $\mathscr{C} := \{Y\{\rho_1, \rho_2, \sigma\} \mid \rho_1, \rho_2, \sigma \in B\}$ *is pure.*
(3) *The reduction of* Y^s *with respect to the covering* $\mathscr{C}$ *consists of proper components.*
(4) *The components of the reduction are in 1-1 correspondance with the stable polyhedra.*

PROOF. The first statement is evident from the construction. The second statement follows from Proposition 7.4 and the fact that the covering $\mathscr{C}_A$ is pure (Proposition 3.2(3)). Statement (3) follows from statement (1) as in [4, proof of 3.6 part 5]. The last statement follows from Propositions 3.2(5) and 7.4.

8. The reduction of Y^s

8.1. In this section we describe the reduction of Y^s. We firstly determine the reduction of Y^s_A with respect to the pure affinoid covering $\mathscr{C}_A$. Then we use the stabilisers of the components of the reduction to determine the reduction of Y^s with respect to $\mathscr{C}$.

DEFINITION 8.2. For a stable polyhedron $\Delta \subset B$ such that the apartment $A \subset B$ determines Δ we define:

$$Y(\Delta)_A := \{x \in Y^s_A \mid I_A(x) = \Delta \cap A\};$$
$$Y(\Delta) := \{x \in Y^s \mid I(x) = \Delta\};$$
$$P(\Delta)^- := \{g \in SU_3(L) \mid g(\Delta) = \Delta\}.$$

The canonical reduction of $Y(\Delta)_A$, $Y(\Delta)$ etcetera will be denoted by $\overline{Y(\Delta)_A}$, $\overline{Y(\Delta)}$, and so on.

Note that if $\Delta \cap A$ is the convex hull of the vertices τ_1 and τ_2, then $Y(\Delta)_A = Y\{\tau_1, \tau_2, \sigma\}_A$. Here the chamber σ is choosen in such a way that $v_A(\Delta \cap A) \in \sigma$.

The following proposition is rather obvious and we omit the proof.

PROPOSITION 8.3. (1) $Y(\Delta)_A \subset Y\{\rho_1, \rho_2, \sigma\}_A$ *if and only if* $\Delta \cap \rho_1 \neq \emptyset$, $\Delta \cap \rho_2 \neq \emptyset$ $v_B(\Delta) \in \sigma$ *and* $\Delta \cap A \subset$ *convex hull of* $\{\rho_1, \rho_2\}$.
(2) $Y(\Delta)_A = \cap Y\{\rho_1, \rho_2, \sigma\}_A$ *with* $\Delta \cap \rho_1 \neq \emptyset$, $\Delta \cap \rho_2 \neq \emptyset$ *and* $v_B(\Delta) \in \sigma$ *and* $\Delta \cap A \subset$ *convex hull of* $\{\rho_1, \rho_2\}$.

(3) *$Y(\Delta)_A \subset Y\{\rho_1, \rho_2, \sigma\}_A$ implies $\overline{Y(\Delta)_A}$ is in the component of $\overline{Y\{\rho_1, \rho_2, \sigma\}_A}$ corresponding to Δ_A. Furthermore $\overline{Y(\Delta)_A}$ is open and affine in $\overline{Y\{\rho_1, \rho_2, \sigma\}_A}$ and $Y(\Delta)_A = R^{-1}(\overline{Y(\Delta)_A})$.*

(4) *$Y(\Delta) = \bigcap_{g\in P(\Delta)^-} g(Y(\Delta)_A) = \bigcap_{g \in P(\Delta)^-} Y(\Delta)_{gA}$.*

(5) *$Y(\Delta) = \{x \in Y(\Delta)_A \mid |g^* x_i / x_i(x)| = 1, i = 1, 2\ \forall g \in P(\Delta)^-\}$.*

(6) *$\overline{Y(\Delta)} \subset \overline{Y(\Delta)_A}$ is open and affine and $R^{-1}(\overline{Y(\Delta)}) = Y(\Delta)$, where $R : Y(\Delta)_A \to \overline{Y(\Delta)_A}$ is the canonical reduction map.*

DEFINITION 8.4. Before describing the reduction of Y_A^s we need some definitions. Let A be the standard apartment and let the vertices correspond with the integers as before.

For an A-stable polyhedron $\Delta_A = [2n, 2m]_A$, the *centre* $v_A(\Delta)$ and its *length* $|\Delta_A|$ are given by $v_A(\Delta_A) = (n + m)$, $\quad |\Delta_A| = |2m - 2n|$.

We put a simplicial structure on the set of A-stable polyhedra. The collection of simplices will be denoted by $\mathscr{P}_A$. The elements of $\mathscr{P}_A$ are the non-empty subsets of the following sets:

$$\{\Delta_A^1, \Delta_A^2, \Delta_A^3 \mid \Delta_A^1 \subset \Delta_A^2 \subset \Delta_A^3, v_A(\Delta_A^1) = v_A(\Delta_A^2), \left|\Delta_A^3\right| = \left|\Delta_A^2\right| + 2 = \left|\Delta_A^1\right| + 4\}$$

Note that the triangles in the picture of $\mathscr{C}_A$ correspond to the maximal simplices.

PROPOSITION 8.5. *Let $\Delta_A = [2i, 2j]_A$ with $i \le j$. The component $X(\Delta_A)$ of the reduction of Y_A^s corresponding to Δ_A is a $\mathbb{P}^2_{\bar{L}}$ with a point blown up for the Δ'_A such that $\{\Delta_A, \Delta'_A\} \in \mathscr{P}_A$ and $|\Delta_A| = \left|\Delta'_A\right| + 4$ or $\left|\Delta'_A\right| = |\Delta_A| - 2$.*

The intersections with the other components of the reduction are:

(1) *$X(\Delta_A) \cap X(\Delta'_A)$ is an exceptional line in $X(\Delta_A)$ if:*

$$\{\Delta_A, \Delta'_A\} \in \mathscr{P}_A \quad \text{and} \quad \left|\Delta'_A\right| = |\Delta_A| + 4 \quad \text{or} \quad \left|\Delta'_A\right| = |\Delta_A| - 2;$$

(2) *$X(\Delta_A) \cap X(\Delta'_A)$ is an ordinary line in $X(\Delta_A)$ if:*

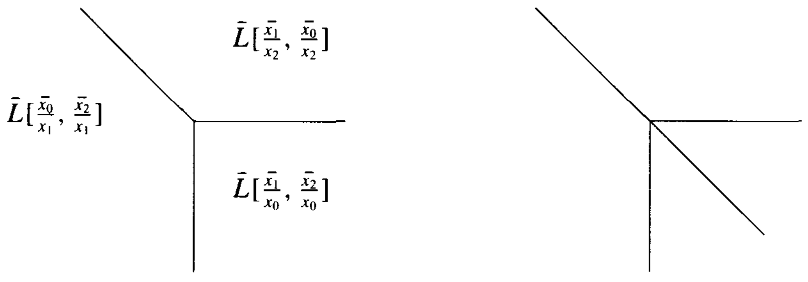
$$\{\Delta_A, \Delta'_A\} \in \mathscr{P}_A \quad \text{and} \quad \left|\Delta'_A\right| = |\Delta_A| - 4 \quad \text{or} \quad \left|\Delta'_A\right| = |\Delta_A| + 2;$$

(3) *$X(\Delta_A) \cap X(\Delta'_A) \cap X(\Delta''_A)$ is a point if and only if $\{\Delta_A, \Delta'_A, \Delta''_A\} \in \mathscr{P}_A$.*

(4) *$X(\Delta_A) \cap X(\Delta'_A) = \emptyset$ if $\{\Delta_A, \Delta'_A,\} \notin \mathscr{P}_A$.*

PROOF. We firstly treat the case $\Delta_A = [0, 0]_A$. One calculates $X(\Delta_A)$ using torus embeddings (see [6]). The picture of the covering $\mathscr{C}_A$ more or less directly gives $(X(\Delta_A)$.

The first picture (see below) shows that the affines of the reduction glue together in a $\mathbb{P}^2_{\bar{L}}$, corresponding to $\text{Proj}(\bar{L}[\bar{x}_0, \bar{x}_1, \bar{x}_2])$. The actual picture at $\Delta_A = [0, 0]_A$ is a subdivision of this picture. This extra line gives a blow up of a point. In our case the point is given by $\bar{x}_1/x_0 = \bar{x}_2/x_0 = 0$. Hence we find $X([0,0]_A)$ is a $\mathbb{P}^2_{\bar{L}}$ with a point blown up.



The line connecting $[0,0]_A$ with $[-2,2]_A$ in the picture (see 3.1) gives the intersection $X([0,0]_A) \cap X([-2,2]_A)$. It corresponds to the exceptional line in $\mathbb{P}^2_{\bar{L}}$. Furthermore $X([0,2]_A) \cap X([0,0]_A)$ and $X([-2,0]_A) \cap X([0,0]_A)$ are ordinary lines in $X([0,0]_A)$.

The intersection $X([0,0]_A) \cap X(\Delta_A) \cap X(\Delta'_A)$ is a point if there exists an affinoid F in $\mathscr{C}_A$ such that $[0,0]_A$, Δ_A and Δ'_A correspond to vertices of F in the picture. Hence $\{[0,0], \Delta_A, \Delta'_A\} \in \mathscr{P}_A$.

If $\{[0,0]_A, \Delta_A\} \notin \mathscr{P}_A$, then there does not exist an affinoid in the covering $\mathscr{C}_A$ having a component corresponding to both A-stable polyhedra. Hence the intersection of the corresponding components is empty.

From this one concludes that the proposition is true for $\Delta_A = [0,0]_A$. For the other Δ_A that are vertices $[2n, 2n]_A$ the situation is exactly the same.

If Δ_A is not a vertex of A then the picture around the component Δ_A has two more lines in it. Hence the component $X(\Delta_A)$ is a $\mathbb{P}^2_{\bar{L}}$ with $(1,0,0)$, $(0,1,0)$ and $(0,0,1)$ blown up. Now one proves the proposition in a similar vein as for $\Delta_A = [0,0]_A$. This concludes the proof.

DEFINITION 8.6. We take $\mathscr{C}'_A := \{Y\{\rho_1, \rho_2, \sigma\} \mid \rho_1, \rho_2, \sigma \in A\}$ and we denote by $X(\Delta)_A$ the component of the reduction with respect to this covering belonging to Δ_A. Here Δ is a stable polyhedron such that A determines Δ. Hence Δ is uniquely determined by $\Delta_A = \Delta \cap A$.

Since $\overline{Y\{\rho_1, \rho_2, \sigma\}} \subset \overline{Y\{\rho_1, \rho_2, \sigma\}_A}$ is open and affine, $X(\Delta)_A \subset X(\Delta \cap A)$ is a Zariski open subset.

The action of $g \in P(\Delta)^-$ on $Y(\Delta)$ induces an action $\bar{g}$ on the reduction $\overline{Y(\Delta)}$ of $Y(\Delta)$. We put $\overline{P(\Delta)}^- := \{\bar{g} \mid g \in P(\Delta)^-\}$. The action of $\overline{P(\Delta)}^-$ on $\overline{Y(\Delta)}$ can be extended to the component $X(\Delta)$. Here $X(\Delta)$ is the component corresponding to Δ of the reduction of Y^s with respect to $\mathscr{C}$. Clearly $\overline{Y(\Delta)} \subset X(\Delta)$. Furthermore one

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has $X(\Delta)_A \supset \overline{Y(\Delta)}$. In fact one has:

PROPOSITION 8.7. $X(\Delta) = \bigcup_{g \in P(\Delta)^-} X(\Delta)_{gA} = \bigcup_{\bar{g} \in \overline{P(\Delta)^-}} \bar{g}(X(\Delta)_A)$.

PROOF. This is clear since $P(\Delta)^-$ acts transitively on the apartments A such that A determines Δ, that is, $|\Delta \cap A|$ is maximal.

PROPOSITION 8.8. *Let Δ be a stable polyhedron and let A determine Δ. Then $X(\Delta)_A \subset X(\Delta \cap A)$ is the open subset obtained by omitting the images of the lines $g^* x_i = 0$, $i = 1, 2$, $g \in P(\Delta)^-$, that do not coincide with the images of $x_1 = 0$ or $x_2 = 0$ in $X(\Delta \cap A)$.*

PROOF. This follows more or less directly from the description of $Y\{\rho_1, \rho_2, \sigma\} \subset Y\{\rho_1, \rho_2, \sigma\}_A$ given in the proof of Proposition 7.4.

DEFINITION 8.9. For a stable polyhedron Δ we put $|\Delta| = \max_A |\Delta \cap A|$. Furthermore we put a simplicial structure on the set of stable polyhedra. The collection $\mathscr{P}$ of simplices has as its elements the non-empty subsets of the sets:

$$\{\Delta_1, \Delta_2, \Delta_3 \mid \exists (A \subset B)\, |\Delta_i \cap A| = |\Delta_i| \quad \text{and} \quad \{\Delta_1 \cap A, \Delta_2 \cap A, \Delta_3 \cap A\} \in \mathscr{P}_A\}.$$

PROPOSITION 8.10. *Let A determine the stable polyhedron Δ. If $\Delta = [2i, 2j]$ with $i \leq j$ then the component of the reduction $X(\Delta)$ belonging to Δ consists of a $\mathbb{P}^2_{\bar{L}}$ with a point blown up for each Δ' such that $\{\Delta, \Delta'\} \in \mathscr{P}$ and $|\Delta'| = |\Delta| + 4$ or $|\Delta'| = |\Delta| - 2$.*

The intersections with other components are as follows:

(1) *$X(\Delta) \cap X(\Delta')$ is an exceptional line in $X(\Delta)$ if:*

$$\{\Delta, \Delta'\} \in \mathscr{P} \quad \textit{and} \quad |\Delta'| = |\Delta| + 4 \quad \textit{or} \quad |\Delta| - 2.$$

(2) *$X(\Delta) \cap X(\Delta')$ is an ordinary line in $X(\Delta)$ if:*

$$\{\Delta, \Delta'\} \in \mathscr{P} \quad \textit{and} \quad |\Delta'| = |\Delta| - 4 \quad \textit{or} \quad |\Delta'| = |\Delta| + 2.$$

(3) *$X(\Delta) \cap X(\Delta') \cap X(\Delta'')$ is a point if and only if $\{\Delta, \Delta', \Delta''\} \in \mathscr{P}$.*

(4) *$X(\Delta) \cap X(\Delta') = \emptyset$ if $\{\Delta, \Delta'\} \notin \mathscr{P}$.*

PROOF. This follows directly from the previous propositions using the fact that $\overline{P(\Delta)^-}$ acts linearly on the $\mathbb{P}^2_{\bar{L}}$.

REMARK 8.11. One can embed $SU_3(L)$ into $SL_3(L)$. The maximal K-split torus $S(K) \cong L^*$ of $SU_3(L)$ is contained in a unique maximal L-split torus $T \subset SL_3(L)$, which again acts diagonally on $\mathbb{P}^2_L$ with respect to the coordinates x_0, x_1, x_2. Hence $S(K)$ determines a unique apartment A in the building B_3 of $SL_3(L)$.

Let $\mathscr{H} := \bigcup_{g \in SU_3(L)} g \cdot A \subset B_3$. Then $\mathscr{H}$ is a convex subcomplex of B_3. The vertices of $\mathscr{H}$ correspond 1-1 with the $SU_3(L)$-images of $[< e_0, \pi^n e_1, \pi^m e_2 >]$, $n, m \in \mathbb{Z}$. The maximal simplices are triangles. The three equivalence classes $[\tilde{N}_i]$, $i = 1, 2, 3$, correspond to a maximal simplex, if there exist representatives $N_i \in [\tilde{N}_i]$ such that $N_1 \supset N_2 \supset N_3 \supset \pi N_1$.

Let $Y_{\mathscr{H}} := \{z \in Y^s \mid \forall (g \in SU_3(L)) \quad g^* x_0(z) \neq 0\}$. Since $\mathscr{H} \subset B_3$ is convex, there exists a formal scheme for $Y_{\mathscr{H}}$ whose closed fibre consists of a proper component for each vertex of $\mathscr{H}$ (See [5]). These components are of the form $\mathbb{P}^2_{\bar{L}}$ with some points blown up.

One can associate to each stable polyhedron Δ a unique equivalence class $[M_\Delta]$ of L^0 modules as follows. Suppose A determines Δ. We can find an $x \in Y^s$ such that $I(x) = \Delta$. Then $n_i := v(x_i/x_0) \in \mathbb{Z}$ for $i = 1, 2$. The integers n_i only depend on Δ. Then we define $M_\Delta := < e_0, \pi^{n_1} e_1, \pi^{n_2} e_2 >$. By construction $n_1 + n_2 \geq 0$. This gives a unique equivalence class $[M_\Delta]$ for Δ. The stabilizer of M_Δ in $SU_3(L)$ is the group $P(\Delta)^-$. One easily sees that our simplicial structure $\mathscr{P}$ on the set of stable polyhedra corresponds with the simplicial structure of the modules $[M_\Delta]$ coming from $\mathscr{H}$.

So we can embed the set of stable polyhedra simplicially into the building B_3 of $SL_3(L)$. One now easily concludes that the affinoids $Y\{\rho_1, \rho_2, \sigma\}$, $\rho_1 \neq \rho_2$, in the covering $\mathscr{C}$ of Y^s that correspond to triangles in the picture (See 3.1) are exactly the same as the affinoids that go with the corresponding chamber in the $SL_3(L)$-building in the affinoid covering of $Y_{\mathscr{H}}$. Moreover the component of the reduction of Y^s corresponding to the stable polyhedron Δ is the same as the component of the reduction of $Y_{\mathscr{H}}$ associated to M_Δ, if Δ is not a vertex of type 0 in B. The components do differ if Δ is a vertex.

REMARK 8.12. The component of the reduction of Y^s belonging to Δ is a $\mathbb{P}^2_{\bar{L}}$ with some points blown up. The number of points blown up is as follows.
If L/K is unramified the number of points blown up is:

$q^2(q^2 - q + 1)$	if	Δ	is a vertex.
$q^4 + q + 1$	if	Δ	is not a vertex.

If L/K is ramified then the numbers are as follows:

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$q(q+1)/2$	if	Δ	is a vertex.		
q^2+q+1	if	Δ	is not a vertex and	$v_B(\Delta)$	is a vertex of type 1.
q^2+2	if	Δ	is not a vertex and	$v_B(\Delta)$	is a vertex of type 0.

For the convenience of the reader we give in the table below the number of simplices $\{\Delta, \Delta'\} \in \mathscr{P}$. We assume $|\Delta| = |\Delta \cap A|$. Each $P(\Delta)^-$ orbit is determined by the difference $|\Delta'| - |\Delta|$. We give the number of elements in the $P(\Delta)^-$ orbit. They differ if L/K is ramified or not.

Δ	$\lvert\Delta'\rvert - \lvert\Delta\rvert$	L/K unramified	L/K ramified
$[2i, 2i] = [2i]$	2	q^3+1	$q+1$
	4	$q^2(q^2-q+1)$	$q(q+1)/2$
$[2i, 2i+2]$	-2	$q+1$	$q+1$
	2	$q^2(q+1)$	$q(q+1)$
	4	q^4	q^2
$[2i, 2j]$, $i < j$	-4	1	1
$i \neq j \mod 2$	-2	$q+1$	$q+1$
$j \neq i+1$	2	$q^2(q+1)$	$q(q+1)$
	4	q^4	q^2
$[2i, 2j]$, $i < j$	-4	1	1
$i = j \mod 2$	-2	$q+1$	2
	2	$q^2(q+1)$	$2q$
	4	q^4	q^2

8.13. The quotient Y^s/Γ. Let $\Gamma \subset SU_3(L)$ be a torsion-free discrete co-compact subgroup. The quotient Y^s/Γ is a separated rigid analytic space. Since Γ has infinitely many orbits on the components of the reduction of Y^s, the quotient is not proper. Moreover I would guess that the quotient itself cannot be compactified. To explain this, consider an easier example.

Let F be the subgroup SL_2 of SU_3 that preserves the quadratic form $g(x) = x_1x_2 + x_0^2$. We also let F act diagonally on $\mathbb{P}^1_{K^0} \times \mathbb{P}^1_{K^0}$. One has an F-equivariant map $\psi : \mathbb{P}^1 \times \mathbb{P}^1 \longrightarrow \mathbb{P}^2$ given by: $\psi(y, z) = (-(y_1z_2 + y_2z_1), 2y_1z_1, 2y_2z_2)$. The map ψ is 2:1 and ramifies along $g(x) = 0$. We will also denote $\psi \otimes K$ by ψ.

Let $S(K) \subset F(K) \subset SU_3(L)$ be a maximal K-split torus. Let $A \subset B$ be the apartment belonging to S. We put $Y_F^s := \bigcap_{g \in F(K)} g(Y_A^s)$ and $Y_F^{ss} := \bigcap_{g \in F(K)} g(Y_A^{ss})$. We also define $\tilde{Y}_F^{ss} := Y_F^{ss} - \{g((1,0,0)) | g \in F(K)\}$, where $(1,0,0)$ is the point fixed by S. Let $\Omega_1 := \mathbb{P}^1 \otimes K - \mathbb{P}^1(K)$ be Drinfeld's symmetric space. One easily proves the following:

(1) $\psi^{-1}(Y_F^s) = \Omega_1 \times \Omega_1$.
(2) $\psi^{-1}(\tilde{Y}_F^{ss}) = (\mathbb{P}_K^1 \times \Omega_1) \cup (\Omega_1 \times \mathbb{P}_K^1)$.
(3) $\psi(\mathbb{P}_K^1 \times \Omega_1) = \tilde{Y}_F^{ss}$.

Let $\Gamma_F \subset F(K)$ be a discrete co-compact subgroup. Then $(\mathbb{P}_K^1 \times \Omega_1)/\Gamma_F$ is a projective variety, whereas Y_F^s/Γ_F is a separated non-proper variety. The restriction of the map ψ to $\psi^{-1}(\tilde{Y}_F^{ss})$ is a finite rigid analytic map. Since $\psi^{-1}(\tilde{Y}_F^{ss})/\Gamma_F$ is not separated, the same is true of $\tilde{Y}_F^{ss}/\Gamma_F$. So it appears very difficult (impossible?) to compactify the quotient Y_F^s/Γ_F. However $\psi^{-1}(Y_F^s)/\Gamma_F$ is an open subspace of $(\mathbb{P}_K^1 \times \Omega_1)/\Gamma_F$.

Let $I_F(x) := \{z \in B_F | \forall (A \subset B_F \wedge A \ni z) z \in I_A(x)\}$, where $B_F := \bigcup_{g \in F(K)} g \cdot A \subset B$ is the building of $F(K)$. Let $\mathscr{Z}_F := \mathbb{P}_K^1 \times \Omega_1 - \{(y, z) \in \Omega_1 \times \Omega_1 | I(y)_{\mathbb{P}^1} = I(z)_{\mathbb{P}^1}\}$, where $I(-)_{\mathbb{P}^1}$ is as in 5.2. The map $(y, z) \longrightarrow (\psi(y, z), I(z)_{\mathbb{P}^1})$ identifies $\mathscr{Z}_F$ with the set of pairs (x, p) with $x \in \tilde{Y}_F^{ss}$ such that $|I_F(x)| > 0$ and p an extremal point of $I_F(x)$. So $\mathscr{Z}_F$ can be constructed from $\tilde{Y}^{ss}$.

The case of Y^s/Γ seems to be similar. One takes $\tilde{Y}^{ss} := Y^{ss} - \{g((1, 0, 0)) | g \in SU_3(L)\}$. One defines a space $\mathscr{Z}$ as consisting of the pairs (x, p) with $x \in \tilde{Y}^{ss}$ and $I(x)$ not a point and p an extremal point of $I(x)$. Now it is hoped that $\mathscr{Z}/\Gamma$ can be compactified (instead of Y^s/Γ).

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https://doi.org/10.1017/S1446788700001075 Published online by Cambridge University Press